Exact Solutions of the Paraxial Wave Dynamical Model in Kerr Media with Truncated M-fractional Derivative using the $(G'/G, 1/G)$-Expansion Method

PIM MALINGAM[1], PAIWAN WONGSASINCHAI[1], SEKSON SIRISUBTAWEE[2,3], SANOE KOONPRASERT[2,3]

[1]Department of Mathematics, Faculty of Science and Technology
Rambhai Barni Rajabhat University, Chanthaburi 22000,
THAILAND

[2]Department of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok, Bangkok 10800,
THAILAND

[3]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400,
THAILAND

Abstract: The main purpose of this article is to use the $(G'/G, 1/G)$-expansion method to derive exact traveling wave solutions of the paraxial wave dynamical model in Kerr media in the sense of the truncated M-fractional derivative. To the best of the authors' knowledge, the solutions of the model obtained using the expansion method are reported here for the first time. The exact solutions are complex-valued functions expressed in terms of hyperbolic, trigonometric, and rational functions. In order to show the physical interpretations of the solutions, the magnitude of selected solutions is plotted in 3D, 2D, and contour plots for a range of values of the fractional-order of the equation. With the aid of a symbolic software package, all of the obtained solutions are substituted back into the relevant equation to verify their correctness. Obtaining the results by this technique confirms the strength and efficacy of the method for generating a variety of exact solutions of the problems arising in applied sciences and engineering.

Key-Words: - Exact solutions, Paraxial wave dynamical model, Kerr media, $(G'/G, 1/G)$-expansion method, Truncated M-fractional derivative, Anti-soliton solution.

Received: February 26, 2023. Revised: November 23, 2023. Accepted: December 11, 2023. Published: December 31, 2023.

1 Introduction

Many physical phenomena arising in nature can be modeled by nonlinear partial differential equations (NPDEs). For example, NPDEs have been used as models in mechanics, [1], fiber optics, [2], oceanography, [3], acoustics, [4], biology, [5], and finance, [6]. Hence, finding solutions of NPDEs has become important for modeling many real-world problems. In particular, finding efficient techniques for deriving exact traveling wave solutions of NPDEs has attracted the interest of many researchers. Efficient methods that have been developed for extracting exact solutions of NPDEs include the modified (G'/G^2)-expansion method, [7], the extended generalized (G'/G)-expansion method, [8], the tanh-coth method, [9], the Bäcklund transformations, [10], the F-expansion method, [11], the modified auxiliary equation method, [12], and the two-variables $(G'/G, 1/G)$-expansion method, [13].

One of the interesting PDEs in applied sciences and engineering is the paraxial wave dynamical equation model for transmission of light through optical fibers of Kerr media, [14], [15], [16], [17]. This model can be used to explain wave dynamics in optical fibers. This behavior includes optical solitons consisting of non-diffractive spatiotemporal and non-dispersive localized wave packets which transmit through the fiber. The paraxial wave equation in Kerr media can be written as, [15], [16], [17], [18],

$$i\frac{\partial W}{\partial y} + \frac{f}{2}\frac{\partial^2 W}{\partial t^2} + \frac{g}{2}\frac{\partial^2 W}{\partial x^2} + k|W|^2 W = 0, \quad (1)$$

where $W = W(x, y, t)$ is the Kerr term which represents the bound of the complex wave, and f, g, k are real constants. Eq. (1) for a monochromatic beam is equivalent to the nonlinear Schrödinger equation (NLSE) of a quantum particle. If f, g in Eq. (1) are such that $fg < 0$, Eq. (1) becomes a hyperbolic NLSE but if $fg > 0$, then Eq. (1) becomes an elliptic NLSE. In recent years, equation (1) has been solved using various analytical methods for its exact traveling wave solutions as follows. In [15] the authors obtained solitons, elliptic function, and other solutions of Eq. (1) via utilizing three analytical techniques, namely, the improved simple equation method, the $\exp(-\Phi(\zeta))$-expansion method and the modified extended direct algebraic technique. Distinct types of structures for the obtained solutions were depicted graphically. According to [16] some singular, periodic, solitary wave, and rational solutions of Eq. (1) were established using the Sardar subequation method (SSM). The modulus, real and imaginary plots of the solutions were demonstrated for manifold implementations in many research fields. In [17] the modified extended mapping technique was employed to assemble the solitons, solitary waves, and rational solutions for Eq. (1). The stability of Eq. (1) was studied via using modulational instability (MI) analysis from which all soliton solutions of the equation were verified to be stable and exact. Moreover, the ϕ^6 model expansion technique was utilized to obtain dark, bright, singular, bright-dark, and periodic solitons for Eq. (1) as discussed in [18]. The obtained solutions were expressed in the form of trigonometric, hyperbolic, and exponential functions. In addition, a recent literature review for solving Eq. (1) by using other different techniques and solving the fractional complex paraxial wave dynamical model with Kerr media in the sense of the conformable fractional derivative with respect to time can be found in [19], and, [20], respectively.

However, a truncated M-fractional derivative, [21], [22], [23], has recently attracted considerable attention from many research scholars. Many partial differential equations equipped with the truncated M-fractional derivatives have been solved for their exact traveling wave solutions which can be found in [24], [25], [26], [27], [28]. An application of such a derivative to the paraxial wave equation in Kerr media for formulating a novel equation is our major motivation. Consequently, it is very interesting to obtain exact solutions of the resulting equation.

In this article, we adapt Eq. (1) by replacing its classical partial derivatives with the truncated M-fractional derivatives. The new equation can be written as:

$$\begin{aligned} & i\,{}_m\partial^{\beta,\gamma}_{M,y}W + \frac{f}{2}\,{}_m\partial^{\beta,\gamma}_{M,t}\left({}_m\partial^{\beta,\gamma}_{M,t}W\right) \\ & + \frac{g}{2}\,{}_m\partial^{\beta,\gamma}_{M,x}\left({}_m\partial^{\beta,\gamma}_{M,x}W\right) + k|W|^2W = 0, \\ & 0<\beta\le 1, \end{aligned} \tag{2}$$

where ${}_m\partial^{\beta,\gamma}_{M,x}$, ${}_m\partial^{\beta,\gamma}_{M,y}$ and ${}_m\partial^{\beta,\gamma}_{M,t}$ are the truncated M-fractional partial derivatives of order β with respect to x, y, and t, respectively, which will be defined in section 2. The main aim of this paper is to use the $(G'/G, 1/G)$-expansion method to extract exact traveling wave solutions of Eq. (2) so that new solutions and their physical behaviors are revealed here for the first time. The remaining parts of this article are organized as follows. Section 2 includes the definition of the truncated M-fractional derivative and its characteristics. The main steps of the $(G'/G, 1/G)$-expansion method are also described in this section. The application of the expansion method to Eq. (2) is described in section 3. Graphical representations of chosen exact solutions are shown in section 4. The conclusions of this research are discussed in the final section.

2 Methoddology

In this section, a definition of the truncated M-fractional derivative, its important properties and a description of the $(G'/G, 1/G)$-expansion method are presented. They are required for constructing exact traveling wave solutions of Eq. (2).

2.1 Truncated M-fractional derivative and its properties

Definition 2.1 *The truncated Mittag-Leffler function with one parameter is defined as, [21], [22], [23],*

$${}_mE_\gamma(z) = \sum_{n=0}^{m} \frac{z^n}{\Gamma(\gamma n+1)}, \tag{3}$$

where $\gamma > 0$ and $z \in \mathbb{C}$.

Definition 2.2 *Let $f : [0,\infty) \to \mathbb{R}$ be a function. Then, the truncated M-fractional derivative of f of order β is defined by, [21], [22], [23],*

$${}_mD^{\beta,\gamma}_{M,t}f(t) = \lim_{\tau\to 0} \frac{f\left(t\,{}_mE_\gamma(\tau t^{-\beta})\right) - f(t)}{\tau}, \tag{4}$$

where $0 < \beta \le 1$ and $\gamma > 0$. If the limit in (4) exists, then we say that the function f is β-truncated M-fractional differentiable, or shortly, β-differentiable.

Moreover, if f is β-differentiable on $(0,a)$, $a > 0$ and $\lim_{t\to 0^+}\left({}_mD^{\beta,\gamma}_{M,t}f(t)\right)$ exists, then we define ${}_mD^{\beta,\gamma}_{M,t}f(0) = \lim_{t\to 0^+}\left({}_mD^{\beta,\gamma}_{M,t}f(t)\right)$. Some useful properties of the truncated M-fractional derivative are as follows, [21], [25], [29], [30], [31], [32]. Let $f(t)$, $g(t)$ be β-differentiable functions for all $t > 0$, $\beta \in (0,1]$, and $\gamma > 0$. Then, we have

(1) ${}_mD^{\beta,\gamma}_{M,t}(\lambda) = 0, \ \forall\lambda \in \mathbb{R}$.

(2) ${}_mD^{\beta,\gamma}_{M,t}(af(t)+bg(t)) = a\,{}_mD^{\beta,\gamma}_{M,t}f(t) + b\,{}_mD^{\beta,\gamma}_{M,t}g(t),\ \forall a,b\in\mathbb{R}$.

(3) ${}_mD^{\beta,\gamma}_{M,t}\left(f(t)g(t)\right) = f(t)\,{}_mD^{\beta,\gamma}_{M,t}g(t) + g(t)\,{}_mD^{\beta,\gamma}_{M,t}f(t)$.

(4) ${}_mD^{\beta,\gamma}_{M,t}\left(\dfrac{f(t)}{g(t)}\right) = \dfrac{g(t)\,{}_mD^{\beta,\gamma}_{M,t}f(t) - f(t)\,{}_mD^{\beta,\gamma}_{M,t}g(t)}{(g(t))^2}$, where $g(t)\neq 0$.

(5) ${}_mD^{\beta,\gamma}_{M,t}(f\circ g)(t) = f'(g(t))\,{}_mD^{\beta,\gamma}_{M,t}g(t)$ when f is differentiable at $g(t)$.

(6) If, in addition, f is differentiable, then ${}_mD^{\beta,\gamma}_{M,t}(f(t)) = \frac{t^{1-\beta}}{\Gamma(\gamma+1)}\frac{df(t)}{dt}$.

Utilizing the definition 2.2, the truncated M-fractional partial derivative of $u=u(x,t)$ with respect to $t>0$ of order $\beta\in(0,1]$ is defined as

$$
\begin{aligned}
&{}_m\partial^{\beta,\gamma}_{M,t}u(x,t) \\
&= \lim_{\tau\to 0}\frac{u\left(x, t\,{}_mE_\gamma(\tau t^{-\beta})\right) - u(x,t)}{\tau}.
\end{aligned} \tag{5}
$$

2.2 The $(G'/G, 1/G)$-expansion Method

Consider the following nonlinear partial differential equation in three independent variables t, x, and y:

$$
\begin{aligned}
&F\Big(u, {}_m\partial^{\alpha,\gamma}_{M,t}u, {}_m\partial^{\beta,\gamma}_{M,x}u, {}_m\partial^{\delta,\gamma}_{M,y}u, {}_m\partial^{\alpha,\gamma}_{M,t}\left({}_m\partial^{\beta,\gamma}_{M,x}u\right), \\
&{}_m\partial^{\alpha,\gamma}_{M,t}\left({}_m\partial^{\delta,\gamma}_{M,y}u\right), {}_m\partial^{\beta,\gamma}_{M,x}\left({}_m\partial^{\delta,\gamma}_{M,y}u\right), \ldots\Big) = 0,
\end{aligned} \tag{6}
$$

where $0<\alpha,\beta,\delta\le 1$, and ${}_m\partial^{\alpha,\gamma}_{M,t}u$, ${}_m\partial^{\beta,\gamma}_{M,x}u$, and ${}_m\partial^{\delta,\gamma}_{M,y}u$ are the truncated M-fractional partial derivatives of a dependent variable u with respect to independent variables t, x, and y. F is a polynomial of the unknown function $u=u(x,y,t)$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. Employing the following traveling wave transformation, [30]

$$
\begin{aligned}
&u(x,y,t) = U(\xi), \\
&\xi = \Gamma(\gamma+1)\left(\frac{kx^\beta}{\beta} + \frac{ly^\delta}{\delta} + \frac{ct^\alpha}{\alpha} + d\right),
\end{aligned} \tag{7}
$$

where k, l, c, and d are constants to be determined later, we can reduce Eq. (6) to the following ODE in $U=U(\xi)$:

$$
P(U, U', U'', \ldots) = 0, \tag{8}
$$

where P is a polynomial of $U(\xi)$ and its various derivatives in which the prime notation $(')$ represents the derivative with respect to ξ. Before we can provide the key steps of the $(G'/G, 1/G)$-expansion method, it is necessary to give the following information, [13], [33], [34], [35], [36]. Consider the following second-order linear ODE:

$$
G''(\xi) + \lambda G(\xi) = \mu, \tag{9}
$$

where λ, μ are constants. Denoting the functions ϕ and ψ as

$$
\phi(\xi) = \frac{G'(\xi)}{G(\xi)},\ \ \psi(\xi) = \frac{1}{G(\xi)}, \tag{10}
$$

we can transform equations (9) and (10) into the following system of two nonlinear ordinary differential equations

$$
\phi' = -\phi^2 + \mu\psi - \lambda,\ \ \psi' = -\phi\psi. \tag{11}
$$

The solutions of Eq. (9) can be separated into the following three cases.

Case 1: If $\lambda<0$, then the general solution of (9) is of the form

$$
\begin{aligned}
G(\xi) = {} & A_1\sinh\left(\xi\sqrt{-\lambda}\right) \\
& + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda},
\end{aligned} \tag{12}
$$

and we have the following relationship

$$
\psi^2 = \frac{-\lambda}{\lambda^2\sigma_1+\mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{13}
$$

where A_1 and A_2 are arbitrary constants and $\sigma_1 = A_1^2 - A_2^2$.

Case 2: If $\lambda>0$, then the general solution of (9) can be written as

$$
G(\xi) = A_1\sin(\xi\sqrt{\lambda}) + A_2\cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{14}
$$

and we have the following associated relation

$$
\psi^2 = \frac{\lambda}{\lambda^2\sigma_2-\mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{15}
$$

where A_1 and A_2 are arbitrary constants and $\sigma_2 = A_1^2 + A_2^2$.

Case 3: If $\lambda=0$, then the general solution of (9) can be displayed as

$$
G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{16}
$$

and the corresponding relation is

$$
\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right), \tag{17}
$$

where A_1 and A_2 are arbitrary constants.

The main steps of the $(G'/G, 1/G)$-expansion method, [13], [33], [34], [35], [36], can be described as follows.

Step 1. Assume that the solution to Eq. (8) can be expressed in terms of a polynomial of the two variables ϕ and ψ as follows

$$U(\xi) = a_0 + \sum_{j=1}^{N} a_j \phi^j + \sum_{j=1}^{N} b_j \phi^{j-1}\psi, \qquad (18)$$

where a_0, a_j, and $b_j, j = 1, 2, \ldots, N$, are constants to be found later with $a_N^2 + b_N^2 \neq 0$ and where the functions $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ are implicitly associated to Eq. (9) through the relations in Eq. (10).

Step 2. We determine the value of the positive integer N in Eq. (18) by balancing the highest order derivative and the nonlinear terms in Eq. (8). Denoting the degree of $U(\xi)$ by $\text{Deg}[U(\xi)] = N$, we can calculate the degree of other terms in the equation using the following relations

$$\begin{aligned} &\text{Deg}\left[\frac{d^q U(\xi)}{d\xi^q}\right] = N + q, \\ &\text{Deg}\left[(U(\xi))^p \left(\frac{d^q U(\xi)}{d\xi^q}\right)^s\right] = Np + s(N + q). \end{aligned} \qquad (19)$$

Step 3. Replacing the solution form (18) with the known value of N into Eq. (8) with the assistance of Eq. (11) and Eq. (13), we can convert the function P in Eq. (8) into a polynomial in ϕ and ψ in which the degree of ψ is one. Equating each coefficient of the resulting polynomial to zero, we obtain a system of algebraic equations. These algebraic equations can then be solved using the Maple software package to obtain values for the unknowns $a_0, a_j, b_j, j = 1, 2, \ldots, N, k, l, c, d, \mu, \lambda (< 0)$. Therefore, the exact solutions of Eq. (6) can be obtained in terms of hyperbolic functions by using Eq. (12) and the transformation in Eq. (7).

Step 4. Similar to Step 3, substituting the result from Eq. (18) into Eq. (8) with the aid of Eq. (11) and Eq. (15) for $\lambda > 0$, we can obtain the exact solutions of Eq. (6) by using the transformation (7). The obtained exact solutions in this step are written in terms of trigonometric functions.

Step 5. In the same manner as Step 3, substituting the result from Eq. (18) into Eq. (8) with the aid of Eq. (11) and Eq. (17) for $\lambda = 0$, we can obtain the traveling wave solutions of Eq. (6) with the help of the transformation (7). The resulting exact solutions in this step are obtained in terms of rational functions.

Remark 1: Particularly, if the balance number N in Step 2 is not a positive integer, then some special transformations must be applied for $U(\xi)$ in (8) so that the equation (8) can be converted into a new equation of a new variable. For example, if $N = \frac{q}{p}$ is a fraction in the lowest terms, then $U(\xi) = V^{\frac{q}{p}}(\xi)$ is inserted in (8). Consequently, the new equation, written in terms of $V(\xi)$, has a positive integer balance number, [37].

Remark 2: When applied, the $(G'/G, 1/G)$-expansion method can provide three types of exact solutions including hyperbolic, trigonometric, and rational function solutions.

Remark 3: The $(G'/G, 1/G)$-expansion method can be reduced to the (G'/G)-expansion method by some special setting, [37]. Thus, the $(G'/G, 1/G)$-expansion method is more effective and more general than the (G'/G)-expansion method.

3 Implementation of the $(G'/G, 1/G)$-expansion Method

In this section, we obtain exact traveling wave solutions of Eq. (2) by using the $(G'/G, 1/G)$-expansion method. First, we assume that the exact solution of (2) has the form

$$W(x, y, t) = U(\chi)e^{i\xi}, \qquad (20)$$

where U is a real-valued function of χ, $i = \sqrt{-1}$, and

$$\begin{aligned} \chi &= \frac{\Gamma(\gamma+1)}{\beta}(d_1 x^\beta + d_2 y^\beta + \rho t^\beta), \\ \xi &= \frac{\Gamma(\gamma+1)}{\beta}(s_1 x^\beta + s_2 y^\beta + \tau t^\beta + \omega), \end{aligned} \qquad (21)$$

where d_1, d_2, ρ, s_1, s_2, τ, and ω are real constants, the order $0 < \beta \leq 1$, and the parameter $\gamma > 0$. Substituting the solution form (20) into the proposed problem (2) and then separating the real and imaginary parts of the resulting equation, we obtain the following equations:

$$\begin{aligned} \text{Re:} \quad & \left(f\rho^2 + g d_1^2\right)U''(\chi) - \left(f\tau^2 + g s_1^2 + 2\,s_2\right)U(\chi) \\ & + 2\,k U^3(\chi) = 0, \qquad (22) \\ \text{Im:} \quad & (d_2 + f\rho\,\tau + g d_1 s_1)\,U'(\chi) = 0. \qquad (23) \end{aligned}$$

Since $U'(\chi)$ in Eq. (23) is not zero, we have

$$d_2 = -(f\rho\,\tau + g d_1 s_1). \qquad (24)$$

Next, balancing the terms U'' and U^3 in Eq. (22) via the formulas in (19), we obtain $N = 1$ and the solution form of Eq. (22) is then

$$U(\chi) = a_0 + a_1\phi(\chi) + b_1\psi(\chi), \qquad (25)$$

where a_0, a_1, and b_1 are constant coefficients which will be determined later such that $a_1^2 + b_1^2 \neq 0$. As explained in equations (12), (14), and (16), there are three cases for the functions $\phi(\chi)$ and $\psi(\chi)$ in (25) depending on the sign of λ.

Case 1 (Hyperbolic function solutions): If $\lambda < 0$, we substitute Eq.(25) with Eqs.(11) and (13) into Eq.(22) and then the left-hand side of Eq. (22) becomes a polynomial in $\phi(\chi)$ and $\psi(\chi)$. Setting all of the coefficients of this resulting polynomial to zero, we obtain the following system of nonlinear algebraic equations in $\lambda, \mu, \omega, \rho, \tau, a_0, a_1, b_1, d_1, s_1$, and s_2:

$$\begin{aligned}
\phi^3 \;:\;& 2\,f\lambda^4\rho^2A_1^4a_1 - 4\,f\lambda^4\rho^2A_1^2A_2^2a_1 \\
&+ 2\,f\lambda^4\rho^2A_2^4a_1 + 2\,g\lambda^4A_1^4a_1d_1^2 \\
&- 4\,g\lambda^4A_1^2A_2^2a_1d_1^2 + 2\,g\lambda^4A_2^4a_1d_1^2 \\
&+ 2\,k\lambda^4A_1^4a_1^3 - 4\,k\lambda^4A_1^2A_2^2a_1^3 \\
&+ 2\,k\lambda^4A_2^4a_1^3 + 4\,f\lambda^2\mu^2\rho^2A_1^2a_1 \\
&- 4\,f\lambda^2\mu^2\rho^2A_2^2a_1 + 4\,g\lambda^2\mu^2A_1^2a_1d_1^2 \\
&- 4\,g\lambda^2\mu^2A_2^2a_1d_1^2 + 4\,k\lambda^2\mu^2A_1^2a_1^3 \\
&- 4\,k\lambda^2\mu^2A_2^2a_1^3 - 6\,k\lambda^3A_1^2a_1b_1^2 \\
&+ 6\,k\lambda^3A_2^2a_1b_1^2 + 2\,f\mu^4\rho^2a_1 \\
&+ 2\,g\mu^4a_1d_1^2 + 2\,k\mu^4a_1^3 - 6\,k\lambda\,\mu^2a_1b_1^2 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\phi^2 \;:\;& 6\,k\lambda^4A_1^4a_0a_1^2 - 12\,k\lambda^4A_1^2A_2^2a_0a_1^2 \\
&+ 6\,k\lambda^4A_2^4a_0a_1^2 + f\lambda^3\mu\,\rho^2A_1^2b_1 \\
&- f\lambda^3\mu\,\rho^2A_2^2b_1 + g\lambda^3\mu\,A_1^2b_1d_1^2 \\
&- g\lambda^3\mu\,A_2^2b_1d_1^2 + 12\,k\lambda^2\mu^2A_1^2a_0a_1^2 \\
&- 12\,k\lambda^2\mu^2A_2^2a_0a_1^2 - 6\,k\lambda^3A_1^2a_0b_1^2 \\
&+ 6\,k\lambda^3A_2^2a_0b_1^2 + f\lambda\,\mu^3\rho^2b_1 \\
&+ g\lambda\,\mu^3b_1d_1^2 + 6\,k\mu^4a_0a_1^2 - 4\,b_1^3\lambda^2k\mu \\
&- 6\,k\lambda\,\mu^2a_0b_1^2 = 0,
\end{aligned}$$

$$\begin{aligned}
\phi^2\psi \;:\;& 2\,f\lambda^4\rho^2A_1^4b_1 - 4\,f\lambda^4\rho^2A_1^2A_2^2b_1 \\
&+ 2\,f\lambda^4\rho^2A_2^4b_1 + 2\,g\lambda^4A_1^4b_1d_1^2 \\
&- 4\,g\lambda^4A_1^2A_2^2b_1d_1^2 + 2\,g\lambda^4A_2^4b_1d_1^2 \\
&+ 6\,k\lambda^4A_1^4a_1^2b_1 - 12\,k\lambda^4A_1^2A_2^2a_1^2b_1 \\
&+ 6\,k\lambda^4A_2^4a_1^2b_1 + 4\,f\lambda^2\mu^2\rho^2A_1^2b_1 \\
&- 4\,f\lambda^2\mu^2\rho^2A_2^2b_1 + 4\,g\lambda^2\mu^2A_1^2b_1d_1^2 \\
&- 4\,g\lambda^2\mu^2A_2^2b_1d_1^2 + 12\,k\lambda^2\mu^2A_1^2a_1^2b_1 \\
&- 12\,k\lambda^2\mu^2A_2^2a_1^2b_1 - 2\,k\lambda^3A_1^2b_1^3 \\
&+ 2\,k\lambda^3A_2^2b_1^3 + 2\,f\mu^4\rho^2b_1 + 2\,g\mu^4b_1d_1^2 \\
&+ 6\,k\mu^4a_1^2b_1 - 2\,k\lambda\,\mu^2b_1^3 = 0,
\end{aligned}$$

$$\begin{aligned}
\phi \;:\;& 2\,f\lambda^5\rho^2A_1^4a_1 - 4\,f\lambda^5\rho^2A_1^2A_2^2a_1 \\
&+ 2\,f\lambda^5\rho^2A_2^4a_1 + 2\,g\lambda^5A_1^4a_1d_1^2 \\
&- 4\,g\lambda^5A_1^2A_2^2a_1d_1^2 + 2\,g\lambda^5A_2^4a_1d_1^2 \\
&- f\lambda^4\tau^2A_1^4a_1 + 2\,f\lambda^4\tau^2A_1^2A_2^2a_1 \\
&- f\lambda^4\tau^2A_2^4a_1 - g\lambda^4A_1^4a_1s_1^2 \\
&+ 2\,g\lambda^4A_1^2A_2^2a_1s_1^2 - g\lambda^4A_2^4a_1s_1^2 \\
&+ 6\,k\lambda^4A_1^4a_0^2a_1 - 12\,k\lambda^4A_1^2A_2^2a_0^2a_1 \\
&+ 6\,k\lambda^4A_2^4a_0^2a_1 + 4\,f\lambda^3\mu^2\rho^2A_1^2a_1 \\
&- 4\,f\lambda^3\mu^2\rho^2A_2^2a_1 + 4\,g\lambda^3\mu^2A_1^2a_1d_1^2 \\
&- 4\,g\lambda^3\mu^2A_2^2a_1d_1^2 - 2\,f\lambda^2\mu^2\tau^2A_1^2a_1 \\
&+ 2\,f\lambda^2\mu^2\tau^2A_2^2a_1 - 2\,g\lambda^2\mu^2A_1^2a_1s_1^2 \\
&+ 2\,g\lambda^2\mu^2A_2^2a_1s_1^2 - 6\,k\lambda^4A_1^2a_1b_1^2 \\
&+ 6\,k\lambda^4A_2^2a_1b_1^2 + 12\,k\lambda^2\mu^2A_1^2a_0^2a_1 \\
&- 12\,k\lambda^2\mu^2A_2^2a_0^2a_1 - 2\,\lambda^4A_1^4a_1s_2 \\
&+ 4\,\lambda^4A_1^2A_2^2a_1s_2 - 2\,\lambda^4A_2^4a_1s_2 \\
&+ 2\,f\lambda\,\mu^4\rho^2a_1 + 2\,g\lambda\,\mu^4a_1d_1^2 \\
&- f\mu^4\tau^2a_1 - g\mu^4a_1s_1^2 - 6\,k\lambda^2\mu^2a_1b_1^2 \\
&+ 6\,k\mu^4a_0^2a_1 - 4\,\lambda^2\mu^2A_1^2a_1s_2 \\
&+ 4\,\lambda^2\mu^2A_2^2a_1s_2 - 2\,\mu^4a_1s_2 = 0,
\end{aligned}$$

$$\begin{aligned}
\phi\psi \;:\;& -3\,f\lambda^4\mu\,\rho^2A_1^4a_1 + 6\,f\lambda^4\mu\,\rho^2A_1^2A_2^2a_1 \\
&- 3\,f\lambda^4\mu\,\rho^2A_2^4a_1 - 3\,g\lambda^4\mu\,A_1^4a_1d_1^2 \\
&+ 6\,g\lambda^4\mu\,A_1^2A_2^2a_1d_1^2 - 3\,g\lambda^4\mu\,A_2^4a_1d_1^2 \\
&+ 12\,k\lambda^4A_1^4a_0a_1b_1 - 24\,k\lambda^4A_1^2A_2^2a_0a_1b_1 \\
&+ 12\,k\lambda^4A_2^4a_0a_1b_1 - 6\,f\lambda^2\mu^3\rho^2A_1^2a_1 \\
&+ 6\,f\lambda^2\mu^3\rho^2A_2^2a_1 - 6\,g\lambda^2\mu^3A_1^2a_1d_1^2 \\
&+ 6\,g\lambda^2\mu^3A_2^2a_1d_1^2 + 12\,k\lambda^3\mu\,A_1^2a_1b_1^2 \\
&- 12\,k\lambda^3\mu\,A_2^2a_1b_1^2 + 24\,k\lambda^2\mu^2A_1^2a_0a_1b_1 \\
&- 24\,k\lambda^2\mu^2A_2^2a_0a_1b_1 - 3\,f\mu^5\rho^2a_1 \\
&- 3\,g\mu^5a_1d_1^2 + 12\,k\lambda\,\mu^3a_1b_1^2 \\
&+ 12\,k\mu^4a_0a_1b_1 = 0,
\end{aligned}$$

$$\begin{aligned}
\psi \;:\;& f\lambda^5\rho^2A_1^4b_1 - 2\,f\lambda^5\rho^2A_1^2A_2^2b_1 \\
&+ f\lambda^5\rho^2A_2^4b_1 + g\lambda^5A_1^4b_1d_1^2 \\
&- 2\,g\lambda^5A_1^2A_2^2b_1d_1^2 + g\lambda^5A_2^4b_1d_1^2 \\
&- f\lambda^4\tau^2A_1^4b_1 + 2\,f\lambda^4\tau^2A_1^2A_2^2b_1 \\
&- f\lambda^4\tau^2A_2^4b_1 - g\lambda^4A_1^4b_1s_1^2 \\
&+ 2\,g\lambda^4A_1^2A_2^2b_1s_1^2 - g\lambda^4A_2^4b_1s_1^2 \\
&+ 6\,k\lambda^4A_1^4a_0^2b_1 - 12\,k\lambda^4A_1^2A_2^2a_0^2b_1 \\
&+ 6\,k\lambda^4A_2^4a_0^2b_1 - 2\,f\lambda^2\mu^2\tau^2A_1^2b_1 \\
&+ 2\,f\lambda^2\mu^2\tau^2A_2^2b_1 - 2\,g\lambda^2\mu^2A_1^2b_1s_1^2 \\
&+ 2\,g\lambda^2\mu^2A_2^2b_1s_1^2 - 2\,k\lambda^4A_1^2b_1^3 \\
&+ 2\,k\lambda^4A_2^2b_1^3 + 12\,k\lambda^3\mu\,A_1^2a_0b_1^2 \\
&- 12\,k\lambda^3\mu\,A_2^2a_0b_1^2 + 12\,k\lambda^2\mu^2A_1^2a_0^2b_1 \\
&- 12\,k\lambda^2\mu^2A_2^2a_0^2b_1 - 2\,\lambda^4A_1^4b_1s_2 \\
&+ 4\,\lambda^4A_1^2A_2^2b_1s_2 - 2\,\lambda^4A_2^4b_1s_2
\end{aligned}$$

$$
\begin{aligned}
& - f\lambda\,\mu^4\rho^2 b_1 - g\lambda\,\mu^4 b_1 d_1^2 - f\mu^4\tau^2 b_1 \\
& - g\mu^4 b_1 s_1^2 + 6\,k\lambda^2\mu^2 b_1^3 + 12\,k\lambda\,\mu^3 a_0 b_1^2 \\
& + 6\,k\mu^4 a_0^2 b_1 - 4\,\lambda^2\mu^2 A_1^2 b_1 s_2 \\
& + 4\,\lambda^2\mu^2 A_2^2 b_1 s_2 - 2\,\mu^4 b_1 s_2 = 0, \\
\phi^0\ : \ & - f\lambda^4\tau^2 A_1^4 a_0 + 2\,f\lambda^4\tau^2 A_1^2 A_2^2 a_0 \\
& - f\lambda^4\tau^2 A_2^4 a_0 - g\lambda^4 A_1^4 a_0 s_1^2 \\
& + 2\,g\lambda^4 A_1^2 A_2^2 a_0 s_1^2 - g\lambda^4 A_2^4 a_0 s_1^2 \\
& + 2\,k\lambda^4 A_1^4 a_0^3 - 4\,k\lambda^4 A_1^2 A_2^2 a_0^3 \\
& + 2\,k\lambda^4 A_2^4 a_0^3 + f\lambda^4\mu\,\rho^2 A_1^2 b_1 \\
& - f\lambda^4\mu\,\rho^2 A_2^2 b_1 + g\lambda^4\mu\,A_1^2 b_1 d_1^2 \\
& - g\lambda^4\mu\,A_2^2 b_1 d_1^2 - 2\,f\lambda^2\mu^2\tau^2 A_1^2 a_0 \\
& + 2\,f\lambda^2\mu^2\tau^2 A_2^2 a_0 - 2\,g\lambda^2\mu^2 A_1^2 a_0 s_1^2 \\
& + 2\,g\lambda^2\mu^2 A_2^2 a_0 s_1^2 - 6\,k\lambda^4 A_1^2 a_0 b_1^2 \\
& + 6\,k\lambda^4 A_2^2 a_0 b_1^2 + 4\,k\lambda^2\mu^2 A_1^2 a_0^3 \\
& - 4\,k\lambda^2\mu^2 A_2^2 a_0^3 - 2\,\lambda^4 A_1^4 a_0 s_2 \\
& + 4\,\lambda^4 A_1^2 A_2^2 a_0 s_2 - 2\,\lambda^4 A_2^4 a_0 s_2 \\
& + f\lambda^2\mu^3\rho^2 b_1 + g\lambda^2\mu^3 b_1 d_1^2 \\
& - f\mu^4\tau^2 a_0 - g\mu^4 a_0 s_1^2 - 4\,b_1^3\lambda^3 k\mu \\
& - 6\,k\lambda^2\mu^2 a_0 b_1^2 + 2\,k\mu^4 a_0^3 \\
& - 4\,\lambda^2\mu^2 A_1^2 a_0 s_2 + 4\,\lambda^2\mu^2 A_2^2 a_0 s_2 \\
& - 2\,\mu^4 a_0 s_2 = 0.
\end{aligned} \tag{26}
$$

Solving the above algebraic system using the Maple package program, we get the following results.

Result 1

$$
\begin{aligned}
& \lambda = \lambda,\ \mu = 0,\ \omega = \omega,\ \rho = \rho,\ \tau = \tau,\ a_0 = 0, \\
& a_1 = \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{k}},\ b_1 = 0, d_1 = d_1, \\
& d_2 = -\left(f\rho\,\tau + g d_1 s_1\right),\ s_1 = s_1, \\
& s_2 = f\lambda\,\rho^2 + g\lambda\,d_1^2 - \frac{f\tau^2}{2} - \frac{g s_1^2}{2},
\end{aligned} \tag{27}
$$

where $\lambda(<0)$, $f,\ g,\ k, d_1,\ \rho,\ s_1,\ \tau,\ \omega$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (12), (20), (25), and (27), we get the solution of Eq. (2) as:

$$
\begin{aligned}
& W(x,y,t) = \\
& \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{k}}\left(\frac{\left(A_1\cosh\chi\sqrt{-\lambda}\right)\sqrt{-\lambda} + A_2\sinh\left(\chi\sqrt{-\lambda}\right)\sqrt{-\lambda}}{A_1\sinh\left(\chi\sqrt{-\lambda}\right) + A_2\cosh\left(\chi\sqrt{-\lambda}\right)}\right) \\
& \times \mathrm{e}^{i\xi},
\end{aligned} \tag{28}
$$

where $A_1,\ A_2$ arbitrary constants and

$$
\chi = \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\,t^\beta\right),
$$

$$
\xi = \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta + \left(f\lambda\,\rho^2 + g\lambda\,d_1^2 - \frac{1}{2}\,f\tau^2 - \frac{1}{2}\,g s_1^2\right) y^\beta + \tau t^\beta + \omega\right).
$$

Result 2

$$
\begin{aligned}
& \lambda = \frac{k b_1^2}{\left(f\rho^2 + g d_1^2\right)\sigma_1},\ \mu = 0,\ \omega = \omega, \\
& \rho = \rho,\ \tau = \tau,\ a_0 = 0,\ a_1 = 0,\ b_1 = b_1, \\
& d_1 = d_1,\ d_2 = -\left(f\rho\,\tau + g d_1 s_1\right), \\
& s_1 = s_1,\ s_2 = -\frac{\sigma_1\left(f\tau^2 + g s_1^2\right) + k b_1^2}{2\sigma_1},
\end{aligned} \tag{29}
$$

where $\sigma_1 = A_1^2 - A_2^2$ and $f,\ g,\ k,\ d_1,\ \rho,\ s_1,\ \tau,\ \omega,\ b_1,$ $A_1,\ A_2$ are arbitrary constants such that $\lambda < 0$. From Eqs. (12), (20), (25), and (29), we obtain the exact solution of Eq. (2) as follows:

$$
\begin{aligned}
& W(x,y,t) = \\
& \frac{b_1}{A_1\sinh\left(\chi\sqrt{-\lambda}\right) + A_2\cosh\left(\chi\sqrt{-\lambda}\right)} \times \mathrm{e}^{i\xi},
\end{aligned} \tag{30}
$$

where

$$
\chi = \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\,t^\beta\right),
$$

$$
\xi = \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta - \left(\frac{\sigma_1\left(f\tau^2 + g s_1^2\right) + k b_1^2}{2\sigma_1}\right) y^\beta + \tau t^\beta + \omega\right).
$$

Result 3

$$
\begin{aligned}
& \lambda = \frac{4 k b_1^2}{\sigma_1\left(f\rho^2 + g d_1^2\right)}, \mu = 0, \omega = \omega, \rho = \rho, \\
& \tau = \tau, a_0 = 0, a_1 = \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{4k}}, \\
& b_1 = b_1,\ d_1 = d_1,\ d_2 = -\left(f\rho\,\tau + g d_1 s_1\right),\ s_1 = s_1, \\
& s_2 = -\frac{\sigma_1\left(f\tau^2 + g s_1^2\right) - 2 k b_1^2}{2\sigma_1},
\end{aligned} \tag{31}
$$

where $\sigma_1 = A_1^2 - A_2^2$ and $f,\ g,\ k,\ d_1,\ \rho,\ s_1,\ \tau,\ \omega,\ b_1,$ $A_1,\ A_2$ are arbitrary constants such that $\lambda < 0$ and $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (12), (20), (25), and (31), we obtain the exact solution of Eq. (2) as follows:

$$
\begin{aligned}
& W(x,y,t) = \\
& \left\{ \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{4k}}\left(\frac{A_1\cosh\left(\chi\sqrt{-\lambda}\right)\sqrt{-\lambda} + A_2\sinh\left(\chi\sqrt{-\lambda}\right)\sqrt{-\lambda}}{A_1\sinh\left(\chi\sqrt{-\lambda}\right) + A_2\cosh\left(\chi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}}\right)\right. \\
& \left. + \frac{b_1}{A_1\sinh\left(\chi\sqrt{-\lambda}\right) + A_2\cosh\left(\chi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}}\right\} \times \mathrm{e}^{i\xi},
\end{aligned} \tag{32}
$$

where

$$
\chi = \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\,t^\beta\right),
$$

$$
\xi = \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta - \left(\frac{\sigma_1\left(f\tau^2 + g s_1^2\right) - 2 k b_1^2}{2\sigma_1}\right) y^\beta + \tau t^\beta + \omega\right).
$$

Result 4

$$\begin{aligned}
&\lambda=\lambda,\ \mu=\mu,\ \omega=\omega,\ \rho=\rho,\ \tau=\tau,\ a_0=0,\\
&a_1=\pm\sqrt{-\frac{f\rho^2+gd_1^2}{4k}},\\
&b_1=\pm\sqrt{\frac{(f\rho^2+gd_1^2)\lambda^2\sigma_1+(f\rho^2+gd_1^2)\mu^2}{4k\lambda}},\\
&d_1=d_1,\ d_2=-\left(f\rho\,\tau+gd_1s_1\right),\ s_1=s_1,\\
&s_2=\frac{1}{4}\lambda\left(f\rho^2+gd_1^2\right)-\frac{1}{2}\left(f\tau^2+gs_1^2\right),
\end{aligned} \tag{33}$$

where $\sigma_1=A_1^2-A_2^2$ and $\lambda(<0)$, μ, f, g, k, d_1, ρ, s_1, τ, ω, A_1, A_2 are arbitrary constants such that $k\left(f\rho^2+gd_1^2\right)<0$ and $b_1\in\mathbb{R}$. From Eqs. (12), (20), (25), and (33), we obtain the exact solution of Eq. (2) as follows:

$$\begin{aligned}
&W(x,y,t)=\\
&\Bigg\{\pm\sqrt{-\frac{f\rho^2+gd_1^2}{4k}}\left(\frac{A_1\cosh\left(\chi\sqrt{-\lambda}\right)\sqrt{-\lambda}+A_2\sinh\left(\chi\sqrt{-\lambda}\right)\sqrt{-\lambda}}{A_1\sinh\left(\chi\sqrt{-\lambda}\right)+A_2\cosh\left(\chi\sqrt{-\lambda}\right)+\frac{\mu}{\lambda}}\right)\\
&\pm\left(\frac{\sqrt{\frac{(f\rho^2+gd_1^2)\lambda^2\sigma_1+(f\rho^2+gd_1^2)\mu^2}{4k\lambda}}}{\left(A_1\sinh\left(\chi\sqrt{-\lambda}\right)+A_2\cosh\left(\chi\sqrt{-\lambda}\right)+\frac{\mu}{\lambda}\right)}\right)\Bigg\}\times e^{i\xi},
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
&\chi=\frac{\Gamma(\gamma+1)}{\beta}\left(d_1x^\beta-\left(f\rho\,\tau+gd_1s_1\right)y^\beta+\rho\,t^\beta\right),\\
&\xi=\frac{\Gamma(\gamma+1)}{\beta}\left(s_1x^\beta+\left(\frac{1}{4}\lambda\left(f\rho^2+gd_1^2\right)-\frac{1}{2}\left(f\tau^2+gs_1^2\right)\right)y^\beta+\tau\,t^\beta+\omega\right).
\end{aligned}$$

Case 2 (Trigonometric function solutions):
If $\lambda>0$, we substitute Eq.(25) with Eqs.(11) and (15) into Eq.(22), so that the left-hand side of Eq.(22) becomes a polynomial in $\phi(\chi)$ and $\psi(\chi)$. Setting all of the coefficients of this resulting polynomial to zero, we obtain the following system of nonlinear algebraic equations in $\lambda,\mu,\omega,\rho,\tau,a_0,a_1,b_1,d_1,s_1$, and s_2:

$$\begin{aligned}
\phi^3\ :\ & 2f\lambda^4\rho^2A_1^4a_1+4f\lambda^4\rho^2A_1^2A_2^2a_1\\
&+2f\lambda^4\rho^2A_2^4a_1+2g\lambda^4A_1^4a_1d_1^2\\
&+4g\lambda^4A_1^2A_2^2a_1d_1^2+2g\lambda^4A_2^4a_1d_1^2\\
&+2k\lambda^4A_1^4a_1^3+4k\lambda^4A_1^2A_2^2a_1^3\\
&+2k\lambda^4A_2^4a_1^3-4f\lambda^2\mu^2\rho^2A_1^2a_1\\
&-4f\lambda^2\mu^2\rho^2A_2^2a_1-4g\lambda^2\mu^2A_1^2a_1d_1^2\\
&-4g\lambda^2\mu^2A_2^2a_1d_1^2-4k\lambda^2\mu^2A_1^2a_1^3\\
&-4k\lambda^2\mu^2A_2^2a_1^3+6k\lambda^3A_1^2a_1b_1^2\\
&+6k\lambda^3A_2^2a_1b_1^2+2f\mu^4\rho^2a_1\\
&+2g\mu^4a_1d_1^2+2k\mu^4a_1^3-6k\lambda\,\mu^2a_1b_1^2=0,
\end{aligned}$$

$$\begin{aligned}
\phi^2\ :\ & 6k\lambda^4A_1^4a_0a_1^2+12k\lambda^4A_1^2A_2^2a_0a_1^2\\
&+6k\lambda^4A_2^4a_0a_1^2-f\lambda^3\mu\,\rho^2A_1^2b_1\\
&-f\lambda^3\mu\,\rho^2A_2^2b_1-g\lambda^3\mu\,A_1^2b_1d_1^2\\
&-g\lambda^3\mu\,A_2^2b_1d_1^2-12k\lambda^2\mu^2A_1^2a_0a_1^2\\
&-12k\lambda^2\mu^2A_2^2a_0a_1^2+6k\lambda^3A_1^2a_0b_1^2\\
&+6k\lambda^3A_2^2a_0b_1^2+f\lambda\,\mu^3\rho^2b_1\\
&+g\lambda\,\mu^3b_1d_1^2+6k\mu^4a_0a_1^2\\
&-4k\lambda^2\mu\,b_1^3-6k\lambda\,\mu^2a_0b_1^2=0,
\end{aligned}$$

$$\begin{aligned}
\phi^2\psi\ :\ & 2f\lambda^4\rho^2A_1^4b_1+4f\lambda^4\rho^2A_1^2A_2^2b_1\\
&+2f\lambda^4\rho^2A_2^4b_1+2g\lambda^4A_1^4b_1d_1^2\\
&+4g\lambda^4A_1^2A_2^2b_1d_1^2+2g\lambda^4A_2^4b_1d_1^2\\
&+6k\lambda^4A_1^4a_1^2b_1+12k\lambda^4A_1^2A_2^2a_1^2b_1\\
&+6k\lambda^4A_2^4a_1^2b_1-4f\lambda^2\mu^2\rho^2A_1^2b_1\\
&-4f\lambda^2\mu^2\rho^2A_2^2b_1-4g\lambda^2\mu^2A_1^2b_1d_1^2\\
&-4g\lambda^2\mu^2A_2^2b_1d_1^2-12k\lambda^2\mu^2A_1^2a_1^2b_1\\
&-12k\lambda^2\mu^2A_2^2a_1^2b_1+2k\lambda^3A_1^2b_1^3\\
&+2k\lambda^3A_2^2b_1^3+2f\mu^4\rho^2b_1\\
&+2g\mu^4b_1d_1^2+6k\mu^4a_1^2b_1\\
&-2k\lambda\,\mu^2b_1^3=0,
\end{aligned}$$

$$\begin{aligned}
\phi\ :\ & 2f\lambda^5\rho^2A_1^4a_1+4f\lambda^5\rho^2A_1^2A_2^2a_1\\
&+2f\lambda^5\rho^2A_2^4a_1+2g\lambda^5A_1^4a_1d_1^2\\
&+4g\lambda^5A_1^2A_2^2a_1d_1^2+2g\lambda^5A_2^4a_1d_1^2\\
&-f\lambda^4\tau^2A_1^4a_1-2f\lambda^4\tau^2A_1^2A_2^2a_1\\
&-f\lambda^4\tau^2A_2^4a_1-g\lambda^4A_1^4a_1s_1^2\\
&-2g\lambda^4A_1^2A_2^2a_1s_1^2-g\lambda^4A_2^4a_1s_1^2\\
&+6k\lambda^4A_1^4a_0^2a_1+12k\lambda^4A_1^2A_2^2a_0^2a_1\\
&+6k\lambda^4A_2^4a_0^2a_1-4f\lambda^3\mu^2\rho^2A_1^2a_1\\
&-4f\lambda^3\mu^2\rho^2A_2^2a_1-4g\lambda^3\mu^2A_1^2a_1d_1^2\\
&-4g\lambda^3\mu^2A_2^2a_1d_1^2+2f\lambda^2\mu^2\tau^2A_1^2a_1\\
&+2f\lambda^2\mu^2\tau^2A_2^2a_1+2g\lambda^2\mu^2A_1^2a_1s_1^2\\
&+2g\lambda^2\mu^2A_2^2a_1s_1^2+6k\lambda^4A_1^2a_1b_1^2\\
&+6k\lambda^4A_2^2a_1b_1^2-12k\lambda^2\mu^2A_1^2a_0^2a_1\\
&-12k\lambda^2\mu^2A_2^2a_0^2a_1-2\lambda^4A_1^4a_1s_2\\
&-4\lambda^4A_1^2A_2^2a_1s_2-2\lambda^4A_2^4a_1s_2\\
&+2f\lambda\,\mu^4\rho^2a_1+2g\lambda\,\mu^4a_1d_1^2\\
&-f\mu^4\tau^2a_1-g\mu^4a_1s_1^2-6k\lambda^2\mu^2a_1b_1^2\\
&+6k\mu^4a_0^2a_1+4\lambda^2\mu^2A_1^2a_1s_2\\
&+4\lambda^2\mu^2A_2^2a_1s_2-2\mu^4a_1s_2=0,
\end{aligned}$$

$$
\begin{aligned}
\phi\psi \;:\;\; & -3\,f\lambda^4\mu\,\rho^2A_1^4a_1-6\,f\lambda^4\mu\,\rho^2A_1^2A_2^2a_1\\
& -3\,f\lambda^4\mu\,\rho^2A_2^4a_1-3\,g\lambda^4\mu\,A_1^4a_1d_1^2\\
& -6\,g\lambda^4\mu\,A_1^2A_2^2a_1d_1^2-3\,g\lambda^4\mu\,A_2^4a_1d_1^2\\
& +12\,k\lambda^4A_1^4a_0a_1b_1+24\,k\lambda^4A_1^2A_2^2a_0a_1b_1\\
& +12\,k\lambda^4A_2^4a_0a_1b_1+6\,f\lambda^2\mu^3\rho^2A_1^2a_1\\
& +6\,f\lambda^2\mu^3\rho^2A_2^2a_1+6\,g\lambda^2\mu^3A_1^2a_1d_1^2\\
& +6\,g\lambda^2\mu^3A_2^2a_1d_1^2-12\,k\lambda^3\mu\,A_1^2a_1b_1^2\\
& -12\,k\lambda^3\mu\,A_2^2a_1b_1^2-24\,k\lambda^2\mu^2A_1^2a_0a_1b_1\\
& -24\,k\lambda^2\mu^2A_2^2a_0a_1b_1-3\,f\mu^5\rho^2a_1\\
& -3\,g\mu^5a_1d_1^2+12\,k\lambda\,\mu^3a_1b_1^2\\
& +12\,k\mu^4a_0a_1b_1=0,\\
\psi \;:\;\; & f\lambda^5\rho^2A_1^4b_1+2\,f\lambda^5\rho^2A_1^2A_2^2b_1\\
& +f\lambda^5\rho^2A_2^4b_1+g\lambda^5A_1^4b_1d_1^2\\
& +2\,g\lambda^5A_1^2A_2^2b_1d_1^2+g\lambda^5A_2^4b_1d_1^2\\
& -f\lambda^4\tau^2A_1^4b_1-2\,f\lambda^4\tau^2A_1^2A_2^2b_1\\
& -f\lambda^4\tau^2A_2^4b_1-g\lambda^4A_1^4b_1s_1^2\\
& -2\,g\lambda^4A_1^2A_2^2b_1s_1^2-g\lambda^4A_2^4b_1s_1^2\\
& +6\,k\lambda^4A_1^4a_0^2b_1+12\,k\lambda^4A_1^2A_2^2a_0^2b_1\\
& +6\,k\lambda^4A_2^4a_0^2b_1+2\,f\lambda^2\mu^2\tau^2A_1^2b_1\\
& +2\,f\lambda^2\mu^2\tau^2A_2^2b_1+2\,g\lambda^2\mu^2A_1^2b_1s_1^2\\
& +2\,g\lambda^2\mu^2A_2^2b_1s_1^2+2\,k\lambda^4A_1^2b_1^3\\
& +2\,k\lambda^4A_2^2b_1^3-12\,k\lambda^3\mu\,A_1^2a_0b_1^2\\
& -12\,k\lambda^3\mu\,A_2^2a_0b_1^2-12\,k\lambda^2\mu^2A_1^2a_0^2b_1\\
& -12\,k\lambda^2\mu^2A_2^2a_0^2b_1-2\,\lambda^4A_1^4b_1s_2\\
& -4\,\lambda^4A_1^2A_2^2b_1s_2-2\,\lambda^4A_2^4b_1s_2\\
& -f\lambda\,\mu^4\rho^2b_1-g\lambda\,\mu^4b_1d_1^2\\
& -f\mu^4\tau^2b_1-g\mu^4b_1s_1^2+6\,b_1^3\lambda^2k\mu^2\\
& +12\,k\lambda\,\mu^3a_0b_1^2+6\,k\mu^4a_0^2b_1\\
& +4\,\lambda^2\mu^2A_1^2b_1s_2+4\,\lambda^2\mu^2A_2^2b_1s_2\\
& -2\,\mu^4b_1s_2=0,\\
\phi^0 \;:\;\; & -f\lambda^4\tau^2A_1^4a_0-2\,f\lambda^4\tau^2A_1^2A_2^2a_0\\
& -f\lambda^4\tau^2{A_2}^4a_0-g\lambda^4A_1^4a_0s_1^2\\
& -2\,g\lambda^4A_1^2A_2^2a_0s_1^2-g\lambda^4A_2^4a_0s_1^2\\
& +2\,k\lambda^4A_1^4a_0^3+4\,k\lambda^4A_1^2A_2^2a_0^3\\
& +2\,k\lambda^4A_2^4a_0^3-f\lambda^4\mu\,\rho^2A_1^2b_1\\
& -f\lambda^4\mu\,\rho^2A_2^2b_1-g\lambda^4\mu\,A_1^2b_1d_1^2\\
& -g\lambda^4\mu\,A_2^2b_1d_1^2+2\,f\lambda^2\mu^2\tau^2A_1^2a_0\\
& +2\,f\lambda^2\mu^2\tau^2A_2^2a_0+2\,g\lambda^2\mu^2A_1^2a_0s_1^2\\
& +2\,g\lambda^2\mu^2A_2^2a_0s_1^2+6\,k\lambda^4A_1^2a_0b_1^2\\
& +6\,k\lambda^4A_2^2a_0b_1^2-4\,k\lambda^2\mu^2A_1^2a_0^3\\
& -4\,k\lambda^2\mu^2A_2^2a_0^3-2\,\lambda^4A_1^4a_0s_2\\
& -4\,\lambda^4A_1^2A_2^2a_0s_2-2\,\lambda^4A_2^4a_0s_2\\
& +f\lambda^2\mu^3\rho^2b_1+g\lambda^2\mu^3b_1d_1^2\\
& -f\mu^4\tau^2a_0-g\mu^4a_0s_1^2-4\,b_1^3\lambda^3k\mu\\
& -6\,k\lambda^2\mu^2a_0b_1^2+2\,k\mu^4a_0^3+4\,\lambda^2\mu^2A_1^2a_0s_2\\
& +4\,\lambda^2\mu^2A_2^2a_0s_2-2\,\mu^4a_0s_2=0.
\end{aligned}
\tag{35}
$$

Solving the above algebraic system using the Maple package program, we get the following results:

Result 1

$$
\begin{aligned}
& \lambda=\lambda,\ \mu=0,\ \omega=\omega,\ \rho=\rho,\ \tau=\tau,\ a_0=0,\\
& a_1=\pm\sqrt{-\frac{f\rho^2+gd_1^2}{k}},\ b_1=0,\ d_1=d_1,\\
& d_2=-\left(f\rho\,\tau+gd_1s_1\right),\ s_1=s_1,\\
& s_2=f\lambda\,\rho^2+g\lambda\,d_1^2-\frac{1}{2}\,f\tau^2-\frac{1}{2}\,gs_1^2,
\end{aligned}
\tag{36}
$$

where $\lambda(>0)$, f, g, k, d_1, ρ, s_1, τ, ω are arbitrary constants such that $k\left(f\rho^2+gd_1^2\right)<0$. From Eqs. (14), (20), (25), and (36), we get the solution of Eq. (2) as follows:

$$
\begin{aligned}
& W(x,y,t)=\pm\sqrt{-\frac{f\rho^2+gd_1^2}{k}}\\
& \times\left(\frac{\left(A_1\cos\left(\chi\sqrt{\lambda}\right)\sqrt{\lambda}-A_2\sin\left(\chi\sqrt{\lambda}\right)\sqrt{\lambda}\right)}{A_1\sin\left(\chi\sqrt{\lambda}\right)+A_2\cos\left(\chi\sqrt{\lambda}\right)}\right)\\
& \times \mathrm{e}^{i\xi},
\end{aligned}
\tag{37}
$$

where A_1, A_2 re arbitrary constants and

$$
\begin{aligned}
& \chi=\frac{\Gamma(\gamma+1)}{\beta}\left(d_1x^\beta-\left(f\rho\tau+gd_1s_1\right)y^\beta+\rho t^\beta\right)\\
& \xi=\frac{\Gamma(\gamma+1)}{\beta}\\
& \times\left(s_1x^\beta+\left(f\lambda\rho^2+g\lambda\,d_1^2-\frac{1}{2}f\tau^2-\frac{1}{2}gs_1^2\right)y^\beta+\tau t^\beta+\omega\right).
\end{aligned}
$$

Result 2

$$
\begin{aligned}
& \lambda=-\frac{kb_1^2}{\sigma_2\left(f\rho^2+gd_1^2\right)},\ \mu=0,\ \omega=\omega,\ \rho=\rho,\\
& \tau=\tau,\ a_0=0,\ a_1=0, b_1=b_1,\ d_1=d_1,\\
& d_2=-\left(f\rho\,\tau+gd_1s_1\right),\ s_1=s_1,\\
& s_2=-\frac{\sigma_2\left(f\tau^2+gs_1^2\right)-kb_1^2}{2\sigma_2},
\end{aligned}
\tag{38}
$$

where $\sigma_2 = A_1^2 + A_2^2$ and $f,\ g,\ k,\ d_1,\ \rho,\ s_1,\ \tau,\ \omega,\ b_1,$ $A_1,\ A_2$ are arbitrary constants such that $\lambda > 0$. From Eqs. (14), (20), (25), and (38), we obtain the exact solution of Eq. (2) as follows:

$$W(x,y,t) = \frac{b_1}{A_1 \sin\left(\chi\sqrt{\lambda}\right) + A_2 \cos\left(\chi\sqrt{\lambda}\right)} \times \mathrm{e}^{i\xi}, \tag{39}$$

where

$$\chi = \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - (f\rho\tau + g d_1 s_1)\, y^\beta + \rho\, t^\beta\right),$$

$$\xi = \frac{\Gamma(\gamma+1)}{\beta} \times \left(s_1 x^\beta - \frac{\sigma_2\left(f\tau^2 + g s_1^2\right) - k b_1^2}{2\sigma_2} y^\beta + \tau\, t^\beta + \omega\right).$$

Result 3

$$\begin{aligned}
&\lambda = \lambda,\ \mu = \mu,\ \omega = \omega,\ \rho = \rho,\ \tau = \tau,\ a_0 = 0,\\
&a_1 = \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{4k}},\\
&b_1 = \pm\sqrt{-\frac{\left(f\rho^2 + g d_1^2\right)\lambda^2\sigma_2 - (f\rho^2 + g d_1^2)\mu^2}{4k\lambda}},\\
&d_1 = d_1,\ d_2 = -\left(f\rho\tau + g d_1 s_1\right),\ s_1 = s_1,\\
&s_2 = \frac{\lambda}{4}\left(f\rho^2 + g d_1^2\right) - \frac{1}{2}\left(f\tau^2 + g s_1^2\right),
\end{aligned} \tag{40}$$

where $\sigma_2 = A_1^2 + A_2^2$ and $\lambda(>0),\ \mu,\ f,\ g,\ k,\ d_1,\ \rho,$ $s_1,\ \tau,\ \omega,\ A_1,\ A_2$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$ and $b_1 \in \mathbb{R}$. From Eqs. (14), (20), (25), and (40), we obtain the exact solution of Eq. (2) as follows:

$$\begin{aligned}
&W(x,y,t)\\
&= \Bigg\{ \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{4k}}\\
&\times \left(\frac{A_1\cos\left(\chi\sqrt{\lambda}\right)\sqrt{\lambda} - A_2 \sin\left(\chi\sqrt{\lambda}\right)\sqrt{\lambda}}{A_1\sin\left(\chi\sqrt{\lambda}\right) + A_2\cos\left(\chi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}}\right)\\
&\pm \frac{\sqrt{-\frac{(f\rho^2+g d_1^2)\lambda^2\sigma_2 - (f\rho^2 + g d_1^2)\mu^2}{4k\lambda}}}{A_1\sin\left(\chi\sqrt{\lambda}\right) + A_2\cos\left(\chi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}}\Bigg\} \times \mathrm{e}^{i\xi}
\end{aligned} \tag{41}$$

where

$$\chi = \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - (f\rho\tau + g d_1 s_1)\, y^\beta + \rho\, t^\beta\right),$$

$$\xi = \frac{\Gamma(\gamma+1)}{\beta} \times \left(s_1 x^\beta + \left(\frac{\lambda}{4}\left(f\rho^2 + g d_1^2\right) - \frac{1}{2}\left(f\tau^2 + g s_1^2\right)\right) y^\beta + \tau\, t^\beta + \omega\right).$$

Case 3 (Rational function solutions): If $\lambda = 0$, we substitute Eq.(25) with Eqs.(11) and (17) into Eq.(22), so that the left-hand side of Eq. Eq.(22) becomes a polynomial in $\phi(\chi)$ and $\psi(\chi)$. Setting all of the coefficients of this resulting polynomial to zero, we obtain the following system of nonlinear algebraic equations in $\mu, \omega, \rho, \tau, a_0, a_1, b_1, d_1, s_1,$ and s_2:

$$\begin{aligned}
\phi^3 \ :\ & 8 f\mu^2\rho^2 A_2^2 a_1 - 8 f\mu\,\rho^2 A_1^2 A_2 a_1\\
& + 2 f\rho^2 A_1^4 a_1 + 8 g\mu^2 A_2^2 a_1 d_1^2\\
& - 8 g\mu\, A_1^2 A_2 a_1 d_1^2 + 2 g A_1^4 a_1 d_1^2\\
& + 8 k\mu^2 A_2^2 a_1^3 - 8 k\mu\, A_1^2 A_2 a_1^3\\
& + 2 k A_1^4 a_1^3 - 12 k\mu\, A_2 a_1 b_1^2\\
& + 6 k A_1^2 a_1 b_1^2 = 0,\\
\phi^2 \ :\ & 24 k\mu^2 A_2^2 a_0 a_1^2 - 24 k\mu\, A_1^2 A_2 a_0 a_1^2\\
& + 6 k A_1^4 a_0 a_1^2 + 2 f\mu^2\rho^2 A_2 b_1\\
& - f\mu\,\rho^2 A_1^2 b_1 + 2 g\mu^2 A_2 b_1 d_1^2\\
& - g\mu\, A_1^2 b_1 {d_1}^2 - 12 k\mu\, A_2 a_0 b_1^2\\
& + 6 k A_1^2 a_0 b_1^2 - 4 k\mu\, b_1^3 = 0,\\
\phi^2\psi \ :\ & 8 f\mu^2\rho^2 A_2^2 b_1 - 8 f\mu\,\rho^2 A_1^2 A_2 b_1\\
& + 2 f\rho^2 A_1^4 b_1 + 8 g\mu^2 A_2^2 b_1 d_1^2\\
& - 8 g\mu\, A_1^2 A_2 b_1 d_1^2 + 2 g A_1^4 b_1 d_1^2\\
& + 24 k\mu^2 A_2^2 a_1^2 b_1 - 24 k\mu\, A_1^2 A_2 a_1^2 b_1\\
& + 6 k A_1^4 a_1^2 b_1 - 4 k\mu\, A_2 b_1^3\\
& + 2 k A_1^2 b_1^3 = 0,\\
\phi \ :\ & -4 f\mu^2\tau^2 A_2^2 a_1 + 4 f\mu\,\tau^2 A_1^2 A_2 a_1\\
& - f\tau^2 A_1^4 a_1 - 4 g\mu^2 A_2^2 a_1 s_1^2\\
& + 4 g\mu\, A_1^2 A_2 a_1 s_1^2 - g A_1^4 a_1 s_1^2\\
& + 24 k\mu^2 A_2^2 a_0^2 a_1 - 24 k\mu\, A_1^2 A_2 a_0^2 a_1\\
& + 6 k A_1^4 a_0^2 a_1 - 8\mu^2 A_2^2 a_1 s_2\\
& + 8\mu\, A_1^2 A_2 a_1 s_2 - 2 A_1^4 a_1 s_2 = 0,\\
\phi\psi \ :\ & -12 f\mu^3\rho^2 A_2^2 a_1 + 12 f\mu^2\rho^2 A_1^2 A_2 a_1\\
& - 3 f\mu\,\rho^2 A_1^4 a_1 - 12 g\mu^3 A_2^2 a_1 d_1^2\\
& + 12 g\mu^2 A_1^2 A_2 a_1 d_1^2 - 3 g\mu\, A_1^4 a_1 d_1^2\\
& + 48 k\mu^2 A_2^2 a_0 a_1 b_1 - 48 k\mu\, A_1^2 A_2 a_0 a_1 b_1\\
& + 12 k A_1^4 a_0 a_1 b_1 + 24 k\mu^2 A_2 a_1 b_1^2\\
& - 12 k\mu\, A_1^2 a_1 b_1^2 = 0,\\
\psi \ :\ & -4 f\mu^3\rho^2 A_2 b_1 + 2 f\mu^2\rho^2 A_1^2 b_1\\
& - 4 f\mu^2\tau^2 A_2^2 b_1 + 4 f\mu\,\tau^2 A_1^2 A_2 b_1\\
& - f\tau^2 A_1^4 b_1 - 4 g\mu^3 A_2 b_1 d_1^2\\
& + 2 g\mu^2 A_1^2 b_1 d_1^2 - 4 g\mu^2 A_2^2 b_1 s_1^2\\
& + 4 g\mu\, A_1^2 A_2 b_1 s_1^2 - g A_1^4 b_1 s_1^2
\end{aligned}$$

$$
\begin{aligned}
& + 24\, k\mu^2 A_2^2 a_0^2 b_1 - 24\, k\mu\, A_1^2 A_2 a_0^2 b_1 \\
& + 6\, k A_1^4 a_0^2 b_1 + 24\, k\mu^2 A_2 a_0 b_1^2 \\
& - 12\, k\mu\, A_1^2 a_0 b_1^2 + 8\, k\mu^2 b_1^3 \\
& - 8\, \mu^2 A_2^2 b_1 s_2 + 8\, \mu\, A_1^2 A_2 b_1 s_2 \\
& - 2\, A_1^4 b_1 s_2 = 0, \\
\phi^0 \ : \ & - 4\, f\mu^2 \tau^2 A_2^2 a_0 + 4\, f\mu\, \tau^2 A_1^2 A_2 a_0 \qquad (42) \\
& - f\tau^2 A_1^4 a_0 - 4\, g\mu^2 A_2^2 a_0 s_1^2 \\
& + 4\, g\mu\, A_1^2 A_2 a_0 s_1^2 - g A_1^4 a_0 s_1^2 \\
& + 8\, k\mu^2 A_2^2 a_0^3 - 8\, k\mu\, A_1^2 A_2 a_0^3 \\
& + 2\, k A_1^4 a_0^3 - 8\, \mu^2 A_2^2 a_0 s_2 \\
& + 8\, \mu\, A_1^2 A_2 a_0 s_2 - 2\, A_1^4 a_0 s_2 = 0.
\end{aligned}
$$

Solving the above algebraic system using the Maple package program, we get the following results.

Result 1

$$
\begin{aligned}
& \mu = 0,\, \omega = \omega,\, \rho = \rho,\, \tau = \tau,\, a_0 = 0, \\
& a_1 = \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{k}},\, b_1 = 0,\, d_1 = d_1, \\
& d_2 = -\left(f\rho\,\tau + g d_1 s_1\right),\, s_1 = s_1, \\
& s_2 = -\frac{1}{2}\left(f\tau^2 + g s_1^2\right),
\end{aligned} \tag{43}
$$

where $f,\, g,\, k, d_1,\, \rho,\, s_1,\, \tau,\, \omega$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (16), (20), (25), and (43), we obtain the exact solution of Eq. (2) as follows:

$$
W(x,y,t) = \pm\frac{\sqrt{-\frac{f\rho^2+g d_1^2}{k}}A_1}{A_1\chi + A_2} \times \mathrm{e}^{i\xi}, \tag{44}
$$

where $A_1,\, A_2$ are arbitrary constants and

$$
\begin{aligned}
\chi &= \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\, t^\beta\right), \\
\xi &= \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta - \frac{1}{2}\left(f\tau^2 + g s_1^2\right) y^\beta + \tau\, t^\beta + \omega\right).
\end{aligned}
$$

Result 2

$$
\begin{aligned}
& \mu = 0,\, \omega = \omega,\, \rho = \rho,\, \tau = \tau,\, a_0 = 0, a_1 = 0, \\
& b_1 = \pm\sqrt{-\frac{f\rho^2 + g d_1^2}{k}}A_1,\, d_1 = d_1, \\
& d_2 = -\left(f\rho\,\tau + g d_1 s_1\right),\, s_1 = s_1, \\
& s_2 = -\frac{1}{2}\left(f\tau^2 + g s_1^2\right),
\end{aligned} \tag{45}
$$

where $f,\, g,\, k, d_1,\, \rho,\, s_1,\, \tau,\, \omega,\, A_1$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (16), (20), (25), and (45), the solution of Eq. (2) is:

$$
W(x,y,t) = \pm\frac{\sqrt{-\frac{f\rho^2+g d_1^2}{k}}A_1}{A_1\chi + A_2} \times \mathrm{e}^{i\xi}, \tag{46}
$$

where A_2 is an arbitrary constant and

$$
\begin{aligned}
\chi &= \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\, t^\beta\right), \\
\xi &= \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta - \frac{1}{2}\left(f\tau^2 + g s_1^2\right) y^\beta + \tau\, t^\beta + \omega\right).
\end{aligned}
$$

Result 3

$$
\begin{aligned}
& \mu = 0,\, \omega = \omega,\, \rho = \rho,\, \tau = \tau,\, a_0 = 0, \\
& a_1 = \pm\frac{1}{2}\sqrt{-\frac{f\rho^2 + g d_1^2}{k}},\, b_1 = a_1 A_1, \\
& d_1 = d_1,\, d_2 = -\left(f\rho\,\tau + g d_1 s_1\right),\, s_1 = s_1, \\
& s_2 = -\frac{1}{2}\left(f\tau^2 + g s_1^2\right),
\end{aligned} \tag{47}
$$

where $f,\, g,\, k, d_1,\, \rho,\, s_1,\, \tau,\, \omega,\, A_1$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (16), (20), (25), and (47), we obtain the exact solution of Eq. (2) as follows:

$$
W(x,y,t) = \pm\frac{\sqrt{-\frac{f\rho^2+g d_1^2}{k}}A_1}{A_1\chi + A_2} \times \mathrm{e}^{i\xi}, \tag{48}
$$

where A_2 is arbitrary constant and

$$
\begin{aligned}
\chi &= \frac{\Gamma(\gamma+1)}{\beta}\left(d_1 x^\beta - \left(f\rho\,\tau + g d_1 s_1\right) y^\beta + \rho\, t^\beta\right), \\
\xi &= \frac{\Gamma(\gamma+1)}{\beta}\left(s_1 x^\beta - \frac{1}{2}\left(f\tau^2 + g s_1^2\right) y^\beta + \tau\, t^\beta + \omega\right).
\end{aligned}
$$

Result 4

$$
\begin{aligned}
& \mu = \frac{\left(f\rho^2 + g d_1^2\right) A_1^2 + 4\, k b_1^2}{2A_2\left(f\rho^2 + g d_1^2\right)},\, \omega = \omega,\, \rho = \rho, \\
& \tau = \tau,\, a_0 = 0,\, a_1 = \pm\frac{1}{2}\sqrt{-\frac{f\rho^2 + g d_1^2}{k}}, \\
& b_1 = b_1,\, d_1 = d_1, d_2 = -\left(f\rho\,\tau + g d_1 s_1\right), \\
& s_1 = s_1,\, s_2 = -\frac{1}{2}\, f\tau^2 - \frac{1}{2}\, g s_1^2,
\end{aligned} \tag{49}
$$

where $f,\, g,\, k, d_1,\, \rho,\, s_1,\, \tau,\, \omega,\, b_1,\, A_1,\, A_2$ are arbitrary constants such that $k\left(f\rho^2 + g d_1^2\right) < 0$. From Eqs. (16), (20), (25), and (49), we get the solution of Eq. (2) as follows:

$$
W(x,y,t) = \left(\frac{\Phi\Psi + \Omega}{\Theta}\right)\mathrm{e}^{i\xi}, \tag{50}
$$

where

$$\Phi = \left(\chi\,\left(f\rho^2+gd_1^2\right)A_1^2+2\,A_1A_2\left(f\rho^2+gd_1^2\right)+4\,\chi\,kb_1^2\right),$$

$$\Psi=\sqrt{-\frac{f\rho^2+gd_1^2}{k}},$$

$$\Omega=4\,b_1A_2\left(f\rho^2+gd_1^2\right),$$

$$\Theta=\chi^2\left(f\rho^2+gd_1^2\right)A_1^2+4\,A_1\chi\,A_2\left(f\rho^2+gd_1^2\right) + \left(4\,f\rho^2+4\,gd_1^2\right)A_2^2+4\,\chi^2kb_1^2,$$

$$\chi=\frac{\Gamma(\gamma+1)}{\beta}\left(d_1x^\beta-\left(f\rho\,\tau+gd_1s_1\right)y^\beta+\rho\,t^\beta\right),$$

$$\xi=\frac{\Gamma(\gamma+1)}{\beta}\left(s_1x^\beta-\frac{1}{2}\left(f\tau^2+gs_1^2\right)y^\beta+\tau\,t^\beta+\omega\right).$$

4 Graphs of Some Exact Solutions

In this section, we show graphs of some of the exact traveling wave solutions of the truncated M-fractional paraxial wave dynamical model in Kerr media in Eq. (2) that we obtained using the $(G'/G, 1/G)$-expansion method. In particular, we show the exact solutions through 3D, 2D, and contour plots for the following range of fractional-order values: $\beta = 0.9$, $\beta = 0.8$, and $\beta = 0.6$. The exact traveling wave solutions in Eq. (28) and Eq. (37) have been chosen to demonstrate how their physical behavior changes in terms of 3D, 2D, and contour plots when values of the fractional-order β are altered. All figures were obtained using the Maple software package.

In Figure 1 (Appendix), magnitudes of the exact traveling wave solution $W(x, y, t)$ in (28) are plotted on the domain

$$D_1 = \{(x, y, t)\,|\,0 \leqslant x \leqslant 60,\ y = 1,\ \text{and}\ 0 \leqslant t \leqslant 30\}$$

for the 3D plots and on the domain

$$D_2 = \{(x, y, t)\,|\,0 \leqslant x \leqslant 60,\ y = 1,\ \text{and}\ t = 1\}$$

for the 2D graphs. In addition, contour plots, which represent a 3D surface by plotting (x, t) contours for a range of fixed $|W|$ values, are also illustrated. The following parameter values: $\lambda = -1$, $\mu = 0$, $a_0 = 0$, $b_1 = 0$, $f = 0.8$, $g = 2$, $k = 0.8$, $d_1 = 2$, $\rho = 0.5$, $s_1 = 5$, $\tau = 2$, $\omega = 3$, $A_1 = 3$, $A_2 = 5$, and $\gamma = 1.5$ are used in this figure. In particular, Figures 1 (a)-(c), (d)-(f), and (g)-(i) (Appendix) show the 3D, 2D, and contour graphs of magnitudes of the exact solution $W(x, y, t)$ in (28) calculated at $\beta = 0.9$, $\beta = 0.8$, and $\beta = 0.6$, respectively. As can be observed from the 3D graphs of Figure 1 (Appendix), the physical behavior of the magnitude of solution (28) can be characterized as an anti-soliton solution. In Figure 1 (Appendix), it is worth noticing that the singular point of $|W(x, t)|$ can be moved as the value of the fractional-order β is changed. This is because the denominator term $A_1 \sinh\left(\chi\sqrt{-\lambda}\right) + A_2 \cosh\left(\chi\sqrt{-\lambda}\right)$ in Eq. (28) can be zero depending upon the value of β, which is embedded in χ. For the given parameter values as mentioned above, the singular point is $x \approx 12.7918$ when $\beta = 0.9$ and $t = 1$ as shown in Figure 1 (b) (Appendix).

In Figure 2 (Appendix), magnitudes of the exact traveling wave solution $W(x, y, t)$ in (37) are plotted on the domain

$$D_3 = \{(x, y, t)\,|\,0 \leqslant x \leqslant 10,\ y = 1,\ \text{and}\ 0 \leqslant t \leqslant 10\}$$

for the 3D plots and on the domain

$$D_4 = \{(x, y, t)\,|\,0 \leqslant x \leqslant 10,\ y = 1,\ \text{and}\ t = 1\}$$

for the 2D graphs. Contour plots, which represent a 3D surface by plotting (x, t) contours for a range of fixed $|W|$ values, are also shown. The following parameter values: $\lambda = 1$, $\mu = 0$, $a_0 = 0$, $b_1 = 0$, $f = 1$, $g = 0.5$, $k = 1$, $d_1 = 2$, $\rho = 0.1$, $s_1 = 5$, $\tau = 0.5$, $\omega = 1$, $A_1 = 3$, $A_2 = 5$, and $\gamma = 0.8$ are used in this figure. In particular, Figures 2 (a)-(c), (d)-(f), and (g)-(i) (Appendix) display the 3D, 2D, and contour plots of magnitudes of the exact solution $W(x, y, t)$ in (37) calculated at $\beta = 0.9$, $\beta = 0.8$, and $\beta = 0.6$, respectively. As can be observed from the 3D graphs of Figure 2 (Appendix), the magnitude of solution (37) can be categorized as a singularly periodic wave solution. In Figure 2 (Appendix), it is worth observing that the singular point of $|W(x, t)|$ can be changed when the value of the fractional-order β is varied. This is because the denominator term $A_1 \sin\left(\chi\sqrt{\lambda}\right) + A_2 \cos\left(\chi\sqrt{\lambda}\right)$ in Eq. (37) can be zero depending upon the value of β, which appears in χ. Specifically, the singular point of $|W(x, t)|$ is obtained when $\chi\sqrt{\lambda} = -\arctan\left(\frac{A_2}{A_1}\right)$. For the given parameter values as described above, some of the singular points are, for instance, $x \approx 2.1327$ and $x \approx 4.0164$ when $\beta = 0.9$ and $t = 1$ as shown in Figure 2 (b) (Appendix).

5 Conclusions

In this paper, the paraxial wave dynamical model in Kerr media with truncated M-fractional derivatives given in (2) has been symbolically solved to obtain exact traveling wave solutions using the $(G'/G, 1/G)$-expansion method. Since the equation has complex-valued solutions, we wrote exact solutions as the product of a real function $U(\chi)$ and $e^{i\xi}$ as shown in (20). The algebraic manipulations required to obtain the exact solutions of the function $U(\chi)$ were carried out using the Maple software package. We found that exact solutions for $U(\chi)$ can be written in terms of either hyperbolic functions, trigonometric functions, or rational functions. From these solutions for $U(\chi)$, we finally obtained the exact traveling wave solutions of the equation (2) via Eq. (20) and the transformation (21).

In [38] the authors used the modified simple equation method (MSEM) and the auxiliary equation method (AEM) to find exact solutions of the M-fractional paraxial wave equation with Kerr media. Their governing equation was equipped with higher-order truncated M-fractional partial derivatives with respect to t and x. This is slightly different from Eq. (2) in which the composite of the truncated M-fractional derivatives of order less than one is used. In addition, the fractional-order β was not inserted as an exponent of the independent variables x, y, and t in their traveling wave transformation. However, the real function $U(\chi)$ of their solutions expressed in terms of the exponential functions were found. In [39] the truncated time M-fractional paraxial wave equation in kerr media was explored for some optical solutions. The unified scheme was implemented to obtain exact traveling wave solutions of the proposed equation. As a result, the solutions were expressed in terms of hyperbolic, trigonometric, and rational functions with some free parameters. Roughly comparing our results to the obtained solutions in [38], [39], some of the exact solutions obtained in this article have not been derived in any previous work because equation (2) and the used method are not the same as in the referred literature.

From our results, the 3D, 2D, and contour plots of magnitudes of selected solutions have been plotted for a range of values of fractional-order β using the Maple package in order to understand the effects of changing the fractional-order on the physical behavior of chosen solutions. From Figure 1 and Figure 2 (Appendix), an anti-soliton solution and a singularly periodic wave solution have been found. Finally, with the assistance of Maple, all of the exact solutions have been verified by substituting them back into the original equation to check their correctness. In summary, since the $(G'/G, 1/G)$-expansion method is an extension of the (G'/G)-expansion method and its relevant methods, [40], the advantage of the proposed method is that it is more productive, efficient, and reliable for generating exact traveling wave solutions of nonlinear real-world problems modeled by NPDEs. This work could be improved by using different fractional order values for the truncated M-fractional partial derivatives with respect to x, y, and t. A promising future work would be to compare the fractional equation and solutions developed in this article with real data obtained from physical phenomena.

Acknowledgments:
The authors are grateful to anonymous referees for their valuable comments and several constructive suggestions, which have significantly improved this manuscript.

References:

[1] Plastino, A.: Entropic aspects of nonlinear partial differential equations: Classical and quantum mechanical perspectives. *Entropy* 19(4), 166 (2017)

[2] Kudryashov, N.A.: Construction of nonlinear differential equations for description of propagation pulses in optical fiber. *Optik* 192, 162964 (2019)

[3] Rashid, S., Ashraf, R., Hammouch, Z.: New generalized fuzzy transform computations for solving fractional partial differential equations arising in oceanography. *Journal of Ocean Engineering and Science* 8(1), 55--78 (2023)

[4] Ali, I., Khan, H., Shah, R., Baleanu, D., Kumam, P., Arif, M.: Fractional view analysis of acoustic wave equations using fractional-order differential equations. *Applied Sciences* 10(2), 610 (2020)

[5] Agarwal, P., Deniz, S., Jain, S., Alderremy, A.A., Aly, S.: A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques. *Physica A: Statistical Mechanics and its Applications* 542, 122769 (2020)

[6] Sharp, K.P.: Stochastic differential equations in finance. *Applied mathematics and Computation* 37(2), 131--148 (1990)

[7] Behera, S., Aljahdaly, N., Virdi, J.: On the modified (G'/G^2)-expansion method for finding some analytical solutions of the traveling waves. *Journal of Ocean Engineering and Science* 7(4), 313--320 (2022)

[8] Mohanty, S.K., Kravchenko, O.V., Deka, M.K., Dev, A.N., Churikov, D.V.: The exact solutions of the (2+1)-dimensional Kadomtsev--Petviashvili equation with variable coefficients by extended generalized (G'/G)-expansion method. *Journal of King Saud University-Science* 35(1), 102358 (2023)

[9] Naowarat, S., Saifullah, S., Ahmad, S., De la Sen, M.: Periodic, singular and dark solitons of a generalized geophysical KdV equation by using the tanh-coth method. *Symmetry* 15(1), 135 (2023)

[10] Gao, X.Y., Guo, Y.J., Shan, W.R.: Symbolically computing the shallow water via a (2+1)-dimensional generalized modified dispersive water-wave system: Similarity reductions, scaling and hetero-Bäcklund transformations. *Qualitative Theory of Dynamical Systems* 22(1), 17 (2023)

[11] Ozisik, M., Secer, A., Bayram, M.: On solitary wave solutions for the extended nonlinear Schrödinger equation via the modified F-expansion method. *Optical and Quantum Electronics* 55(3), 215 (2023)

[12] Zainab, I., Akram, G.: Effect of β-derivative on time fractional Jaulent--Miodek system under modified auxiliary equation method and exp(-g(Ω))-expansion method. *Chaos, Solitons & Fractals* 168, 113147 (2023)

[13] Sadaf, M., Arshed, S., Akram, G., Iqra: Exact soliton and solitary wave solutions to the Fokas system using two variables $(G'/G, 1/G)$-expansion technique and generalized projective Riccati equation method. *Optik* 268, 169713 (2022)

[14] Weinberger, P.: John kerr and his effects found in 1877 and 1878. *Philosophical Magazine Letters* 88(12), 897--907 (2008)

[15] Arshad, M., Seadawy, A.R., Lu, D., Khan, F.U.: Optical solitons of the paraxial wave dynamical model in Kerr media and its applications in nonlinear optics. *International Journal of Modern Physics B* 34(09), 2050078 (2020)

[16] Rehman, H.U., Seadawy, A.R., Younis, M., Yasin, S., Raza, S.T., Althobaiti, S.: Monochromatic optical beam propagation of paraxial dynamical model with Kerr media. *Results in Physics* 31, 105015 (2021)

[17] Arshad, M., Seadawy, A.R., Lu, D., Saleem, M.S.: Elliptic function solutions, modulation instability and optical solitons analysis of the paraxial wave dynamical model with Kerr media. *Optical and Quantum Electronics* 53, 1--20 (2021)

[18] Rehman, H.U., Awan, A.U., Allahyani, S.A., Tag-ElDin, E.M., Binyamin, M.A., Yasin, S.: Exact solution of paraxial wave dynamical model with Kerr media by using ϕ^6 model expansion technique. *Results in Physics* 42, 105975 (2022)

[19] Ullah, N.: Exact solutions of paraxial wave dynamical model with kerr law non-linearity using analytical techniques. *Open Journal of Mathematical Sciences* 7(1), 172--179 (2023)

[20] Khater, M.M., Alabdali, A.M., Mashat, A., Salama, S.A.: Optical soliton wave solutions of the fractional complex paraxial wave dynamical model along with kerr media. *Fractals* 30(05), 2240153 (2022)

[21] Sousa, J., de Oliveira, E.C.: A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties. arXiv preprint arXiv:1704.08187 (2017)

[22] Yao, S.W., Manzoor, R., Zafar, A., Inc, M., Abbagari, S., Houwe, A.: Exact soliton solutions to the Cahn--Allen equation and predator--prey model with truncated M-fractional derivative. *Results in Physics* 37, 105455 (2022)

[23] Riaz, M.B., Awrejcewicz, J., Jhangeer, A.: Optical solitons with beta and M-truncated derivatives in nonlinear negative-index materials with Bohm potential. *Materials* 14(18), 5335 (2021)

[24] Yusuf, A., İnç, M., Baleanu, D.: Optical solitons with M-truncated and beta derivatives in nonlinear optics. *Frontiers in Physics* 7, 126 (2019)

[25] Ozdemir, N.: M-truncated soliton solutions of the fractional (4+1)-dimensional Fokas equation. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)* 13(1), 123--129 (2023)

[26] Özkan, A., Özkan, E.M., Yildirim, O.: On exact solutions of some space--time fractional differential equations with M-truncated derivative. *Fractal and Fractional* 7(3), 255 (2023)

[27] Esen, H., Ozdemir, N., Secer, A., Bayram, M., Sulaiman, T.A., Ahmad, H., Yusuf, A., Albalwi, M.D.: On the soliton solutions to the density-dependent space time fractional reaction--diffusion equation with conformable and M-truncated derivatives. *Optical and Quantum Electronics* 55(10), 923 (2023)

[28] Hamza, A.E., Alshammari, M., Atta, D., Mohammed, W.W.: Fractional-stochastic shallow water equations and its analytical solutions. *Results in Physics* 53, 106953 (2023)

[29] Roshid, H.O., Roshid, M.M., Hossain, M.M., Hasan, M.S., Munshi, M.J.H., Sajib, A.H.: Dynamical structure of truncated M-fractional Klein--Gordon model via two integral schemes. *Results in Physics* 46, 106272 (2023)

[30] Onder, I., Cinar, M., Secer, A., Bayram, M.: Analytical solutions of simplified modified Camassa-Holm equation with conformable and M-truncated derivatives: A comparative study. Journal of Ocean Engineering and Science (2022)

[31] Raheel, M., Razzaq, W., Alsharidi, A.K., Zafar, A.: Exact solitons to M-fractional (2+1)-dimensional CNLSE based on three different methods. *Results in Physics* 42, 105983 (2022)

[32] Mohammed, W.W., Cesarano, C., Al-Askar, F.M.: Solutions to the $(4+1)$-dimensional time-fractional Fokas equation with M-truncated derivative. *Mathematics* 11(1), 194 (2022)

[33] Alurrfi, K.A., Shahoot, A.M., Elhasadi, O.I.: Exact solutions for the GKdV--mKdV equation with higher-order nonlinear terms using the generalized $(G'/G, 1/G)$-expansion method and the generalized Liénard equation. Ricerche di Matematica, 1--19 (2021)

[34] Sirisubtawee, S., Koonprasert, S., Sungnul, S.: Some applications of the $(G'/G, 1/G)$-expansion method for finding exact traveling wave solutions of nonlinear fractional evolution equations. *Symmetry* 11(8), 952 (2019)

[35] Miah, M.M., Ali, H.S., Akbar, M.A., Seadawy, A.R.: New applications of the two variable $(G'/G, 1/G)$-expansion method for closed form traveling wave solutions of integro-differential equations. *Journal of Ocean Engineering and Science* 4(2), 132--143 (2019)

[36] Demiray, S., Ünsal, Ö., Bekir, A.: Exact solutions of nonlinear wave equations using $(\frac{G'}{G}, \frac{1}{G})$-expansion method. *Journal of the Egyptian Mathematical Society* 23(1), 78--84 (2015)

[37] Zayed, E.M.E., Alurrfi, K.A.E.: The $(\frac{G'}{G}, \frac{1}{G})$-expansion method and its applications to two nonlinear schrödinger equations describing the propagation of femtosecond pulses in nonlinear optical fibers. *Optik* 127(4), 1581--1589 (2016)

[38] Tariq, K.U., Zainab, H., Seadawy, A.R., Younis, M., Rizvi, S., Mousa, A.A.A.: On some novel optical wave solutions to the paraxial M-fractional nonlinear schrödinger dynamical equation. *Optical and Quantum Electronics* 53, 1--14 (2021)

[39] Roshid, M.M., Uddin, M., Mostafa, G.: Dynamic optical soliton solutions for M-fractional paraxial wave equation using unified technique. *Results in Physics* 51, 106632 (2023)

[40] Sirisubtawee, S., Koonprasert, S., Sungnul, S., Leekparn, T.: Exact traveling wave solutions of the space--time fractional complex Ginzburg--Landau equation and the space-time fractional Phi-4 equation using reliable methods. *Advances in Difference Equations* 2019(1), 1--23 (2019)

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Pim Malingam and Paiwan Wongsasinchai: Conceptualization, data curation, funding acquisition, investigation, methodology, software, visualization, writing-original draft, and writing-review and editing. Sekson Sirisubtawee (Corresponding author): Conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, supervision, validation, visualization, writing-original draft, and writing-review and editing. Sanoe Koonprasert: Supervision, validation, and writing-review and editing.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

Pim Malingam (pim.m@rbru.ac.th) would like to acknowledge the financial support from the Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Appendix

Figures described in section 4 are shown in this section.

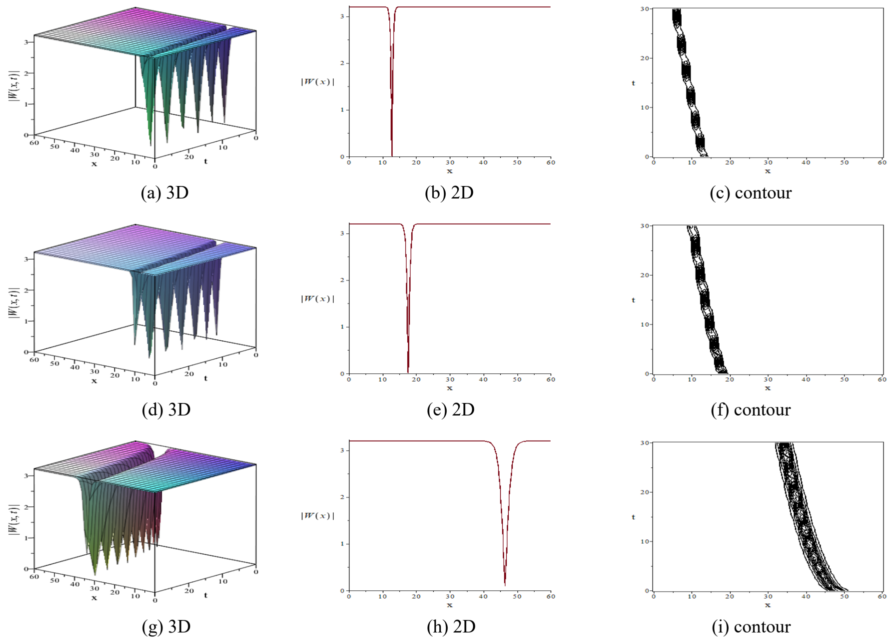

Fig. 1: Graphs for $|W(x, y, t)|$ where $W(x, y, t)$ is expressed in (28) and obtained using the $(G'/G, 1/G)$-expansion method: (a)-(c) when $\beta = 0.9$; (d)-(f) when $\beta = 0.8$; (g)-(i) when $\beta = 0.6$.

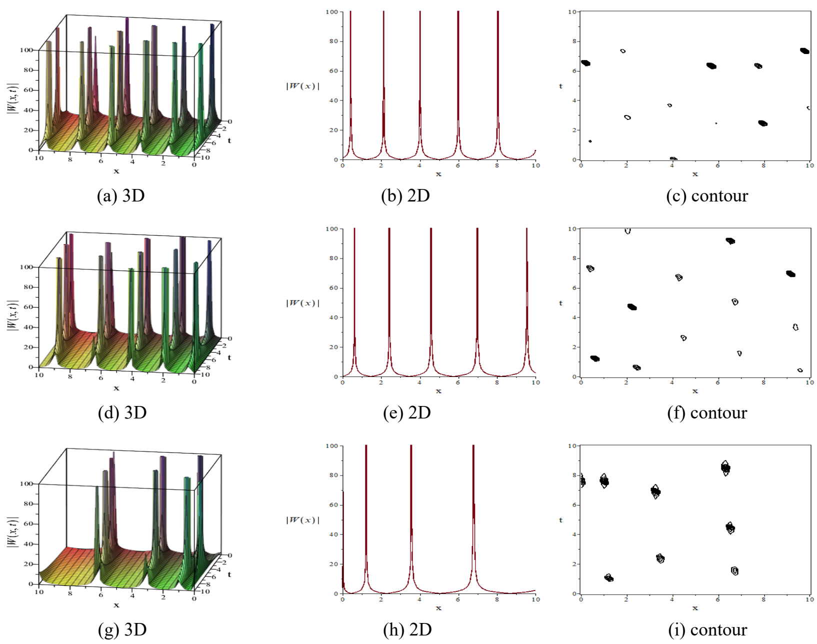

Fig. 2: Graphs for $|W(x, y, t)|$ where $W(x, y, t)$ is expressed in (37) and obtained using the $(G'/G, 1/G)$-expansion method: (a)-(c) when $\beta = 0.9$; (d)-(f) when $\beta = 0.8$; (g)-(i) when $\beta = 0.6$.