Ostensible Metzler Linear Uncertain Systems: Goals, LMI Synthesis, Constraints and Quadratic Stability

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Abstract: - This paper deals with the design problem for a class of linear continuous systems with dynamics prescribed by the system matrix of an ostensible Metzler structure. The novelty of the proposed solution lies in the diagonal stabilization of the system, which uses the idea of decomposition of the ostensible Metzler matrix, preserving the incomplete positivity of the system during the synthesis. The proposed approach creates a unified framework that covers compactness of interval system parameter representation, Metzler parametric constraints, and quadratic stability. Combining these extensions, all of the conditions and constraints are expressed as linear matrix inequalities. Implications of the results, both for design and for research directions that follow from the proposed method, are discussed at the end of the paper. The efficiency of the method is illustrated by a numerical example.

Key-Words: - positive and incomplete positive systems, strictly Metzler systems, ostensible Metzler matrices, state feedback, interval state observers, linear matrix inequalities.

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1 Introduction

Linear time invariant systems offer many properties for their adaptation to specific control problems and give potential conditions under which they will behave in a predetermined manner. Since in practice there are often requirements for the positivity of system states, [1], [2], the synthesis of their control must take into account such state restrictions. However, the specific nature of the problem results in a design procedure tolerating the system positiveness by additional constraints, [3], [4], when focusing on the dynamical systems. Concerning on a class of linear dynamical systems with positive states, the Metzler matrix theory, [5], due to its particular structure, provides an alternative solution in the analysis and synthesis of linear positive systems, [6]. Setting the Jacobian matrix to a Metzler structure for cooperative systems, this property stays a key candidate for the use in interval observers, [7], [8].

Because of the strong nonnegative property, there are remarkable impacts that are valid only for linear positive systems. Above all, unlike general linear systems, the positive systems asymptotic or quadratic stability have to be lossless reflected by considering linear matrix inequality (LMI) principle, using positive definite diagonal matrices, [9]. These particular forms simplify stabilization analysis and allow in the same vein the design of structured and decentralized controllers and observers, [10]. A strictly LMI-based approach for design under Metzler constraints, reflecting the diagonal stabilization principle (DSP), is given in [11]. Motivated by the problem of incomplete positive observation and control design, additional insights into the analysis with Metzler parametric constraints is provided in [12], [13].

The LMIs compatibility in design of ostensible Metzler systems is presented in the paper, summarizing an algorithmic platform with relationships to system stability, incomplete internal positivity and the ostensible Metzler parametric constraints. Instead of using algebraic techniques, the approach is based on the Lyapunov matrix inequality and diagonal matrix variables, when constructing LMIs for an equivalent ostensible Metzler system matrix representation. Generalized duality in the controller and observer design task is proven to be modifiable for uncertain linear incomplete positive systems with interval ostensible Metzler parameters. To the best of the author's knowledge, this approach represents a new LMI synthesis method for this class of systems.

Online: In Section 2 the parametrisation principles of positive systems is outlined and in Section 3 short characterization of systems with ostensible Metzler system matrices is presented. Design conditions to ostensible interval observer synthesis are derived in Section 4, while the approach is illustrated by usage to a model with interval ostensible Metzler system matrices in Section 5. The presented approach and the application example are finally discussed in Section 6.

Notations: Throughout the paper $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{x}$, and the matrix $\boldsymbol{X}$, respectively, diag $[\cdot]$ marks a (block) diagonal matrix, for a square symmetric matrix $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is negative definite matrix, $\boldsymbol{I}_n$ labels the n-th order unit matrix, $\boldsymbol{X}^{-1}$, $\rho(\boldsymbol{X})$ signifies the inverse and the eigenvalue spectrum of a square matrix $\boldsymbol{X}$, the maximum real part of the eigenvalues of $\boldsymbol{X}$ is denoted by $\lambda_0 := \max_{\lambda \in \rho(X)} \Re(\lambda)$, indicator $\mathbb{R}$ ($\mathbb{R}_+$) marks the set of (nonnegative) real numbers, $\mathbb{R}^{n \times n}$ ($\mathbb{R}_+^{n \times n}$) refers to the set of (nonnegative) real matrices and $\mathbb{M}_{-+}^{n \times n}$ indicates the set of matrices with Metzler structure, for any matrix $\boldsymbol{X} = \{x_{ij}\} \in \mathbb{R}_+^{n \times n}$ then $\boldsymbol{X} > 0$, ($\boldsymbol{X} \geq 0$) denotes that $x_{ij} > 0$ ($x_{ij} \geq 0$) for all i, j.

2 Internal System Positivity

This section considers a class linear time-invariant systems described by:

$$\Xi: \quad \begin{array}{l} \dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{D}\boldsymbol{d}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \end{array} \tag{1}$$

where $\boldsymbol{q}(t) \in \mathbb{R}_+^n$, $\boldsymbol{y}(t) \in \mathbb{R}_+^m$ are positive, $\boldsymbol{d}(t) \in \mathbb{R}_+^d$ is positive and bounded, $\boldsymbol{B} \in \mathbb{R}_+^{n \times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m \times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n \times d}$ are nonnegative and $\boldsymbol{A} \in \mathbb{M}_{-+}^{n \times n}$ is Metzler. Then Ξ is noted as internally positive.

A strictly Metzler matrix $\boldsymbol{A} \in \mathbb{M}_{-+}^{n \times n}$ is characterized by its negative diagonal elements and by its strictly positive off diagonal elements. Consequently, a strictly Metzler $\boldsymbol{A}$ is so limited by n^2 parametric constraints

$$a_{ii} < 0, \quad a_{ij} > 0, i \neq j, \ \forall\, i, j \in \langle 1, n \rangle \tag{2}$$

and, to guaranty (2) in design, DSPs have to be used.

Theorem 1 (Metzler matrix parametrisation, [11]). *Let $\boldsymbol{A} \in \mathbb{M}_{-+}^{n \times n}$ be strictly Metzler, then the following two expressions for parametrization are equivalent.*

$$\begin{array}{ll} (i) & \boldsymbol{A} = \sum\limits_{h=0}^{n-1} \boldsymbol{A}(\nu, \nu+h)\boldsymbol{L}^{h\mathrm{T}} \\ (ii) & \boldsymbol{A} = \sum\limits_{h=0}^{n-1} \boldsymbol{L}^h \boldsymbol{A}(\nu+h, \nu) \end{array} \tag{3}$$

where for $\boldsymbol{A} = \{a_{ij}\} \in \mathbb{M}_{-+}^{n \times n}$, $h = 0, \ldots, n-1$,

$$\begin{array}{l} \boldsymbol{A}_\Theta(\nu, \nu+h) = \\ \mathrm{diag}\left[a_{1,1+h} \cdots a_{n-h,n}\, a_{n-h+1,1} \cdots a_{n,h}\right] \end{array} \tag{4}$$

$$\begin{array}{l} \boldsymbol{A}_\Theta(\nu+h, \nu) = \\ \mathrm{diag}\left[a_{1+h,1} \cdots a_{n,n-h}\, a_{1,n-h+1} \cdots a_{h,n}\right] \end{array} \tag{5}$$

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix} \tag{6}$$

$\boldsymbol{L} \in \mathbb{R}^{n \times n}$ is the circulant permutation matrix, [14].

Remark 1 *If a strictly Metzler $\boldsymbol{A} \in \mathbb{M}_{-+}^{n \times n}$ is represented in the following circulant rhombic forms*

$$(i)\ \boldsymbol{A}_\Theta = \begin{bmatrix} -a_{11} & a_{12} \cdots & a_{1n} & & & & \\ & -a_{22} \cdots & a_{2n} & a_{21} & & & \\ & \ddots & \vdots & \vdots & \ddots & & \\ & & -a_{nn} & a_{n1} & \cdots & a_{n,n-1} \end{bmatrix}$$

$$(ii)\ \boldsymbol{A}_\Theta = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ & a_{12} & \cdots & a_{1n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{bmatrix} \tag{7}$$

then diagonals of (7(i)), (7(ii)) imply the diagonal matrix structure (4), (5), respectively.

Evidently, (2) can be interpreted as

$$\begin{array}{l} (i) \begin{cases} \boldsymbol{A}(\nu, \nu+h) \prec 0, h = 0 \\ \boldsymbol{A}(\nu, \nu+h) \succ 0, h = 1, \ldots, n-1 \end{cases} \\ (ii) \begin{cases} \boldsymbol{A}(\nu, \nu+h) \prec 0, h = 0 \\ \boldsymbol{A}(\nu+h, \nu) \succ 0, h = 1, \ldots, n-1 \end{cases} \end{array} \tag{8}$$

or, equivalently,

$$\begin{array}{l} (i) \begin{cases} \boldsymbol{L}^h \boldsymbol{A}(\nu, \nu+h)\boldsymbol{L}^{h\mathrm{T}} \prec 0, h = 0 \\ \boldsymbol{L}^h \boldsymbol{A}(\nu, \nu+h)\boldsymbol{L}^{h\mathrm{T}} \succ 0, h = 1, \ldots, n-1 \end{cases} \\ (ii) \begin{cases} \boldsymbol{L}^h \boldsymbol{A}(\nu, \nu+h)\boldsymbol{L}^{h\mathrm{T}} \prec 0, h = 0 \\ \boldsymbol{L}^h \boldsymbol{A}(\nu+h, \nu)\boldsymbol{L}^{h\mathrm{T}} \succ 0, h = 1, \ldots, n-1 \end{cases} \end{array} \tag{9}$$

since for any $\boldsymbol{Z} = \mathrm{diag}\,[z_1\, z_2 \cdots z_n]$, $\boldsymbol{Z} \in \mathbb{R}_+^{n \times n}$, the following circular shift operation

$$\boldsymbol{L}\mathrm{diag}\,[z_1 \cdots z_n]\,\boldsymbol{L}^{\mathrm{T}} = \mathrm{diag}\,[z_n\, z_1 \cdots z_{n-1}] \tag{10}$$

doesn't change its definiteness.

Theorem 2 *, [15]. Consider the autonomous closed loop control structure*

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{q}(t) = \boldsymbol{A}_c\boldsymbol{q}(t) \tag{11}$$

where $\boldsymbol{A} \in \mathbb{M}_{-+}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}_+^{n \times r}$, $\boldsymbol{K} \in \mathbb{R}_+^{r \times n}$.

Using the set of diagonal matrices $\boldsymbol{A}_\Theta(\nu, \nu+h) \in \mathbb{R}_+^{n \times n}$, as well as $\boldsymbol{B}_l \in \mathbb{R}_+^{n \times n}$, $\boldsymbol{K}_l \in \mathbb{R}_+^{n \times n}$, $\boldsymbol{K}_{hl} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{K}_l\boldsymbol{L}^h \in \mathbb{R}_+^{n \times n}$ constructed such that

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{k}_r^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{B} = [\boldsymbol{b}_1 \cdots \boldsymbol{b}_r] \tag{12}$$

$$\begin{array}{l} \boldsymbol{K}_l = \mathrm{diag}\left[\boldsymbol{k}_l^{\mathrm{T}}\right] = \mathrm{diag}\,[k_{l1} \cdots k_{ln}] \\ \boldsymbol{B}_l = \mathrm{diag}\,[\boldsymbol{b}_l] = \mathrm{diag}\,[b_{l1} \ \cdots \ b_{ln}] \end{array} \tag{13}$$

then the closed-loop system matrix $\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} \in \mathbb{M}_{-+}^{n \times n}$ can be parameterized as

$$\boldsymbol{A}_c = \sum_{h=0}^{n-1} \Big(\boldsymbol{A}_\Theta(\nu, \nu+h) - \sum_{l=0}^{r} \boldsymbol{B}_l\boldsymbol{K}_{lh}\Big)\boldsymbol{L}^{h\mathrm{T}} \tag{14}$$

Theorem 3 , *[16]. Consider the autonomous observer dynamics*

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t) = \boldsymbol{A}_e\boldsymbol{e}(t) \tag{15}$$

where $\boldsymbol{A} \in \mathbb{M}^{n\times n}_{-+}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}_{+}$, $\boldsymbol{J} \in \mathbb{R}^{n\times m}_{+}$.

Using the set of diagonal matrices $\boldsymbol{A}_\Theta(\nu+h,\nu) \in \mathbb{R}^{n\times n}_{+}$*, as well as* $\boldsymbol{C}_l \in \mathbb{R}^{n\times n}_{+}$, $\boldsymbol{J}_l \in \mathbb{R}^{n\times n}_{+}$, $\boldsymbol{J}_{lh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_l\boldsymbol{L}^h \in \mathbb{R}^{n\times n}_{+}$ *constructed such that*

$$\boldsymbol{J} = [\boldsymbol{j}_1 \cdots \boldsymbol{j}_m], \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_m^{\mathrm{T}} \end{bmatrix} \tag{16}$$

$$\begin{aligned} \boldsymbol{J}_l &= \mathrm{diag}\,[\boldsymbol{j}_l] = \mathrm{diag}\,[j_{l1} \ \cdots \ j_{ln}] \\ \boldsymbol{C}_l &= \mathrm{diag}\,[\boldsymbol{c}_l^{\mathrm{T}}] = \mathrm{diag}\,[c_{l1} \cdots c_{ln}] \end{aligned} \tag{17}$$

then the observer matrix $\boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C} \in \mathbb{M}^{n\times n}_{-+}$ *can be parameterized as*

$$\boldsymbol{A}_e = \sum_{h=0}^{n-1} \boldsymbol{L}^h\big(\boldsymbol{A}_i(\nu+h,\nu) - \sum_{l=0}^{m} \boldsymbol{J}_{lh}\boldsymbol{C}_l\big) \tag{18}$$

To generalize the above results, the parameterizations duality can be found.

Theorem 4 *Used parameterisations* (14) *and* (18) *are dual of each other.*

Proof: The proof is by direct verification. For parametrisation (18), the matrix transpose implies

$$\begin{aligned} &\big(\sum_{h=0}^{n-1} \boldsymbol{L}^h\big(\boldsymbol{A}_\Theta(\nu+h,\nu) - \sum_{l=0}^{r} \boldsymbol{J}_{lh}\boldsymbol{C}_l\big)^{\mathrm{T}} \\ &= \sum_{h=0}^{n-1} \big(\boldsymbol{A}_\Theta^{\mathrm{T}}(\nu+h,\nu) - \sum_{l=0}^{m} \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{J}_{lh}^{\mathrm{T}}\big)\boldsymbol{L}^{h\mathrm{T}} \end{aligned} \tag{19}$$

Constructing by (ii) $\boldsymbol{A}_\Theta$ from (7) a rhombic matrix (ii) $\boldsymbol{A}_\Theta^{\mathrm{T}}$ for the transposed matrix $\boldsymbol{A}^{\mathrm{T}}$, then

$$(ii) \quad \boldsymbol{A}_\Theta^{\mathrm{T}} = \begin{bmatrix} a_{11} & & & \\ a_{12} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \\ & a_{21} & \cdots & a_{n1} \\ & & \ddots & \vdots \\ & & & a_{n,n-1} \end{bmatrix} \tag{20}$$

and, evidently, for $h = 0, \dots, n-1$

$$\boldsymbol{A}_\Theta^{\mathrm{T}}(\nu+h,\nu) = \boldsymbol{A}_\Theta(\nu,\nu+h) \tag{21}$$

Thus, with these changes of diagonal matrices

$$\begin{aligned} \boldsymbol{A}_\Theta^{\mathrm{T}}(\nu+h,\nu) &\Leftarrow \boldsymbol{A}_\Theta(\nu,\nu+h) \\ \boldsymbol{C}_l^{\mathrm{T}} \Leftarrow \boldsymbol{B}_l, &\quad \boldsymbol{J}_{lh}^{\mathrm{T}} \Leftarrow \boldsymbol{K}_{lh} \end{aligned} \tag{22}$$

it can get the equivalent dual formulation, which is the same as (14). This concludes the proof. ■

All presented theoretical coberings result from the principle of diagonal stabilization of positive Metzler systems. The interested readers can find more details on this topic in the papers, [11], [15], [16].

3 Incomplete System Positivity

The system is described by (1), where $\boldsymbol{B} \in \mathbb{R}^{n\times r}_{+}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}_{+}$, $\boldsymbol{D} \in \mathbb{R}^{n\times d}_{+}$ are nonnegative matrices but $\boldsymbol{A} \in \mathbb{M}^{n\times n}_{-\odot}$ is ostensible Metzler. Since $\boldsymbol{A}$ is not strictly Metzler, such a system Ξ is not internally positive, which means that only some of its state variables are non-negative.

Definition 1 *The matrix* $\boldsymbol{A} \in \mathbb{M}^{n\times n}_{-\ominus}$ *is ostensible Metzler if all diagonal elements of* $\boldsymbol{A}$ *are negative, at least one of the off-diagonal of elements is negative, while the number of non-negative off-diagonal elements* $\boldsymbol{A}$ *is predominant.*

When working with the system Ξ (1), covering the ostensible Metzler matrix $\boldsymbol{A} \in \mathbb{M}^{n\times n}_{-\ominus}$, the proposed idea of the parametrization is to separate $\boldsymbol{A}$ so that

$$\boldsymbol{A} = \boldsymbol{A}_p + \boldsymbol{A}_m, \quad \boldsymbol{A} = \{a_{ij}\}_{i,j=1}^{n} \tag{23}$$

where $\boldsymbol{A}_p \in \mathbb{M}^{n\times n}_{-+}$ is strictly Metzler and $\boldsymbol{A}_m \in \mathbb{R}^{n\times n}_{-}$ is element-wise negative and Hurwitz. The basic implication is related to the next theorem.

Theorem 5 , *[17]. If for* $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$ *it can be set* $\boldsymbol{Y} = c\boldsymbol{X} + d\boldsymbol{I}_n$ *with scalars* $c, d \in \mathbb{R}$, $c \neq 0$ *and* $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$*, then eigenvalues* η_k, $k = 1, \dots, n$ *of* $\boldsymbol{Y}$ *are*

$$\eta_k = c\lambda_k + d \tag{24}$$

where λ_k *runs over the eigenvalues of* $\boldsymbol{X}$ *and the eigenvectors of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are identical.*

Theorem 6 , *[12]. If there are positive scalars* $\eta, \delta \in \mathbb{R}_+$ *such that for ostensible Metzler* $\boldsymbol{A} \in \mathbb{M}^{n\times n}_{-\ominus}$ *there exists a strictly Metzler* $\boldsymbol{A}_p \in \mathbb{M}^{n\times n}_{-+}$ *satisfying (23) as well as an element-wise negative and Hurwitz* $\boldsymbol{A}_m \in \mathbb{R}^{n\times n}_{-}$ *thus hold*

$$\boldsymbol{A}_p = \boldsymbol{A}_d + \boldsymbol{A}^{+} + \eta\boldsymbol{\Sigma} + p\boldsymbol{I}_n = \boldsymbol{A}_p^{\circ} + p\boldsymbol{I}_n \tag{25}$$

$$\boldsymbol{A}_m = \boldsymbol{A}_m^{\circ} - p\boldsymbol{I}_n, \quad \boldsymbol{A}_m^{\circ} = \boldsymbol{A}^{-} - \eta\boldsymbol{\Sigma} \tag{26}$$

$$\lambda_o = \max_k \big(\lambda_k^{+} \,|\, \lambda_k^{+} = \mathrm{real}(\lambda_k) > 0\big) \tag{27}$$

$$\boldsymbol{A}_d = diag\,[-a_{11} \, -a_{22} \cdots - a_{nn}] \tag{28}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \quad \begin{aligned} &\lambda_k \in \rho(\boldsymbol{A}_m^{\circ}) \\ &p = \lambda_o + \delta \\ &\boldsymbol{A}_d + p\boldsymbol{I}_n \prec 0 \end{aligned} \tag{29}$$

$$\boldsymbol{A}^{+} = \begin{bmatrix} 0 & a_{12}^{+} & \cdots & a_{1,n-1}^{+} & a_{1n}^{+} \\ a_{21}^{+} & 0 & \cdots & a_{2,n-1}^{+} & a_{2n}^{+} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}^{+} & a_{n-1,2}^{+} & \cdots & 0 & a_{n-1,n}^{+} \\ a_{n1}^{+} & a_{n2}^{+} & \cdots & a_{n-1,n-1}^{+} & 0 \end{bmatrix} \tag{30}$$

$$\boldsymbol{A}^- = \begin{bmatrix} 0 & a_{12}^- & \cdots & a_{1,n-1}^- & a_{1n}^- \\ a_{21}^- & 0 & \cdots & a_{2,n-1}^- & a_{2n}^- \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}^- & a_{n-1,2}^- & \cdots & 0 & a_{n-1,n}^- \\ a_{n1}^- & a_{n2}^- & \cdots & a_{n-1,n-1}^- & 0 \end{bmatrix} \tag{31}$$

$$a_{ij}^+ = \begin{cases} a_{ij} \text{ if } a_{ij} > 0 \\ 0 \ \text{ if } a_{ij} < 0 \end{cases} \quad a_{ij}^- = \begin{cases} a_{ij} \text{ if } a_{ij} < 0 \\ 0 \ \text{ if } a_{ij} > 0 \end{cases} \tag{32}$$

applying for all $i, j \in \langle 1, n \rangle, i \neq j$, where $\rho(\boldsymbol{A}_m^\circ)$ is the spectrum of eigenvalues of $\boldsymbol{A}_m^\circ$.

It can be noted that the positive system represented by $\boldsymbol{A}_p \in \mathbb{M}_{-+}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times d}$, can be used in a design with Metzler parametric constraints, parametrizing analogously $\boldsymbol{A}_p \in \mathbb{M}_{-+}^{n\times n}$ by its rhombic representation as presented above and using the principle of duality in relation to specific design tasks.

If $\boldsymbol{q}(0)$ and an ostensible Metzler $\boldsymbol{A} \in \mathbb{M}_{-\ominus}^{n\times n}$ are intervally given as, [18],

$$0 \le \underline{\boldsymbol{q}}(0) \le \boldsymbol{q}(0) \le \overline{\boldsymbol{q}}(0)\,, \quad \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \overline{\boldsymbol{A}} \tag{33}$$

decoupling of $\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}} \in \mathbb{M}_{-\ominus}^{n\times n}$, $\underline{\boldsymbol{A}} = \left\{\underline{a}_{ij}\right\}_{i,j=1}^n$, $\overline{\boldsymbol{A}} = \{\overline{a}_{ij}\}_{i,j=1}^n$ has to be done that

$$\underline{\boldsymbol{A}} = \underline{\boldsymbol{A}}_p + \underline{\boldsymbol{A}}_m, \ \overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}_p + \overline{\boldsymbol{A}}_m \tag{34}$$

where $\underline{\boldsymbol{A}}_p, \overline{\boldsymbol{A}}_p \in \mathbb{M}_{-+}^{n\times n}$, $\underline{\boldsymbol{A}}_m, \overline{\boldsymbol{A}}_m \in \mathbb{R}_-^{n\times n}$.

The parametrization for ostensible interval Metzler systems can be generalized as follows:

Corollary 1 *If there are positive scalars $\underline{\eta}, \overline{\eta}, \underline{\delta}, \overline{\delta} \in \mathbb{R}_+$ such that for the ostensible Metzler $\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}} \in \mathbb{M}_{-\ominus}^{n\times n}$ there exist strictly Metzler $\underline{\boldsymbol{A}}_p, \overline{\boldsymbol{A}}_p \in \mathbb{M}_{-+}^{n\times n}$ satisfying (34) as well as element-wise negative and Hurwitz $\underline{\boldsymbol{A}}_m, \overline{\boldsymbol{A}}_m \in \mathbb{R}_-^{n\times n}$ thus hold*

$$\begin{aligned} \underline{\boldsymbol{A}}_p &= \underline{\boldsymbol{A}}_d + \underline{\boldsymbol{A}}^+ + \underline{\eta}\boldsymbol{\Sigma} + \underline{p}\boldsymbol{I}_n = \underline{\boldsymbol{A}}_p^\circ + \underline{p}\boldsymbol{I}_n \\ \overline{\boldsymbol{A}}_p &= \overline{\boldsymbol{A}}_d + \overline{\boldsymbol{A}}^+ + \overline{\eta}\boldsymbol{\Sigma} + \overline{p}\boldsymbol{I}_n = \overline{\boldsymbol{A}}_p^\circ + \overline{p}\boldsymbol{I}_n \end{aligned} \tag{35}$$

$$\begin{aligned} \underline{\boldsymbol{A}}_m &= \underline{\boldsymbol{A}}^- - \underline{\eta}\boldsymbol{\Sigma} - \underline{p}\boldsymbol{I}_n = \underline{\boldsymbol{A}}_m^\circ - \underline{p}\boldsymbol{I}_n \\ \overline{\boldsymbol{A}}_m &= \overline{\boldsymbol{A}}^- - \overline{\eta}\boldsymbol{\Sigma} - \overline{p}\boldsymbol{I}_n = \overline{\boldsymbol{A}}_m^\circ - \overline{p}\boldsymbol{I}_n \end{aligned} \tag{36}$$

$$\begin{aligned} \underline{\lambda}_o &= \max_k(\underline{\lambda}_k^+ \,|\, \underline{\lambda}_k^+ = \text{real}(\underline{\lambda}_k) > 0) \\ \overline{\lambda}_o &= \max_k(\overline{\lambda}_k^+ \,|\, \overline{\lambda}_k^+ = \text{real}(\overline{\lambda}_k) > 0) \end{aligned} \tag{37}$$

$$\begin{aligned} \underline{\boldsymbol{A}}_d &= \text{diag}\left[-\underline{a}_{11} \ -\underline{a}_{22} \cdots -\underline{a}_{nn}\right] \\ \overline{\boldsymbol{A}}_d &= \text{diag}\left[-\overline{a}_{11} \ -\overline{a}_{22} \cdots -\overline{a}_{nn}\right] \end{aligned} \tag{38}$$

$$\begin{array}{ll} \underline{\lambda}_k \in \rho(\underline{\boldsymbol{A}}_m^\circ) & \overline{\lambda}_k \in \rho(\overline{\boldsymbol{A}}_m^\circ) \\ \underline{p} = \underline{\lambda}_o + \underline{\delta} & \overline{p} = \overline{\lambda}_o + \overline{\delta} \\ \underline{\boldsymbol{A}}_d + \underline{p}\boldsymbol{I}_n \prec 0 & \overline{\boldsymbol{A}}_d + \overline{p}\boldsymbol{I}_n \prec 0 \end{array} \tag{39}$$

where $\boldsymbol{\Sigma}$ is from (29) and $\underline{\boldsymbol{A}}^+$, $\overline{\boldsymbol{A}}^+$, $\underline{\boldsymbol{A}}^-$, $\overline{\boldsymbol{A}}^-$ are constructed as in (30)–(32).

4 Ostensible Interval Observer

Using all the interval system parameters listed above, in relation to the system input and output related data, the interval observer equations are

$$\begin{aligned} \dot{\underline{\boldsymbol{q}}}_e(t) &= \underline{\boldsymbol{A}}_e\overline{\boldsymbol{q}}_e(t) + \boldsymbol{Bu}(t) + \boldsymbol{Jy}(t) \\ \dot{\overline{\boldsymbol{q}}}_e(t) &= \overline{\boldsymbol{A}}_e\overline{\boldsymbol{q}}_e(t) + \boldsymbol{Bu}(t) + \boldsymbol{Jy}(t) \end{aligned} \tag{40}$$

where $\underline{\boldsymbol{q}}_e(t) \in \mathbb{R}^n$, $\overline{\boldsymbol{q}}_e(t) \in \mathbb{R}^n$ are respectively the lower and the upper system state vector estimates. Thus, using the observer parameters in (40) $\boldsymbol{J} \in \mathbb{R}_+^{n\times n}$, $\underline{\boldsymbol{A}}_e, \overline{\boldsymbol{A}}_e \in \mathbb{M}_{+\odot}^{n\times n}$ with connection to system (1) it evident that

$$\overline{\boldsymbol{A}}_e = \overline{\boldsymbol{A}} - \boldsymbol{JC}, \quad \underline{\boldsymbol{A}}_e = \underline{\boldsymbol{A}} - \boldsymbol{JC} \tag{41}$$

$$\begin{array}{ll} \underline{\boldsymbol{y}}(t) = \boldsymbol{C}\underline{\boldsymbol{q}}(t), & \underline{\boldsymbol{y}}_e(t) = \boldsymbol{C}\underline{\boldsymbol{q}}_e(t) \\ \overline{\boldsymbol{y}}(t) = \boldsymbol{C}\overline{\boldsymbol{q}}(t), & \overline{\boldsymbol{y}}_e(t) = \boldsymbol{C}\overline{\boldsymbol{q}}_e(t) \end{array} \tag{42}$$

Using the observation errors, [21],

$$\underline{\boldsymbol{e}}(t) = \boldsymbol{q}(t) - \underline{\boldsymbol{q}}_e(t), \quad \overline{\boldsymbol{e}}(t) = \boldsymbol{q}(t) - \overline{\boldsymbol{q}}_e(t) \tag{43}$$

and substituting the system and observer equations into (43) it follows that

$$\begin{aligned} \dot{\underline{\boldsymbol{e}}}(t) &= \underline{\boldsymbol{A}}_{pe}\underline{\boldsymbol{e}}(t) + \underline{\boldsymbol{A}}_m\underline{\boldsymbol{e}}(t) + \boldsymbol{Dd}(t) \\ \dot{\overline{\boldsymbol{e}}}(t) &= \overline{\boldsymbol{A}}_{pe}\underline{\boldsymbol{e}}(t) + \overline{\boldsymbol{A}}_m\underline{\boldsymbol{e}}(t) + \boldsymbol{Dd}(t) \end{aligned} \tag{44}$$

when constructing

$$\begin{array}{ll} \underline{\boldsymbol{A}}_{pe} = \underline{\boldsymbol{A}}_p - \boldsymbol{JC}, & \overline{\boldsymbol{A}}_{pe} = \overline{\boldsymbol{A}}_p - \boldsymbol{JC} \\ \underline{\boldsymbol{A}}_e = \underline{\boldsymbol{A}}_{pe} + \underline{\boldsymbol{A}}_m, & \overline{\boldsymbol{A}}_e = \overline{\boldsymbol{A}}_{pe} + \overline{\boldsymbol{A}}_m \end{array} \tag{45}$$

which predefine the conditions for interval observer quadratic stability.

It is need to impose $\underline{\boldsymbol{A}}_{pe}, \overline{\boldsymbol{A}}_{pe} \in \mathbb{M}_{-+}^{n\times n}$ to be strictly Metzler and Hurwitz as well as to impose $\underline{\boldsymbol{A}}_m, \overline{\boldsymbol{A}}_m \in \mathbb{R}_-^{n\times n}$ to be element-wise negative and Hurwitz when implementing for ostensible Metzler lower and upper matrices $\underline{\boldsymbol{A}} = \underline{\boldsymbol{A}}_p + \underline{\boldsymbol{A}}_m \in \mathbb{M}_{-\ominus}^{n\times n}$, $\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}_p + \overline{\boldsymbol{A}}_m \in \mathbb{M}_{-\ominus}^{n\times n}$, whilst the matrices $\underline{\boldsymbol{A}}_e, \overline{\boldsymbol{A}}_e \in \mathbb{M}_{-\ominus}^{n\times n}$ need to be ostensible Metzler and Hurwitz.

Note that in both cases, the necessity of the system matrix separation approach has to be preserved.

Using the equivalent procedure for the system parametrization, then

$$\begin{aligned} \underline{\boldsymbol{A}}_{pe} &= \sum_{h=0}^{n-1} \boldsymbol{L}^h(\underline{\boldsymbol{A}}_p(\nu+h,\nu) - \sum_{j=0}^{r} \boldsymbol{J}_{jh}\boldsymbol{C}_j) \\ \overline{\boldsymbol{A}}_{pe} &= \sum_{h=0}^{n-1} \boldsymbol{L}^h(\overline{\boldsymbol{A}}_p(\nu+h,\nu) - \sum_{j=0}^{r} \boldsymbol{J}_{jh}\boldsymbol{C}_j) \end{aligned} \tag{46}$$

when applying appropriately the rhombic diagonal principle. These representations are captured by generalization of inequalities (9).

A statement of ostensible Metzler interval observer design procedure is provided by the following theorem.

Theorem 7 *The matrices $\underline{\boldsymbol{A}}_{ep}$, $\overline{\boldsymbol{A}}_{ep} \in \mathbb{R}_{-+}^{n\times n}$ are strictly Metzler and Hurwitz and the matrices $\overline{\boldsymbol{A}}_e$,*

$\underline{\boldsymbol{A}}_e \in \mathbb{R}_{-\oplus}^{n\times n}$ are ostensible Metzler and Hurwitz if for ostensible Metzler $\underline{\boldsymbol{A}} = \underline{\boldsymbol{A}}_p + \underline{\boldsymbol{A}}_m \in \mathbb{R}_{-\oplus}^{n\times n}$, $\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}_p + \overline{\boldsymbol{A}}_m \in \mathbb{R}_{-\oplus}^{n\times n}$ and non-negative $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_l \in \mathbb{R}_+^{n\times n}$ and positive scalars $\underline{\mu}, \overline{\mu} \in \mathbb{R}_+$ such that for $h = 1, \ldots, n-1$, $\boldsymbol{l}^{\mathrm{T}} = [1 \ \cdots \ 1] \in \mathbb{R}_+^n$

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0 \tag{47}$$

$$\begin{bmatrix} \underline{\boldsymbol{\Pi}} & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} - \underline{\mu}\boldsymbol{I}_d & * \\ \boldsymbol{C} & \boldsymbol{0} & -\underline{\mu}\boldsymbol{I}_m \end{bmatrix} \prec 0, \begin{bmatrix} \overline{\boldsymbol{\Pi}} & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} - \overline{\mu}\boldsymbol{I}_d & * \\ \boldsymbol{C} & \boldsymbol{0} & -\overline{\mu}\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{48}$$

$$\begin{aligned} &\boldsymbol{P}\overline{\boldsymbol{A}}_p(\nu,\nu) - \sum_{l=1}^{m} \boldsymbol{R}_l\boldsymbol{C}_l \prec 0 \\ &\boldsymbol{P}\underline{\boldsymbol{A}}_p(\nu,\nu) - \sum_{l=1}^{m} \boldsymbol{R}_l\boldsymbol{C}_l \prec 0 \end{aligned} \tag{49}$$

$$\begin{aligned} &\boldsymbol{P}\boldsymbol{L}^h\underline{\boldsymbol{A}}_p(l+h,l)\boldsymbol{L}^{h\mathrm{T}} - \sum_{l=1}^{m} \boldsymbol{R}_l\boldsymbol{L}^h\boldsymbol{C}_l\boldsymbol{L}^{h\mathrm{T}} \succ 0 \\ &\boldsymbol{P}\boldsymbol{L}^h\overline{\boldsymbol{A}}_p(l+h,l)\boldsymbol{L}^{h\mathrm{T}} - \sum_{l=1}^{m} \boldsymbol{R}_l\boldsymbol{L}^h\boldsymbol{C}_l\boldsymbol{L}^{h\mathrm{T}} \succ 0 \end{aligned} \tag{50}$$

$$\begin{aligned} \underline{\boldsymbol{\Pi}} =& \boldsymbol{P}\underline{\boldsymbol{A}}_p + \underline{\boldsymbol{A}}_p^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\underline{\boldsymbol{A}}_m + \underline{\boldsymbol{A}}_m^{\mathrm{T}}\boldsymbol{P} - \\ &- \sum_{l=1}^{r}(\boldsymbol{R}_l\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_l + \boldsymbol{C}_l\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_l) \\ \overline{\boldsymbol{\Pi}} =& \boldsymbol{P}\overline{\boldsymbol{A}}_p + \overline{\boldsymbol{A}}_p^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\overline{\boldsymbol{A}}_m + \overline{\boldsymbol{A}}_m^{\mathrm{T}}\boldsymbol{P} - \\ &- \sum_{k=1}^{r}(\boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_k + \boldsymbol{C}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k) \end{aligned} \tag{51}$$

A feasible task for $\boldsymbol{J} \in \mathbb{R}_+^{n\times m}$ implies

$$\boldsymbol{J}_l = \boldsymbol{P}^{-1}\boldsymbol{R}_l, \quad \boldsymbol{j}_l = \boldsymbol{J}_l\boldsymbol{l}, \quad \boldsymbol{J} = [\boldsymbol{j}_1 \cdots \boldsymbol{j}_m] \tag{52}$$

Hereafter, $*$ is the symmetric item in a symmetric matrix.

Proof: Choosing Lyapunov function in the following form

$$\begin{aligned} v(\underline{\boldsymbol{e}}(t)) =& \underline{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\underline{\boldsymbol{e}}(t) - \underline{\eta}\int_0^t \boldsymbol{d}^{\mathrm{T}}(\tau)\boldsymbol{d}(\tau)\mathrm{d}\tau + \\ &+ \underline{\eta}^{-1}\int_0^t \underline{\boldsymbol{e}}_y^{\mathrm{T}}(\tau)\underline{\boldsymbol{e}}_y(\tau)\mathrm{d}\tau \\ &> 0 \end{aligned} \tag{53}$$

where $\boldsymbol{P} \in \mathbb{R}_+^{n\times m}$ is a diagonal positive definite matrix and $\underline{\eta} \in \mathbb{R}_+$ is a positive scalar, then along all stable lover errors

$$\begin{aligned} \dot{v}(\overline{\boldsymbol{e}}(t)) =& \dot{\overline{\boldsymbol{e}}}^{\mathrm{T}}(t)\boldsymbol{P}\overline{\boldsymbol{e}}(t) + \overline{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\overline{\boldsymbol{e}}}(t) + \\ &+ \eta^{-1}\overline{\boldsymbol{e}}_y^{\mathrm{T}}(t)\overline{\boldsymbol{e}}_y(t) - \eta\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t) \\ &< 0 \end{aligned} \tag{54}$$

and applying in (54) the observer error dynamics it gets to

$$\begin{aligned} &\dot{v}(\underline{\boldsymbol{e}}(t)) \\ &= \underline{\boldsymbol{e}}^{\mathrm{T}}(t)(\underline{\boldsymbol{A}}_e^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\underline{\boldsymbol{A}}_e)\underline{\boldsymbol{e}}(t) + \\ &+ \underline{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{D}\boldsymbol{d}(t) + \boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\underline{\boldsymbol{e}}(t)) + \\ &+ \underline{\mu}^{-1}\underline{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\underline{\boldsymbol{e}}(t) - \underline{\mu}\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t) \\ &< 0 \end{aligned} \tag{55}$$

The equality (55) can be compactly written constructing a common notation $\underline{\boldsymbol{e}}_d(t)$ as follows

$$\underline{\boldsymbol{e}}_{ed}^{\mathrm{T}}(t) = \begin{bmatrix} \overline{\boldsymbol{e}}^{\mathrm{T}}(t) & \boldsymbol{d}^{\mathrm{T}}(t) \end{bmatrix} \tag{56}$$

then there is a reasonable ground to conclude that the following have to yield

$$\dot{v}(\underline{\boldsymbol{e}}_{ed}(t)) = \underline{\boldsymbol{e}}_{ed}^{\mathrm{T}}(t)\,\underline{\boldsymbol{\Pi}}^{\circ}\underline{\boldsymbol{e}}_{ed}(t) < 0 \tag{57}$$

where, evidently,

$$\underline{\boldsymbol{\Pi}}^{\circ} = \begin{bmatrix} \underline{\boldsymbol{A}}_e^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\underline{\boldsymbol{A}}_e + \underline{\mu}^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{P}\boldsymbol{D} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\underline{\mu}\boldsymbol{I}_{r_d} \end{bmatrix} \prec 0 \tag{58}$$

After applying the property of Schur complement with relation to the element $\underline{\mu}^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}$ then

$$\begin{bmatrix} \boldsymbol{P}\underline{\boldsymbol{A}}_e + \underline{\boldsymbol{A}}_e^{\mathrm{T}}\boldsymbol{P} & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\underline{\mu}\boldsymbol{I}_d & * \\ \boldsymbol{C} & \boldsymbol{0} & -\underline{\mu}\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{59}$$

and it can be set

$$\begin{aligned} \boldsymbol{P}\underline{\boldsymbol{A}}_{pe} &= \boldsymbol{P}(\underline{\boldsymbol{A}}_p - \boldsymbol{J}\boldsymbol{C}) \\ &= \boldsymbol{P}\underline{\boldsymbol{A}}_p - \sum_{k=1}^{m} \boldsymbol{P}\boldsymbol{j}_k\boldsymbol{c}_k^{\mathrm{T}} \\ &= \boldsymbol{P}\underline{\boldsymbol{A}}_p - \sum_{k=1}^{m} \boldsymbol{P}\boldsymbol{J}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_k \end{aligned} \tag{60}$$

where vector $\boldsymbol{l} \in \mathbb{R}_+^n$ is used to uncover the diagonal matrix structures.

Thus, (59) implies (48), (51) when using the substitutions

$$\boldsymbol{P}\boldsymbol{J}_k = \boldsymbol{R}_k, \quad \underline{\boldsymbol{A}}_e = \underline{\boldsymbol{A}}_{pe} + \underline{\boldsymbol{A}}_m \tag{61}$$

According to the parametrization, pre-multiplying the left side by $\boldsymbol{P}$ and post-multiplying the right side by $\boldsymbol{L}^{h\mathrm{T}}$ then, with $\boldsymbol{J}_{lh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_l\boldsymbol{L}^h \in \mathbb{R}_+^{n\times n}$, (9), (46) gives

$$\sum_{h=0}^{n-1}(\boldsymbol{P}\boldsymbol{L}^h\underline{\boldsymbol{A}}_p(\nu+h,\nu)\boldsymbol{L}^{h\mathrm{T}} - \sum_{l=0}^{r}\boldsymbol{P}\boldsymbol{J}_l\boldsymbol{L}^h\boldsymbol{C}_l\boldsymbol{L}^{h\mathrm{T}}) \tag{62}$$

and using (61) then (62) implies for $h = 0$ the lower part of (49) and for $h > 0$ (62) implies the lower part of (50).

Since analogously can be set the LMIs working on $\overline{\boldsymbol{e}}(t)$, this closes the proof. ■

5 Illustrative Example

The considered system (1), (2) is built on the parameters of linearized dynamic model of F-404 engine, [19], with $\underline{\boldsymbol{A}} \leq \overline{\boldsymbol{A}}$

$$\underline{\boldsymbol{A}} = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ -0.8357 & -2.4 & -0.3788 \\ 0.3107 & 0 & -2.1300 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overline{\boldsymbol{A}} = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ -0.3357 & -1.4 & -0.3788 \\ 0.3107 & 0 & -2.1300 \end{bmatrix}, \ \boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.4182 & 5.2030 \\ 0.3901 & -0.1245 \\ 0.5186 & 0.0236 \end{bmatrix}$$

and the derived design parameters are

$$\underline{\boldsymbol{A}}_d = \operatorname{diag}\left[-1.46 \ -2.4 \ -2.13\right]$$
$$\overline{\boldsymbol{A}}_d = \operatorname{diag}\left[-1.46 \ -1.4 \ -2.13\right]$$

$$\underline{\boldsymbol{A}}^+ = \overline{\boldsymbol{A}}^+ = \begin{bmatrix} 0 & 0 & 2.4280 \\ 0 & 0 & 0 \\ 0.3107 & 0 & 0 \end{bmatrix}, \ \boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{\boldsymbol{A}}^- = \begin{bmatrix} 0 & 0 & 0 \\ -0.8357 & 0 & -0.3788 \\ 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}^- = \begin{bmatrix} 0 & 0 & 0 \\ -0.3357 & 0 & -0.3788 \\ 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{L} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

With $\underline{\boldsymbol{A}}_m^\circ = \underline{\boldsymbol{A}}^- - \eta\boldsymbol{\Sigma}$, $\overline{\boldsymbol{A}}_m^\circ = \overline{\boldsymbol{A}}^- - \eta\boldsymbol{\Sigma}$, where $\eta = 0.005$ then

$$\underline{\boldsymbol{A}}_m^\circ = \begin{bmatrix} 0 & -0.0050 & -0.0050 \\ -0.8407 & 0 & -0.3838 \\ -0.0050 & -0.0050 & 0 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}_m^\circ = \begin{bmatrix} 0 & -0.0050 & -0.0050 \\ -0.3407 & 0 & -0.3838 \\ -0.0050 & -0.0050 & 0 \end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_m^\circ) = \{-0.0808 \ 0.0150 \ 0.0758\}$$
$$\rho(\overline{\boldsymbol{A}}_m^\circ) = \{-0.0627 \ 0.0050 \ 0.0577\}$$

and $\underline{\lambda}_0 = 0.0758$, $\overline{\lambda}_0 = 0.0577$.

To define D-stability regions it is set $\underline{\delta} = 0.003$, $\overline{\delta} = 0.03$, which define $\underline{p} = \underline{\lambda}_0 + \underline{\delta}$, $\overline{p} = \overline{\lambda}_0 + \overline{\delta}$ the Hurwitz matrices $\underline{\boldsymbol{A}}_m = \underline{\boldsymbol{A}}_m^\circ - \underline{p}\boldsymbol{I}_n$, $\overline{\boldsymbol{A}}_m = \overline{\boldsymbol{A}}_m^\circ - \overline{p}\boldsymbol{I}_n$ and the strictly Metzler $\underline{\boldsymbol{A}}_p = \underline{\boldsymbol{A}}_d + \underline{\boldsymbol{A}}^+ + \eta\boldsymbol{\Sigma} + \underline{p}\boldsymbol{I}_n$, $\overline{\boldsymbol{A}}_p = \overline{\boldsymbol{A}}_d + \overline{\boldsymbol{A}}^+ + \eta\boldsymbol{\Sigma} + \overline{p}\boldsymbol{I}_n$ so that $\underline{\boldsymbol{A}}_p \leq \overline{\boldsymbol{A}}_p$,

$$\underline{\boldsymbol{A}}_m = \begin{bmatrix} -0.0788 & -0.0050 & -0.0050 \\ -0.8407 & -0.0788 & -0.3838 \\ -0.0050 & -0.0050 & -0.0788 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}_m = \begin{bmatrix} -0.0807 & -0.0050 & -0.0050 \\ -0.3407 & -0.0807 & -0.3838 \\ -0.0050 & -0.0050 & -0.0777 \end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_m) = \{-0.1596 \ -0.0738 \ -0.0030\}$$
$$\rho(\overline{\boldsymbol{A}}_m) = \{-0.1504 \ -0.0827 \ -0.0300\}$$

$$\underline{\boldsymbol{A}}_p = \begin{bmatrix} -1.3812 & 0.0050 & 2.4330 \\ 0.0050 & -2.3212 & 0.0050 \\ 0.3157 & 0.0050 & -2.1512 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}_p = \begin{bmatrix} -1.3723 & 0.0050 & 2.4330 \\ 0.0050 & -1.3123 & 0.0050 \\ 0.3157 & 0.0050 & -2.1423 \end{bmatrix}$$

Using $\underline{\boldsymbol{A}}_p$, $\overline{\boldsymbol{A}}_p$ it can be found that the matrix variables, which provide a solution by SeDuMi, [20], are

$$\boldsymbol{P} = \operatorname{diag}\left[2.2680 \ 3.3453 \ 2.4098\right],$$
$$\boldsymbol{R}_1 = \operatorname{diag}\left[1.4454 \ 0.0062 \ 0.2474\right], \quad \underline{\gamma} = 4.2335$$
$$\boldsymbol{R}_2 = \operatorname{diag}\left[2.8826 \ 0.0060 \ 1.1582\right], \quad \overline{\gamma} = 4.4557$$

$$\boldsymbol{J} = \begin{bmatrix} 0.6373 & 1.2710 \\ 0.0018 & 0.0018 \\ 0.1027 & 0.4806 \end{bmatrix}$$

These infuse the strictly Metzler and Hurwitz matrices $\underline{\boldsymbol{A}}_{pe} = \underline{\boldsymbol{A}}_p - \boldsymbol{JC}$, $\overline{\boldsymbol{A}}_{pe} = \overline{\boldsymbol{A}}_p - \boldsymbol{JC}$ and the ostensible Metzler and Hurwitz matrices $\underline{\boldsymbol{A}}_e = \underline{\boldsymbol{A}} - \boldsymbol{JC}$, $\overline{\boldsymbol{A}}_e = \overline{\boldsymbol{A}} - \boldsymbol{JC}$

$$\underline{\boldsymbol{A}}_{pe} = \begin{bmatrix} -2.0185 & 0.0050 & 1.1620 \\ 0.0032 & -2.3212 & 0.0032 \\ 0.2130 & 0.0050 & -2.6318 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}_{pe} = \begin{bmatrix} -2.0096 & 0.0050 & 1.1620 \\ 0.0032 & -1.3123 & 0.0032 \\ 0.2130 & 0.0050 & -2.6229 \end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_{pe}) = \{-1.7406 \ -2.3213 \ -2.9096\}$$
$$\rho(\overline{\boldsymbol{A}}_{pe}) = \{-1.3122 \ -1.7319 \ -2.9007\}$$

$$\underline{\boldsymbol{A}}_e = \begin{bmatrix} -2.0973 & 0 & 1.1570 \\ -0.8375 & -2.4 & -0.3806 \\ 0.2080 & 0 & -2.6106 \end{bmatrix}$$

$$\overline{\boldsymbol{A}}_e = \begin{bmatrix} -2.0973 & 0 & 1.1570 \\ -0.3375 & -1.4 & -0.3806 \\ 0.2080 & 0 & -2.6106 \end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_e) = \{-1.8003 \ -2.4000 \ -2.9076\}$$
$$\rho(\overline{\boldsymbol{A}}_e) = \{-1.4000 \ -1.8003 \ -2.9076\}$$

Note, the positions of negative off-diagonal elements in $(\underline{\boldsymbol{A}}, \underline{\boldsymbol{A}}_e)$, $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{A}}_e)$ are preserved.

Simulating the defined uncertain system within constraints $\underline{\boldsymbol{A}} \leq \boldsymbol{A} \leq \overline{\boldsymbol{A}}$, $\underline{\boldsymbol{q}}_e(0) \leq \boldsymbol{q}(0) \leq \overline{\boldsymbol{q}}_e(0)$, $\sigma_d^2 = 0.01^2$, where

$$\boldsymbol{A} = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ -0.5857 & -1.9 & -0.3788 \\ 0.3107 & 0 & -2.1300 \end{bmatrix}, \ \boldsymbol{q}(0) = \begin{bmatrix} 0.250 \\ 3.750 \\ 0.025 \end{bmatrix}$$

and utilizing the system forced mode

$$\boldsymbol{u}(t) = \boldsymbol{W}\boldsymbol{w}(t), \ \boldsymbol{W} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{w}(t) = \begin{bmatrix} 0.352 \\ 0.076 \end{bmatrix}$$

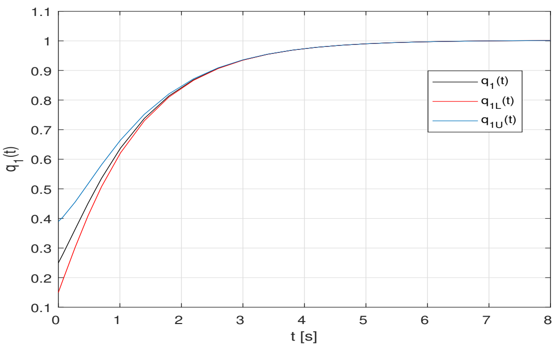

Figure 1: Convergence of the first state variable

simulation results with the initial conditions

$$\boldsymbol{q}(0) = \begin{bmatrix} 0.25 \\ 3.75 \\ 0.02 \end{bmatrix}, \ \underline{\boldsymbol{q}}_e(0) = \begin{bmatrix} 0.15 \\ 0.00 \\ 0.00 \end{bmatrix}, \ \overline{\boldsymbol{q}}_e(0) = \begin{bmatrix} 0.4 \\ 0.0 \\ 0.1 \end{bmatrix}$$

are given in Figure 1. These simulation results show the performance of the proposed interval observer, where the black color curved line denotes the first state variable trajectory and the blue and red curved lines denote its upper and lower estimations. Since the first state variable is positive, it can be seen that its behavior is correctly intervally estimated, guaranteing exponential convergence of the state variable estimation error.

6 Concluding Remarks

The main objectives in this paper are parameterisations approaches in design for ostensible Metzler systems with interval-specified dynamics and bounded system disturbances. The diagonal matrix variables and the proposed LMI structures reflect the key idea to obtain auxiliary Metzler and Hurwitz matrix structures of $\underline{\boldsymbol{A}}_p, \overline{\boldsymbol{A}}_p$ while Lyapunov function and the related LMIs form the base of the quadratic stability. Despite the design conditions complexity, state estimation using ostensible Metzler interval observers is robust to the changes covered in plant dynamics by given interval bounds, taking into account that the positivity of the lower positive state estimation need to be keep.

In the synthesis it is simple to define sequentially different D-regions of stability for $\underline{\boldsymbol{A}}_m, \overline{\boldsymbol{A}}_m$ by using the parameters $\underline{p}, \overline{p} > 0$ and so to guaranty consequently that $\underline{\boldsymbol{A}}_p \leq \overline{\boldsymbol{A}}_p$, when forcing interval bounds on positive state variables, as well as to find via LMIs the acceptable rate of convergence of estimation errors. Although these tasks are parametrical dependent, their interactive predefinition is possible as a rule.

Since scalar variables $\underline{\mu}$, $\overline{\mu}$ are related to the system dependency on the parameters and can be tuned in LMIs, they can be used for attenuation when guaranteing interval observer quadratic stability under unknown disturbance.

The approach certainly requires further investigation the ostensible Metzler system matrices in dependence on the un-structural set of negative off-diagonal elements. It is worth highlighting that there are also many unexplored theoretical and applied aspects of problems in systems with non-strictly Metzler matrix structures and ostensible Metzler matrix representation in control, e.g., of drones and unmanned aerial vehicles, [22], [23], or constructing Metzler matrix representations to match the properties of the interval observers, [24].

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Dušan Krokavec newly addressed the duality principle and the incidence of ostensible Metzler matrix separation as well as LMIs for interval observer quadratic stability and converted these tasks to an LMI problem. The author has read and agreed to the proposed version of the manuscript.

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Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.