On Boundary Value Problems for Liénard Type Equation

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Abstract: The generalized Liénard type differential equation is studied together with the two-point linear boundary conditions of the Sturm-Liouville type. The existence and multiplicity of solutions are considered. The existence under suitable conditions is shown to follow from the lower and upper functions theory. For multiplicity, the polar coordinates approach is used. The multiplicity results are based on the comparison between behavior of solutions near the trivial one, and solutions near the special one, which is preassumed to be non-oscillatory. The existence of the latter is required. It is shown also, that these conditions are fulfilled for a relatively broad class of equations. Some examples are constructed, which are supplied by comments and illustrations.

Key-Words: - Ordinary differential equations, multiple solutions, existence of solutions, Liénard type equations, phase portrait, oscillatory behaviors, boundary value problems, variational equations, heteroclinic trajectories, homoclinic trajectories.

Received: January 27, 2023. Revised: September 29, 2023. Accepted: November 13, 2023. Published: December 6, 2023.

1 Introduction

Boundary value problems (BVPs) for ordinary differential equations appear often in theoretical studies, [1], [2], [3], [4], and mathematical modelling of real-world processes. The existence of solutions is the main question. The existence of a solution should be confirmed before performing some numerical experiments. The linear theory provides answers to the main theoretical questions as to the existence and uniqueness of a solution. Nonlinear problems can be more difficult. The existence of a solution should be proved in many cases. Moreover, multiple non-similar solutions can appear in many practically oriented studies. The answers typically should be obtained for particular cases, where the general theory does not provide recommendations. The equations of Liènard type can be double-nonlinear (nonlinearities at *f(x)* and *g(x)* as in (1)) and there is space for rich dynamics of solutions. The classical Liènard equation is well-studied qualitatively, focusing on periodic solutions and bifurcations. The generalized Liènard type equations are general, and the behavior of solutions may be quite different. Generalized equations of Liénard type have been studied in the works, [5], [6], [7].

In the articles, [2], [8], have studied the boundary value problems of the form $x'' + ax -$

$b(t)x^n = 0$, $x(0) = x(T) = 0$ or $x'(0) = x'(T) = 0$, where n is a positive integer (most results concern the cases *n = 2, n = 3, n = 5*). The exact estimates of the number of solutions were obtained for autonomous equations of the form $x'' + ax - bx^n$=0 and some results (mostly of a computational nature) were stated for the case of $b = b(t)$ being piece-wise constant function, [9]. The phase plane method was used extensively. Equations of the form:

$$x'' + f(x)x' + g(x) = 0 \tag{1}$$

are a classical object for investigation. The Liènard and Van-der-Pol equations fall into this class. Both arose from practice. Equations of the form (1) are rich in oscillatory behaviors. They are known to have (under suitable conditions) isolated periodic solutions. The problem of estimating the number of limit cycles for the case of polynomial functions *f(x)* and *g(x)* has attracted the attention of prominent researchers. In contrast, equation (3) can be reduced to a conservative equation. The comparison of equations (3) and the shorter equation:

$$x'' + g(x) = 0. \tag{2}$$

was made in the work, [10]. We focused on the case $g(x) = ax - bx^3$, *a > 0, b > 0,* and considered two-point boundary conditions for both equations. We intended to compare the number of solutions to the respective BVPs. For this, we made use of the special change of variables resulting in eliminating the middle term in (1). This technique was proposed, [11], when studying isochronous problems. An equation in new variables has a simpler form and can be (formally) integrated. This transformation keeps the trivial solution. This is important because, in various sources devoted to the study of multiple solutions of BVP, the following idea was exploited. Imagine that the oscillatory behavior of solutions can be measured around the trivial solution. If a comparison can be made with solutions far away from the trivial one, some conclusions can be made about the number of solutions for two-point boundary value problems. After the reduction of equation (1) to form (2) using the above-mentioned variable change, another comparison can be made, namely, the equation in question versus the reduced equation. This approach will be considered in the next sections. In the article, [10], the equation:

$$x'' + f(x)x'^2 + g(x) = 0 \tag{3}$$

was considered together with the two-point boundary conditions of the Dirichlet and Neumann type. The existence of solutions and estimates of the number of solutions were in focus. The behavior of solutions, and as a consequence, the number of solutions heavily depends on the function *f(x)*. Three types of *f(x)* were considered, and for all cases, the comparison was made of the number of solutions to certain BVP for equation (3) and Newtonian equation (2). The main conclusion made in, [10], is that generally, the number of solutions to the Dirichlet and Neumann problems for equation (3) is not less than that for equation (2). In this article, we consider more general the Sturm-Liouville-type conditions of the form:

$$\begin{aligned} a_1 x(0) - b_1 x'(0) = 0, \\ a_2 x(1) + b_2 x'(1) = 0, \end{aligned} \tag{4}$$

where all four coefficients are nonnegative but at least one coefficient in any equation is not zero. Of course, the Dirichlet and Neumann boundary conditions $x(0) = x(1) = 0$, $x'(0) = x'(1) = 0$ are included. After the division of the first line in (4) by $\sqrt{a_1^2 + b_1^2}$ one obtains the first condition in the form $\cos\alpha\, x(0) - \sin\alpha\, x'(0) = 0$, where $\alpha = \arctan\left(\frac{b_1}{a_1}\right) \in \left[0, \frac{\pi}{2}\right]$. Similarly, the second condition in (4) can be written in the form $\cos\beta x(1) - \sin\beta\, x'(1) = 0$, where $tan\,\beta = b_2/a_2, \quad \beta \in [\pi/2, \pi]$.

For instance, the boundary conditions:

$$x(0) - x'(0) = 0,$$
$$x(1) + x'(1) = 0$$

will be written as:

$$cos\left(\frac{\pi}{4}\right)x(0) - sin(\pi/4)\,x'(0) = 0,$$
$$cos\left(\frac{3\pi}{4}\right)x(1) - sin\left(\frac{3\pi}{4}\right)x'(1) = 0.$$

Therefore, our objects of investigation in this paper are equations (1) and (3) given together with the boundary conditions of the form:

$$\begin{aligned} &cos\,\alpha\, x(0) - sin\,\alpha\, x'(0) = 0,\\ &cos\,\beta\, x(1) - sin\,\beta\, x'(1) = 0, \end{aligned} \tag{5}$$

where $\alpha \in [0, \pi/2]$, $\beta \in [\pi/2, \pi]$. The Dirichlet and Neumann boundary conditions are included.

In this article, we study equations of the type:

$$x'' + f(x)x' + g(x) = 0 \quad \text{and}$$
$$x'' + f(x)x'^2 + g(x) = 0.$$

We are interested in the existence of solutions and the multiplicity. Tools from the general theory, as well as some specific instruments, are used. Visualizations in a phase plane are helpful to understand and explain results.

2 Existence

Suppose all functions in (1) and (3) are continuous. Sometimes continuous differentiability is needed, but these cases are commented on consequently. Since the highest degree of the first order derivative in (1) and (3) is two, the Bernstein condition (the quadratic growth with respect to x'), which ensures boundedness of the first derivative of a solution, is always fulfilled. The existence of a solution to the Dirichlet problem:

$$x(a) = A,\ x(b) = B \tag{6}$$

follows immediately, if the upper and lower functions $\beta(t)$ and $\alpha(t)$ exist such that:

$$\alpha(a) \le A \le \beta(a), \alpha(b) \le B \le \beta(b), \alpha(t) \le \beta(t) \tag{7}$$

and the inequalities:

$$\alpha'' + f(\alpha)\,\alpha'^2 + g(\alpha) \ge 0, \tag{8}$$

$$\beta'' + f(\beta)\,\beta'^2 + g(\beta) \le 0 \tag{9}$$

hold in the interval *[a,b]* (Theorem 4 and Remark 1 in, [12].

This criterion is effective in many cases. For instance, let the function *g(x)* be an odd-degree polynomial with the principal term $-x^{2n+1}$. Then any sufficiently large positive constant β serves as the upper function ($\beta'' + f(\beta)\,\beta'^2 + g(\beta) = g(\beta) \le 0$) and, consequently, $-\beta$ is the lower function. On the other hand, it is difficult often to find α and β such that $\alpha < \beta$ in the interval *[a,b]*. For instance, the problem $x'' = -(\pi + 1)^2 x$, $x(0) = 0 = x(1)$ has only he trivial solution $x(t) \equiv 0$. However, the functions α and β, satisfying (8) and (9) other than the trivial solution, do not exist. This was shown in, [12]. As to multiple solutions for BVPs, the method of upper and lower functions can be used, if pairs of upper and lower functions can be constructed. These are rare cases.

The following result follows from known existence theorems and uses the specific form of the equations (1) and (3).

Theorem 1. Suppose $xg(x) < 0$ for x such that $|x| \ge M > 0$. Then the BVPs (1), (4) and (3), (4) have solutions.

The proof follows from the fact that the functions $\alpha = -M$ and $\beta = M$ are the lower and the upper functions for these problems. It is essentially that the straight line

$$cos\,\alpha\, x(0) - sin\,\alpha\, x'(0) = 0$$

either is vertical or connects the segments $\{x = -M, x' < 0\}$ and $\{x = M, x' > 0\}$, while the straight line $cos\,\beta\, x(1) - sin\,\beta\, x'(1) = 0$ either is vertical or connects the segments $\{x = -M, x' > 0\}$ and $\{x = M, x' < 0\}$. It is also taken into account, that both equations (1) and (3) satisfy the Bernstein condition (at most the quadratic growth with respect to x' in both equations).

3 Multiplicity

Let us pass to polar coordinates using formulas:

$$\begin{cases} x(t) = r(t)\sin\varphi(t) \\ x'(t) = r(t)\cos\varphi(t) \end{cases} \quad (10)$$

the polar system for the equation (3) is:

$$\begin{cases} \varphi'(t) = \cos^2\varphi(t) + f(x)r(t)\sin\varphi(t)\cos^2\varphi(t) + \frac{1}{r(t)}g(x)si\,n\,\varphi(t), \\ r'(t) = \frac{1}{2}r(t)\sin 2\varphi(t) - f(x)r^2(t)\cos^3\varphi(t) - g(x)\cos\varphi(t). \end{cases} \quad (11)$$

The boundary conditions (5) in polar coordinates are:

$$\varphi(0) = \alpha,\ \varphi(1) = \beta + \pi k,\ k = 0,1, \ldots\ . \quad (12)$$

In the work, [8], multiplicity results were formulated in terms of the variational equations for the Dirichlet and Neumann boundary conditions. We wish to do the same for the boundary conditions (5), or, equivalently, (11).

First, let us introduce the variational equations for the nonlinear equations (1) and (3), with respect to the trivial solution $x \equiv 0$.

The variational equations for (1) and (3), provided that the trivial solution exists (the necessary condition for this is $g(0) = 0$) are:

$$y'' + f(0)y' + g_x(0)y = 0 \quad (13)$$

and

$$y'' + g_x(0)y = 0, \quad (14)$$

respectively. Introduce the polar coordinates for the variational equations using the formulas:

$$y = \rho(t)\sin\Theta(t), \qquad y' = \rho(t)\cos\Theta(t).$$

The following results are true.

Theorem 2. Let the following conditions for the equation (1) hold:

1) $g(0) = 0$;
2) $\Theta(0) = \alpha, \Theta(1) \in \big(\beta + \pi i, \beta + \pi(i+1)\big)$, *where* $\Theta(t)$ *corresponds to the variational equation (13), i=0,1, …;*
3) *there exists a solution x(t) of the Cauchy problem (1), ϕ(0)=α such that ϕ(1)< β.*

Then there exist at least ***i*** nontrivial solutions of the BVP (1), (5).

Sketch of the proof. The variational equation around the trivial solution of equation (1) (it exists due to the condition 1)) is equation (13). Consider it in polar coordinates together with the initial conditions $\Theta(0) = \alpha$, $\rho(0) = 1$. It follows from condition 2) that the angular function $\Theta(t)$ attains the values of the form $\beta + \pi k$ at least *i* times. Look at the main equation (1) written in the polar coordinates (10). For small r(t) solutions behave similarly to the solutions of the variational equation. Therefore, the polar function $\varphi(t)$ takes *i* values of the form $\beta + \pi k$ while *r(0)* is small. Increase *r(0)* until the value R corresponds to the solution *x(t)* from condition 3). The angular function $\varphi(t)$ changes continuously, and $\varphi(1)$ is in the interval $\big(\beta + \pi i, \beta + \pi(i+1)\big)$ for *r(0)* small enough, while $\varphi(1)$ is less than β for large enough values *r(0)*. Therefore $\varphi(1)$ takes all the intermediate values from β to $\beta + \pi i$. Therefore *i* nontrivial solutions of the BVP (1), (5) appear.

Remark. If the additional condition similar to condition 3) is added with the text replacement

"*$\phi(0)=\alpha+\pi$ such that $\phi(1)<\beta+\pi$*" then the additional i solutions to the BVP are obtained by considering the initial value $\varphi(0)=\beta+\pi$ and repeating the reasoning.

Theorem 3. Let the following conditions for the equation (3) hold:

1) $g(0)=0$;
2) $\theta(0)=\alpha, \theta(1)\in(\beta+\pi i, \beta+\pi(i+1))$, *where $\theta(t)$ corresponds to the variational equation (14);*
3) *there exists a solution x(t) of the Cauchy problem (3), $\phi(0)=\alpha$ such that $\phi(1)<\beta$.*

Then there exist at least $\boldsymbol{i}$ nontrivial solutions of the BVP (3), (5).

The proof can be conducted similarly to the proof of Theorem 2.

In the authors' previous work, the difference between these two results was discussed considering the Dirichlet boundary conditions.

4 Corollaries

Let the equation (3) be:

$$x''+f(x)x'^2+(ax-bx^3)=0, \qquad (15)$$

where *f(x)* is either a positive constant or *x*.

Theorem 4. Let the following condition for the equation (15) hold:

$\theta(0)=\alpha,\ \theta(1)\in(\beta+\pi i, \beta+\pi(i+1))$, *where $\theta(t)$ corresponds to the variational equation* $y''+ay=0$.

Then there exist at least $\boldsymbol{i}$ nontrivial solutions of the BVP (15), (5).

Proof. The condition 1) of Theorem 3 is fulfilled. Condition 3) of Theorem 3 is fulfilled also since there exists the homoclinic solution with slowly changing angular function $\theta(t)$. This function with $\theta(0)=\alpha$ for the time *[0,1]* does not reach the value $\theta(1)=\beta>\alpha$. We assume that α is in the interval $[0,\pi)$ or β is in the interval $\left(\frac{\pi}{2},\pi\right]$. So the Neumann problem is excluded.

If equation (3) has a phase portrait similar to Figure 5, where two heteroclinic trajectories form a bounded region, a result similar to Theorem 4, can be formulated. Such cases are multiple.

5 Conclusion

Equations of the for (3) may have regions, surrounded by two heteroclinic or one homoclinic trajectories. These regions may have complicated structures, containing the hierarchy of embedded period annuli, [13]. In simple cases, inside there is a unique critical point of the type center. Trajectories near this critical point rotate, and this rotation can be described in terms of the linearized variational equation. On the other hand, trajectories passing by the boundary, slow down and this behavior may be very different from the behavior near the critical point. In such cases, boundary value problems may have multiple solutions. Our examples above are of this kind. The Dirichlet and Neumann-type problems have multiple solutions for the appropriate choice of parameters in the equations.

References:

[1] S. R. Bernfeld. *An Introduction to Nonlinear Boundary Value Problems*, Paperback April 19, 2012

[2] W. G. Kelley, A. C. Peterson. *The Theory of Differential Equations: Classical and Qualitative, (Universitext)*, 2nd ed. 2010 Edition.

[3] M. A. Krasnosel'skiy, A. I. Perov, A. I. Povolotskiy. *Plane vector fields*, Academic Press, 1966.

[4] R. Reissig, G. Sansone, R. Conti. *Qualitative theory of nonlinear differential equations, Moscow, "Nauka"*, 1974. (Russian).

[5] M. T. de Bustos, Z. Diab, J. L. G. Guirao, M. A. López, R. Martínez. Existence of Periodic Solutions for a Class of the Generalized Liénard Equations. *Symmetry*, 2022, 14, 944, https://doi.org/10.3390/sym14050944.

[6] H. Lucas Koudahoun et al. Exact Classical and Quantum Mechanics of a Generalized Singular Equation of Quadratic Liénard Type. *Journal of Mathematics and Statistics*, 2018, vol 14: 183.192, doi: 10.3844/jmssp.2018.183.192.

[7] Jean Akande et al. Theory of Exact Trigonometric Periodic Solutions to Quadratic Liénard Type Equations. *Journal of Mathematics and Statistics*, 2018, vol. 14: 232.240, doi: 10.3844/jmssp.2018.232.240.

[8] A. Kirichuka, The number of solutions to the boundary value problem with linear-quintic and linear-cubic-quintic nonlinearity, *WSEAS Transactions on Mathematics*, vol. 19(64), pp.589-597, 2020, http://dx.doi.org/10.37394/23206.2020.19.64.

[9] E.Ellero and F.Zanolin. Homoclinic and heteroclinic solutions for a class of second-order non-autonomous ordinary differential equations: multiplicity results for stepwise potentials. *Boundary Value Problems*, 2013, vol.167, https://doi.org/10.1186/1687-2770-2013-167.

[10] A. Kirichuka, F. Sadyrbaev. Boundary Value Problems for Liènard-Type Equations with Quadratic Dependence on the 'Velocity', *Abstract and Applied Analysis*, Vol. 2022, Article ID 9228511, https://doi.org/10.1155/2022/9228511.

[11] M. Sabatini. On the period function of $x'' + f(x)x'^2 + g(x) =$. *Journal of Differential Equations*, vol.196 (2004), pp.151-168, https://doi.org/10.1016/S0022-0396(03)00067-6.

[12] Y. Klokov, N. Vasil'ev. Foundations of the theory of Boundary Value Problems for Ordinary Differential Equations, Riga, Zinatne, 1978 (in Russian).

[13] A. Gritsans, I. Yermachenko. On the Maximum Number of Period Annuli for Second Order Conservative Equations, Mathematical Modelling and Analysis, Vol. 26, Issue 4, 612–630, 2021, https://doi.org/10.3846/mma.2021.13979.

APPENDIX

We provide here some visual descriptions of equations of the form (3), which can be studied by the above-used approach. It is significant that following the half-ray $\theta(0) = \alpha$ (the first of the boundary conditions) one meat a trajectory entering a saddle point. Then slowly changing solutions of equation (3) exist in the vicinity of such trajectory. For the estimates of the number of solutions to BVPs the properties of the equations of variations are also essential.

Consider equation (15) with different functions $f(x)$, let us represent the phase portraits.

The phase portrait of equation (15), $a = 50, b = 25$, $f(x) = \mu = 1$ is depicted in Figure 1.

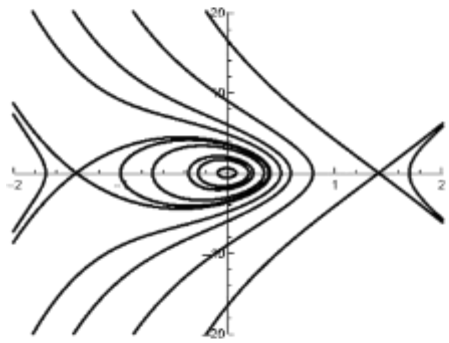

Fig. 1: *The phase portrait of* $x''+f(x)x'^2+(ax-bx^3)=0$, $a = 50$, $b = 25$, $f(x) = \mu = 1$.

The phase portrait of equation (15), $a = 50$, $b = 25$,
$f(x) = (x^2 - 1)$ is depicted in Figure 2.

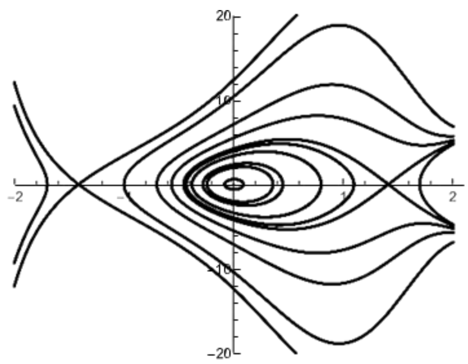

Fig. 2: *The phase portrait of* $x''+f(x)x'^2+(ax-bx^3)=0$, $a = 50$, $b = 25$, $f(x) = x^2 - 1$.

The phase portrait of equation (15), $a = 50, b = 25$, $f(x) = 1-x^3$ is depicted in Figure 3.

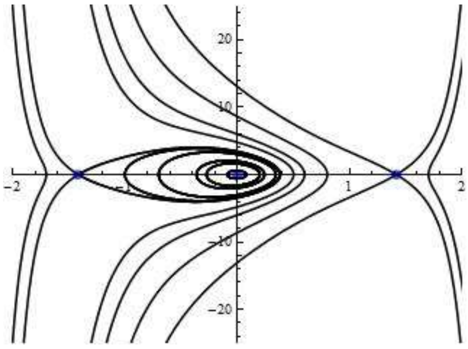

Fig. 3: *The phase portrait of* $x'' + f(x)x'^2 + (ax - bx^3) = 0$, $a = 50$, $b = 25$, $f(x) = 1-x^3$.

The phase portrait of equation (15), $a = 50, b = 25$, $f(x) = e^x$ is depicted in Figure 4.

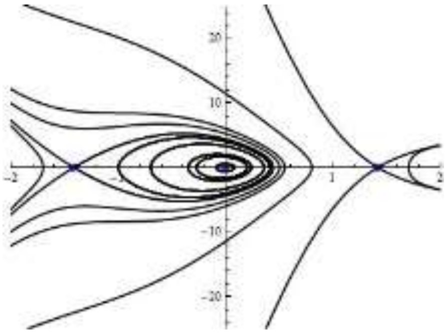

Fig. 4: *The phase portrait of* $x'' + f(x)x'^2 + (ax - bx^3) = 0$, $a = 50$, $b = 25$, $f(x) = e^x$.

The phase portrait of equation (15), $a = 50, b = 25$, $f(x) = x$ is depicted in Figure 5.

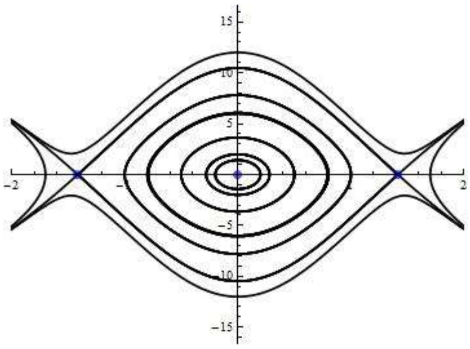

Fig. 5: *The phase portrait of* $x'' + f(x)x'^2 + (ax - bx^3) = 0$, $a = 50$, $b = 25$, $f(x) = x$.

The phase portrait of equation (15), $a = 50, b = 25$,
$f(x) = x^2$ is depicted in Figure 6.

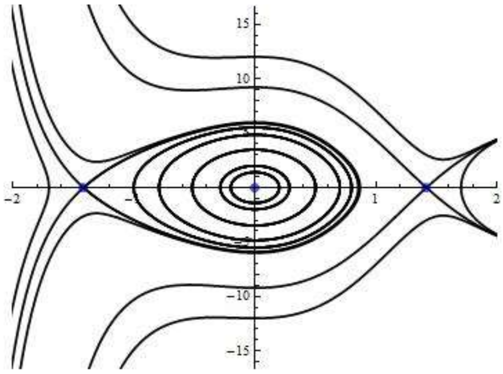

Fig. 6: *The phase portrait of* $x'' + f(x)x'^2 + (ax - bx^3) = 0$, $a = 50$, $b = 25$, $f(x) = x^2$.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

All authors have contributed equally to the creation of this article.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflict of interest to declare.