No regret control for Heat equation with delay and incomplete data

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Abstract: The objective of this paper is to study the optimal control of distributed systems with incomplete data. Particularly control for the heat equation with missing initial condition and delay parameter. The low-regret control seems to be the best method to solve this kind of problems which characterized by an optimality systems.

Key-Words: Distributed system ; Incomplete data ; Low-regret control ; Optimality system.

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1 Statement of problem

This paper is devoted to study the notion of no-regret control for the heat equation with delay time and missing data. First, we proof the existence and the uniqueness of the solution of this problem by the notion of semigroup. Then, we show the approximate optimality system of the low-regret control.

Let Ω be an open bounded set of R^n with smooth boundary $\partial\Omega$. Consider the heat equation with Dirichlet boundary condition, with $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times [0, T]$ and $Q_\tau = \Omega \times (0, \tau)$:

$$\begin{cases} Ly(x,t) - By(x,t-\tau) = v\chi_\omega + f, in\ Q, \\ y(x,t) = 0, on\ \Sigma, \\ y(x,0) = y_0(x), in\ \Omega, \\ By(x,t-\tau) = g(x,t), in\ Q_\tau, \end{cases} \quad (1)$$

Where —

$L = \frac{\partial}{\partial t} - \Delta$,

y is the variable in the state space H, t is the time and v is a control function and $f \in L^2$ is a source function,

χ_ω is the characteristic function of ω a bounded open of Ω and $\tau > 0$ is a delay parameter,

The intial data $y_0 \in H$, $g \in C([0,T]; H)$,

B is fixed bounded operator of H in itself.

We then have a possible state for which , we attach a cost function given by :

$J(v,g) = \int_0^T \|cy(v,g) - y_d\|_H^2 dt + N\int_0^T \|v\|_{\mathcal{U}}^2 dt$,

where

$v \in \mathcal{U} = L^2(\omega)$ and $g \in G$ such that G is non empty closed subspace in $L^2(Q_\tau)$,

$y_d \in L^2(Q)$, $N > 0$ and $c \in \mathcal{L}(V, H)$ $(V isa Hilbert space)$.

$y = y(v, g)$ is the unique solution of the problem (1) in $L^2(Q)$.

The idea is to find a solution for the following inf sup problem :

$\inf(\sup(v,g))$, $v \in \mathcal{U}$ and $g \in G$, but J is not bounded i.e. $\sup(v,g) = +\infty, g \in G$.

Question

Can we controlled the optimal control problem (1) by the no regret control ?

Objective : Our goal is to show the existence and uniqueness of the solution of problem(1) .The main results are given in section (3.4) , and introduce the notion of low-regret control.

Finally, in section (3.6) and (3.7) we give the perturbed optimality system and by passing to the limit in the optimality system associated to the perturbed problem we obtain the optimality system singular.

2 The existence and the uniqueness of the solution

To study the existence and uniqueness of the solution for a heat equation with delay, based on the work of ([20]) and ([36]) .

We consider an operator $A = \Delta$ from H into itself that generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable i.e. there exist two positive constants M and w such that $\mathrm{S}(t) \leq M \exp(-wt), \forall t \geq 0$.

For any initial $y_0 \in H$, there exists a unique solution $y \in C(L^2(\Omega \times [0,T), H))$ of problem (1) .

Moreover,

$$\begin{cases} y(t) = S(t)y_0 + \int_0^t S(t-s) \begin{bmatrix} v(s)\chi_\omega \\ +By(s-\tau) + f(s) \end{bmatrix} ds, \\ By(t-\tau) = g(t)\ in\ (0,\tau). \end{cases} \quad (2)$$

We use an iterative argument. Namely in the interval $(0, \tau)$, problem (1) can be seen

as an inhomogeneous evolution problem :
$$\begin{cases} y' - \Delta y - g_1(t) = v\chi_\omega + f, inQ, \\ y = 0, on\Sigma, \\ y(0) = y_{0,} in\Omega, \end{cases}$$
where $g_1(t) = g(t)$ in $(0,\tau)$.

For all $y_0 \in L^2(\Omega)$ and $v \in L^2(0,\tau;L^2(\Omega))$ this problem has a unique solution $y \in C(L^2(\Omega\times(0,\tau)),H)$ $(see"Th.1.2, Ch.6 of Pazy([20])")$ satisfying :
$y(t) = S(t)y_0 + \int_0^t S(t-s)[v(s)\chi_\omega + g_1(s) + f(s)]\,ds$, $\forall t \in [0,\tau]$,

this solution $y(t)$ is obtained, for $t \in [0,\tau]$ therefore on $(\tau,2\tau)$, problem (1) can be seen as an inhomogeneous evolution problem :
$$\begin{cases} y' - \Delta y - g_2(t) = v\chi_\omega + f, inQ, \\ y = 0, on\Sigma, \\ y(\tau) = y_\tau, \end{cases}$$
where $g_2(t) = By(t-\tau)$ in $(\tau,2\tau)$.

hence, this problem has a unique solution $y \in C(L^2(\Omega\times[\tau,2\tau]),H)$ given by
$y(t) = S(t)y_\tau + \int_0^t S(t-s)[v(s)\chi_\omega + g_2(s) + f(s)]\,ds$, $\forall t \in [\tau,2\tau]$,

by iteration, we obtain a global solution y satisfying (2).

And so, we make sure the existence and the uniqueness of the solution for each continuous initial function.

If $g \in C(0,T;H)$, with H a Hilbert space, then the solution $y \in C((0,T);H)$ of problem (1) satisfies (2).

Let be a classical solution of problem (1) and $(S(t))_{t\geq 0}$ a C_0−semigroup.

For all $t \in [0,T]$, consider the function $\Psi : (0,t) \longrightarrow H$ defined by : $\Psi(s) = S(t-s)y(s), 0 < s < t$.

Since $y \in H^2(\Omega)$ the function $\tau \longrightarrow S(\tau)y(\tau)$ is differentiable for any $\tau > 0$.

Therefore, Ψ is differentiable on $(0,t)$ and we have : $\Psi'(s) = -S'(t-s)y(s) + S(t-s)y'(s)$
$= -S(t-s)\Delta y(s)$ [1] $+$
$S(t-s)[\Delta y(s) + By(s-\tau) + v(s)\chi_\omega + f(s)]$
$= -S(t-s)\Delta y(s) + S(t-s)\Delta y(s) +$
$S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]$
$= S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]$.

By integrating with respect to s, we finds
$\int_0^t \Psi'(s)\,ds = \int_0^t S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]\,ds$,
so,
$\Psi(t) - \Psi(0) = \int_0^t S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]\,ds$,

this is equivalent to,
$\Psi(t) = \Psi(0) + \int_0^t S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]\,ds$,
as a result, $(\Psi(s) = S(t-s)y(s))$
$S(t-t)y(t) = S(t-0)y(0) + \int_0^t S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]\,ds$,
this implies that, $(S(0) = I)$
$y(t) = S(t)y(0) + \int_0^t S(t-s)[By(s-\tau) + v(s)\chi_\omega + f(s)]\,ds$.

3 No regret control

In this instant, we are interested in the study of the existence and characterization of no regret control. For this, we assume that : $[0,T] = \cup[(l-1)\tau, l\tau], l = \overline{1,n}$.

We study the no regret control for $t \in (0,\tau)$ of the following problem :
$$\begin{cases} y_1' - \Delta y_1 = v\chi_\omega + g + f, inQ, \\ y_1 = 0, on\Sigma, \\ y_1(0) = y_0, in\ \Omega. \end{cases} \quad (3)$$

For a choice of v and g, the problem (3) admits a unique solution denoted $y_1(v,g) \in L^2(0,\tau;V)$.

For $t \in (0,\tau)$ fixed, and for all $g \in G$, a closed vector subspace of $L^2(Q)$, we pose :
$J_1(v,g) = \int_0^\tau \left(\|cy_1(v,g) - y_d\|_H^2 + N\|v\|_{\mathcal{U}}^2\right)dt$,
where
$N > 0$, $C \in \mathcal{L}(L^2(0;\tau,V),H)$ and $y_d \in H$ fixed.

Now, we study the optimality system (existence and characterization) of the no regret control of the problem (3), in the case where $G \neq \{0\}$, i.e., find the control for the problem
$$\inf\sup\left(J_1(v,g) - J_1(0,g)\right), \quad (4)$$

We introduce low-regret control for developing the no regret control of problem (3). low-regret control which is an approximation of no regret control.

4 low-regret control

In the previous chapter, we define the low-regret control by relaxed the problem (4) as follows :
$$\inf\sup\left(J_1(v_1,g) - J_1(0,g) - \gamma\|g\|_G^2, \gamma > 0\right). \quad (5)$$

With the low-regret control, we admit the possibility of making a choice v_1 controls, the best possible choice of v is then given by :

starting by a linear case by relaxing the function
$$J_1(v_1,g) - J_1(0,g) = J_1(v_1,0) - J_1(0,0) + 2(S(0,v_1),g)_{V,G}, \forall g \in G,$$

[1] $S(t) = \exp(\Delta t)$
$S(t-s) = \exp\Delta(t-s)$
$S(t-s)y = \exp\Delta(t-s)y$
$-S'(t-s)y = -\Delta\exp(t-s)y = -S'(t-s)\Delta y$

let be $\gamma > 0$ a fixed number, then the relaxed problem (5) has the following formulation $: \inf\sup\begin{pmatrix} J_1(v_1,0)-J_1(0,0) \\ +2(S(0,v_1),g)_{V,G}-\gamma\|g\|_G^2\end{pmatrix}$,it is clear,

$\sup\left(2(S(0,v_1),g)_{V,G}-\gamma\|g\|_G^2\right) = \frac{1}{\gamma}\|S(v_1)\|_V^2,$

the problem (5) we obtain :

$$\inf\left(J_1(v_1,0)-J_1(0,0)+\frac{1}{\gamma}\|S(v_1)\|_V^2,\gamma>0\right). \tag{6}$$

Hence we obtain a standard control optimal.

As for the linear case, (6) also admits a solution noted u_γ, that is the low-regret control.

Let $y_1(v,g)$ be the solution to the problem (3). Then,

$y_1(v,g)=y_1(v_1,0)+y_1(0,g)-y_1(0,0),$

with simple calculation, we obtain :

$$J_1(v_1,0)-J_1(0,0)+2\Xi=J_1(v_1,g)-J_1(0,g), \tag{7}$$

such that

$\Xi=\int_0^\tau(\xi_1(v_1),y_1(0,g)-y_1(0,0))_{V,G}\,dt,$

and $c^*c(y_1(.,0)-y_1(0,0))=\xi_1(.)$.

The adjoint state $\xi_1(v_1)$ defined by $\xi_1(v_1)=c^*c(y_1(v_1,0)-y_1(0,0))$. So, (7) becomes

$$J_1(v_1,g)-J_1(0,g)=J_1(v_1,0)-J_1(0,0)+(\xi(v_1),g)_{V,G}. \tag{8}$$

Where $\xi(v_1)=S(v_1)$ and ξ_1 is the solution of problem :

$-\xi_1'-\Delta\xi_1=c^*c(y_1(v_1,0)-y_1(0,0)),\xi_1(\tau,v_1)=0.$

Hence, the problem (8) becomes for all $\gamma>0$

find $u_\gamma\in\mathcal{U}$ such that$J_1^\gamma(u_{1\gamma})=\inf J_1^\gamma(v_1),v\in\mathcal{U},$

where the now cost function is given by

$J_1^\gamma(v_1)=J_1(v_1,0)-J_1(0,0)+\frac{1}{\gamma}\|S(v_1)\|_V^2.$

5 Approximate optimal system

The low-regret control $u_{1\gamma}$ is characterized by the unique solution$\{y_{1\gamma},\xi_{1\gamma},\rho_{1\gamma},p_{1\gamma}\}$ of the optimality system :

$$\left\{\begin{aligned} y_{1\gamma}'-\Delta y_{1\gamma} &= u_{1\gamma}\chi_\omega+g_1+f,\\ y_{1\gamma}(0) &= y_0,\\ -\xi_{1\gamma}'-\Delta\xi_{1\gamma} &= c^*c(y_{1\gamma}-y_1(0,0)),\\ \xi_{1\gamma}(\tau) &= 0,\\ \rho_{1\gamma}'-\Delta\rho_{1\gamma} &= 0,\\ \rho_{1\gamma}(\tau) &= \tfrac{1}{\gamma}\xi_{1\gamma},\\ -p_{1\gamma}'-\Delta p_{1\gamma} &= c^*c(y_{1\gamma}-y_d)+c^*c\rho_{1\gamma},\\ p_{1\gamma}(\tau) &= 0,\\ p_{1\gamma}+Nu_{1\gamma} &= 0, in\ \mathcal{U}.\end{aligned}\right. \tag{9}$$

The solution $u_{1\gamma}$ satisfies the first order optimality conditions gives

$J_1'^\gamma(u_{1\gamma})(v_1-u_{1\gamma})\geq 0,\forall v_1\in\mathcal{U},$

where

$J_1'^\gamma(u_{1\gamma})(v_1-u_{1\gamma}) = \lim\frac{1}{t}(J_1^\gamma(u_{1\gamma}+t(v_1-u_{1\gamma}))-J_1^\gamma(u_{1\gamma}))\geq 0.$

Simple calculations gives

$\frac{1}{t}(J_1^\gamma(u_{1\gamma}+t(v_1-u_{1\gamma}))-J_1^\gamma(u_{1\gamma})) = t\|cy_{1\gamma}(v_1-u_{1\gamma},0)\|_H^2$
$+2\int_0^\tau(cy_{1\gamma}(u_{1\gamma},0)-y_d,cy_{1\gamma}(v_1-u_{1\gamma},0))_{H\times H}\,dt$
$+tN\|v_1-u_{1\gamma}\|_{\mathcal{U}}^2 + 2N\int_0^\tau(u_{1\gamma},v_1-u_{1\gamma})_{\mathcal{U}\times\mathcal{U}}\,dt$
$+\frac{t}{\gamma}\|S(v_1-u_{1\gamma})\|_V^2 + 2\left(\frac{1}{\gamma}S(u_{1\gamma}),S(v_1-u_{1\gamma})\right)_{G\times G}.$

Make t tend to 0 to get

$J_1'^\gamma(u_{1\gamma})(v_1-u_{1\gamma}) =$
$2\int_0^\tau(cy_{1\gamma}(u_{1\gamma},0)-y_d,cy_{1\gamma}(v_1-u_{1\gamma},0))_{H\times H}\,dt$
$+2N\int_0^\tau(u_{1\gamma},v_1-u_{1\gamma})_{\mathcal{U}\times\mathcal{U}}\,dt + 2\left(\frac{1}{\gamma}S(u_{1\gamma}),S(v_1-u_{1\gamma})\right)_{G\times G},$

thanks to linearity

$y_{1\gamma}(v_1-u_{1\gamma},0)=y_{1\gamma}(v_1,0)-y_{1\gamma}(u_{1\gamma},0)$
$=y_{1\gamma}(v_1-u_{1\gamma},0)-y_{1\gamma}(0,0).$

Then,

$J_1'^\gamma(u_{1\gamma})(v_1-u_{1\gamma}) =$
$2\int_0^\tau\begin{pmatrix} cy_{1\gamma}(u_{1\gamma},0)-y_d, \\ c(y_{1\gamma}(v_1-u_{1\gamma},0)-y_{1\gamma}(0,0))\end{pmatrix}_{H\times H}dt$
$+2N\int_0^\tau(u_{1\gamma},v_1-u_{1\gamma})_{\mathcal{U}\times\mathcal{U}}\,dt + 2\left(\frac{1}{\gamma}S(u_{1\gamma}),S(v_1-u_{1\gamma})\right)_{G\times G}$
$=2\int_0^\tau\begin{pmatrix} c^*c(y_{1\gamma}(u_{1\gamma},0)-y_{1\gamma}(0,0)), \\ y_{1\gamma}(v_1-u_{1\gamma},0)-y_{1\gamma}(0,0)\end{pmatrix}_{H\times H}dt$
$+2N\int_0^\tau(u_{1\gamma},v_1-u_{1\gamma})_{\mathcal{U}\times\mathcal{U}}\,dt + 2\left(\frac{1}{\gamma}S(u_{1\gamma}),S(v_1-u_{1\gamma})\right)_{G\times G},$

we recall the adjoint state defined previously by $\xi_1(u_{1\gamma}) = c^*c(y_1(u_{1\gamma},0)-y_1(0,0))$ then $\left(\frac{1}{\gamma}S(u_{1\gamma}),S(v_1-u_{1\gamma})\right)_{G\times G} = \left(\frac{1}{\gamma}\xi(u_{1\gamma}),\xi(v_1-u_{1\gamma})\right)_{G\times G}.$

6 Singular Optimality system (SOS)

we now establish the optimality system for the no regret control. For this, we need an additional hypothesis.

$Hypothesis$: Let $(\rho_{1\gamma},\delta_{1\gamma})\in(L^2(0;\tau,V))^2$ defined by : $\left\{\begin{array}{l}\rho_{1\gamma}'-\Delta\rho_{1\gamma}=0,\\ \rho_{1\gamma}(0)=g_1,g_1\in G,\end{array}\right.$

and $\left\{\begin{array}{rcl}-\delta_{1\gamma}'-\Delta\delta_{1\gamma} &=& c^*c\rho_{1\gamma},\\ \delta_{1\gamma}(\tau) &=& 0,\end{array}\right.$ and we defended the continuous operator $\mathcal{R}:\mathcal{F}\longrightarrow\mathcal{U}$ by

$\mathcal{R}g = \delta$, such that : $\|\mathcal{R}g\|_{\mathcal{U}_1} \geq c\,\|g\|_{\mathcal{F}}\,, c > 0, g \in G.$ where $\mathcal{F}$ is a vector subspace of G.

Under the hypothesis, then the no regret control u_1 for the system (3) is characterizated by the unique solution $\{y_1, \lambda_1, \rho_1, p_1\}$ of the optimality system :

$$\begin{cases} y_1' - \Delta y_1 = & u_1\chi_\omega + g_1 + f, \\ y_1(0) = & y_0, \\ -\xi_1' - \Delta\xi_1 = & c^*c\left(y_1 - y_1(0,0)\right), \\ \xi_1(\tau) = & 0, \\ \rho_1' - \Delta\rho_1 = & 0, \\ \rho_1(0) = & \lambda, \\ -p_1' - \Delta p_1 = & c^*c\left(y_1 - y_d\right) + c^*c\rho_1, \\ p_1(\tau) = & 0, \\ p_1 + Nu_1 = & 0, in\,\mathcal{U}. \end{cases} \quad (10)$$

With $\lambda \in \hat{G}$ is the completeness of G in F, containing the elements $\mathcal{R}g$.

By the following theorem : "The unique low-regret control u_γ is converge weakly when γ tends to 0 to the unique no-regret control u in $\mathcal{U}_{ad}$" the unique low-regret control $u_{1\gamma}$ converge weakly when γ tend to 0 to the unique no regret control u_1 in the approximation of the optimal system (9) we find the system (10).

We repeat the same steps on each time interval $[(l-1)\tau, l\tau]$, $l = \overline{1,n}$ we obtain a low-regret control $u_{l\gamma}$ converge weakly to the unique no regret control u_l.

Finally, the low-regret control of problem (1) is given by $u_\gamma = u_{i\gamma} \in \mathcal{U}, \forall t \in [(i-1)\tau, i\tau]\,, i = \overline{1,n}$ solution of the following problem :

$\inf J^\gamma(v) = \inf J_i^\gamma(v_i)\,, t \in [(i-1)\tau, i\tau]\,, v \in \mathcal{U}, i = \overline{1,n},$

where

$J_i^\gamma(v_i) = J_i(v_i, 0) - J_i(0,0) + \frac{1}{\gamma}\|S(v_i)\|_V^2\,,$

the low-regret control is characterizated by the unique solution $\{y_{i\gamma}, \xi_{i\gamma}, \rho_{i\gamma}, p_{i\gamma}\}$ of the approximate optimal system :

$$\begin{cases} y_{i\gamma}' - \Delta y_{1\gamma} = & u_{i\gamma}\chi_\omega + z_{i\gamma}(t,1) + f, \\ y_{i\gamma}(0) = & y_0, \\ -\xi_{i\gamma}' - \Delta\xi_{i\gamma} = & c^*c\left(y_{i\gamma} - y_i(0,0)\right), \\ \xi_{i\gamma}(\tau) = & 0, \\ \rho_{i\gamma}' - \Delta\rho_{i\gamma} = & 0, \\ \rho_{i\gamma}(\tau) = & \frac{1}{\gamma}\xi_{i\gamma}, \\ -p_{i\gamma}' - \Delta p_{i\gamma} = & c^*c\left(y_{i\gamma} - y_d\right) + c^*c\rho_{i\gamma}, \\ p_{i\gamma}(\tau) = & 0, \\ p_{i\gamma} + Nu_{i\gamma} = & 0, in\ \mathcal{U}. \end{cases} \quad (11)$$

The low-regret control u_γ converge weakly to the unique no regret control u when $\gamma \longrightarrow 0$, the no regret control u for the system (1), is characterizated by the unique solution $\{y, \lambda, \rho, p\}$ of the Singular Optimality system (SOS) :

$$\begin{cases} y' - \Delta y & = & u\chi_\omega + z(t,1) + f, \\ y(0) & = & y_0, \\ -\xi' - \Delta\xi & = & c^*c\left(y - y(0,0)\right), \\ \xi_1(i\tau) & = & 0, \\ \rho' - \Delta\rho & = & 0, \\ \rho(0) & = & \lambda, \\ -p' - \Delta p & = & c^*c\left(y - y_d\right) + c^*c\rho, \\ p(i\tau) & = & 0, \\ p + Nu & = & 0, in\mathcal{U}. \end{cases} \quad (12)$$

Where $y = y_i$, $\rho = \rho_i$, $z = z_i$, $p = p_i$, $t \in [(i-1)\tau, i\tau]$.

7 Conclusion

This paper is devoted to study the characterization of the no regret control for the heat equation with delay and missing data. First, we establish the existence and the uniqueness of the weak solution for the heat equation by introducing the notion of semigroup. Then, we associate the no-regret control by a sequence of low-regret control defined by a quadratic perturbation. We show here that the perturbed system which characterize the sequence of low-regret control converges to the no-regret control for which we get a singular optimality system. Finally, we can construct and characterize the no-regret control for the heat equation with missing data and delay time from the no-regret control method already established on each interval contained in this work.

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References:

[1] A. Ayadi, & A. Hafdallah. Averaged optimal control for some evolution problems with missing parameter. Technical Report. 2016. DOI:10.13140/RG.2.1.3002.8563.

[2] F. Achab, C. Boukouba, A. Hafdallah. Optimal control of varioud linear PDEs with incomplete data. MÃ©moire de Master, University Larbi tebessi-Tebessa. 2020.

[3] A. Benoua, T.E. Oussaief, I. Rezzoug. Unique solvability of a Dirichlet problem for a fractional parabolic equation using energy-inequality method. Methods of Functional Analysis and Topology, 2020.

[4] A. BRAIK. Sur quelques problÃ¨mes de contrÃ´le pour l'Ã©quation de la chaleur avec retard et de stabilitÃ©. MÃ©moire de magister UniversitÃ© d'Oran. 2018.

[5] A. Berhail and I. Rezzoug. Identification of the source term in Navier-Stokes system with incomplete data. AIMS Mathematics, 4(3): 516–526. DOI:10.3934/math.2019.3.516. 2019.

[6] M. Bergounioux. Optimisation dans R^n et introduction au contrÃ´le optimal des systÃ¨mes linÃ©aires. Cours et exercices. 2001.

[7] A. Bouziad, J. Calbrix. ThÃ©orie de la mesure et de l'integration. Publications de l'universitÃ© de Rouen. 1993.

[8] A. Braik. ContrÃ´le sans regret pour les systeÃ¨mes distribuÃ©s Ã donnÃ©es manquantes. mÃ©moire de magister, UniversitÃ© d'Oran. 2018.

[9] H. Brezis. Analyse fonctionnelle : ThÃ©orie et applications. Masson. 2005.

[10] A. Hafdallah., & A. Ayadi. Optimal control of some hyperbolic equations with missing data. Doctoral dissertation, university of Oum-El-Bouaghi. 2018.

[11] A. Hafdhallah. Introduction Ã la thÃ©orie de controle des systÃ¨mes. Gruadate course, university Larbi Tebessi of Tebessa. 2019.

[12] A. Hafdallah. On the optimal control of linear systems depending upon a parameter and with missing data. Nonlinear Studies, 27(2), 457-469. 2020.

[13] B. Jacob., & Omrane, A. Optimal control for age-structured population dynamics of incomplete data. Journal of mathematical analysis and applications, 370, 42-48. 2018.

[14] I. Kaarer and A. Ayadi and I. Rezzoug. Weak Controllability and the New Choice of Actuators. Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 14, Number 2 (2018), pp. 325–330.

[15] J. L. Lions. Optimal control of systems governed by partial differential equations problÃ¨mes aux limites. 1971.

[16] J. L. Lions. ContrÃ´le Ã moindres regrets des systÃ¨mes distribuÃ©s. Comptes rendus de l'AcadÃ©mie des sciences. SÃ©rie 1, MathÃ©matique, 315(12), 1253-1257. 1992.

[17] T. Laib, I. Rezzoug, T.E. Oussaief. Instantaneous Sentinel for the Identification of the pollution term in Navier-Stokes system. Proyecciones (Antofagasta, On line), 2021.

[18] O. Nakoulima. A. Omrane, & J. Velin. On the pareto control and no-regret control for distributed systems with incomplete data. SIAM journal on control and optimization, 42(4), 1167-1184. 2003.

[19] O. Nakoulima. A. Omrane, & J. Velin. Perturbations Ã moinders regrets dans les systÃ¨mes distribuÃ©s Ã donnÃ©es manquantes. SÃ©rie 1, p. 801-806, 2000.

[20] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Springer Verlag, New York, Berlin, 1983.

[21] I. Rezzoug and A. Ayadi. Weakly Sentinel involving a Navier-Stokes Problem and Detecting Pollution. General Letters in Mathematics Vol. 5, No. 2, Oct 2018, pp.93-104. https://doi.org/10.31559/glm2018.5.2.4

[22] I. Rezzoug, T.E. Oussaief. Solvability of a solution and controllability of partial fractional differential systems. Journal of Interdisciplinary Mathematics, 2021.

[23] I. Rezzoug, T.E. Oussaief and A. Benbrahim. Solvability of a solution and controllability for nonlinear fractional differential equation. Bulletin of the Institute of Mathematics Vol. 15

(2020), No. 3, pp. 237-249 DOI: 10.21915/BIMAS.2020303

[24] I. Rezzoug, T.E. Oussaief. Approximate Controllability. WSEAS TRANSACTIONS on SYSTEMS. Volume 19, 2020 . DOI: 10.37394/23202.2020.19.3

[25] I. Rezzoug, S. Dehilis and T.E. Oussaief. Boundary control of a heat equation. Asia Mathematika. Volume: 5 Issue: 1 , (2021) Pages: 28-43. DOI: doi.org/10.5281/zenodo.4722088

[26] I. Rezzoug and A. Ayadi. Controllability for the parabolic equations .PJM.Vol. 6(2) (2017) , 638–648.

[27] I. Rezzoug and A. Ayadi. Approximate Controllability for the Parabolic Equations of the Fourth Order. Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4495-4509.

[28] I. Rezzoug and A. Ayadi. Sentinels for the identification of pollution in domains with missing data. Advances in Dynamical Systems and Applications. ISSN 0973-5321, Volume 8, Number 3, (2013) pp. 439-449.

[29] I. Rezzoug and A. Ayadi. Weakly Sentinels for the Distributed System with Missing Terms and with Pollution in the Boundary Conditions. Int. Journal of Math. Analysis, Vol. 6, 2012, no. 45, 2245-2256.

[30] I. Rezzoug. Cours : thÃ©orie du contrÃ´le. UniversitÃ© d'Oum El Bouaghi. 2017.

[31] I. Rezzoug. Ã2030̈tude thÃ©orique et numÃ©rique des problÃ¨mes d'identification des systÃ¨mes gouvernÃ©s par des Ã©quations aux dÃ©rivÃ©es partielles, ThÃ¨se de doctorat, UniversitÃ© de Oum El Bouaghi-AlgÃ©rie. 2014.

[32] I. Rezzoug. Identification d'une partie de la frontiÃ¨re inconnue d'une membrane, ThÃ¨se de magister, UniversitÃ© de Oum El Bouaghi-AlgÃ©rie. 2009.

[33] I. Rezzoug and A. Ayadi. identification of an unknown part of the boundary of an Navier-Stokes system by punctual sentinel. Sciences & Technologie A – N°35, Juin. (2012), pp. 31-35.

[34] I. Rezzoug and A. Ayadi. Weakly sentinels for the distributed system with pollution terms in the boundary. FEJAM. Volume 63, Issue 1 (February 2012). Page 25-37.

[35] L. J. Savage. The foundations of statistics. Courier Corporation. 1972.

[36] S. Nicaise, C. Pignotti. Exponential stability of abstract evolution equation with time delay. arXiv : 1405.1850v2. Math. AP, 330, 2014.

[37] L. Teldja and I. Rezzoug and B. Amel. Approximate Controllability of the Stokes system. Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 14, Number 6 (2018), pp. 775-785.

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