First-hitting Place Optimization Problems for Two-dimensional Degenerate Diffusion Processes

MARIO LEFEBVRE
Department of Mathematics and Industrial Engineering,
Polytechnique Montréal,
2500, chemin de Polytechnique, Montréal (Québec) H3T 1J4,
CANADA

Abstract: Optimal control problems for degenerate two-dimensional diffusion processes are considered. The processes could serve as models for wear processes. The objective is to make the controlled process leave the continuation region through a given part of the boundary. Explicit and exact solutions are obtained for important processes such as the geometric Brownian motion and the Ornstein-Uhlenbeck process.

Key-Words: Wear processes, Kolmogorov backward equation, homing problem, dynamic programming, special functions, geometric Brownian motion, Ornstein-Uhlenbeck process.

Received: April 29, 2024. Revised: October 9, 2024. Accepted: November 6, 2024. Published: December 5, 2024.

1 Introduction

Let $(X(t), Y(t))$ be the controlled two-dimensional process defined by

$$\begin{aligned} \mathrm{d}X(t) &= f[X(t), Y(t)]\,\mathrm{d}t, && (1)\\ \mathrm{d}Y(t) &= b[Y(t)]\,u(t)\mathrm{d}t + m[Y(t)]\,\mathrm{d}t \\ &\quad + \{v[Y(t)]\}^{1/2}\,\mathrm{d}W(t), && (2) \end{aligned}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion. The functions $m(\cdot) \in \mathbb{R}$ and $v(\cdot) > 0$ are such that the uncontrolled process $(X_0(t), Y_0(t))$ obtained by setting $u(t) \equiv 0$ in Eq. (2) is a (degenerate) two-dimensional diffusion process.

Assume that $(X(0), Y(0)) = (x, y) \in C \subset \mathbb{R}^2$. We define the *first-passage time*

$$\tau(x, y) = \inf\{t > 0 : (X(t), Y(t)) \in D \subset \mathbb{R}^2\}, \qquad (3)$$

where $D = C^c$; that is, D is the complement of C in $\mathbb{R}^2$.

If the function $f(\cdot, \cdot)$ in Eq. (1) is such that it is always positive in the region C, then the above process could be an appropriate model for the wear $X(t)$ of a certain device at time t. Indeed, in reality, wear should increase with time. We assume that the wear depends on a variable $Y(t)$ that evolves according to a diffusion process; [1].

The aim is to find the control $u^*(t)$ that minimizes the expected value of the cost function

$$J(x, y) = \int_0^{\tau(x,y)} \frac{1}{2}\, q_0\, u^2(t)\,\mathrm{d}t + K[X(\tau), Y(\tau)], \qquad (4)$$

where q_0 is a positive constant and $K(\cdot, \cdot)$ is the terminal cost function.

This type of problem, when the final time is a first-passage time, is called an *LQG homing problem*. Such problems have been treated extensively by the author; see, for instance, [2] and, [3]. Other papers on this topic are, [4], [5] and [6]. In general, the optimizer seeks to minimize or maximize the time spent by the controlled process in the continuation region C. Here, we are interested in the place where the controlled process will leave C.

The current paper is more realistic than other papers published on the optimal control of wear processes because the controlled process is defined in such a way that wear is strictly increasing with time, and the final time is a random variable, rather than being fixed.

To solve our problem, we shall use dynamic programming. We define the value function

$$F(x, y) = \inf_{u(t),\, 0 \leq t < \tau(x)} E[J(x, y)]. \qquad (5)$$

We find that the optimal control can be expressed as follows:

$$u^*(x, y) = -\frac{b(y)}{q_0} F_y(x, y), \qquad (6)$$

where $F_y(x, y) = \partial F(x, y)/\partial y$.

Remark. We wrote $u(t)$ for the control variable in Eq. (2), as most authors do. Actually, it would be more accurate to write $u[X(t), Y(t)]$.

Next, assume that there exists a positive constant θ such that

$$v(y) = \theta\, b^2(y)/q_0. \tag{7}$$

Then, we can show ([7] or [8]) that the function

$$\Phi(x,y) := e^{-F(x,y)/\theta} \tag{8}$$

satisfies the linear partial differential equation (PDE)

$$\frac{1}{2}v(y)\,\Phi_{yy} + m(y)\,\Phi_y + f(x,y)\,\Phi_x = 0, \tag{9}$$

and is subject to the boundary condition

$$\Phi(x,y) = e^{-K(x,y)/\theta} \quad \text{if } (x,y) \in D. \tag{10}$$

Moreover, we can write that

$$\Phi(x,y) = E\left[e^{-K[X_0(T),Y_0(T)]/\theta}\right], \tag{11}$$

where $T(= T(x,y))$ is the same as $\tau(x,y)$, but for the *uncontrolled* process $(X_0(t), Y_0(t))$. Hence, if the relation in Eq. (7) holds, it is possible to determine the optimal control by computing a mathematical expectation for the uncontrolled process.

In the next section, the function $F(x,y)$ will be computed explicitly for important diffusion processes, such as the Ornstein-Uhlenbeck process, from which the optimal control follows at once my making use of Eq. (6).

2 Optimal Control Problems

Case 1. Suppose first that Eq. (2) is given by

$$\mathrm{d}Y(t) = b_0\,u(t)\,\mathrm{d}t - \alpha Y(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \tag{12}$$

where b_0, α and σ are positive constants. Then, if $u(t) \equiv 0$, $\{Y(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process, which is one of the most important diffusion processes. Notice that Eq. (7) is satisfied by taking $\theta = \sigma^2 q_0/b_0^2$.

Let

$$\tau(x,y) = \inf\{t > 0 : Y(t) - X(t) = k_1 \text{ or } k_2\}, \tag{13}$$

where $x \geq 0$, $y > 0$ and $0 \leq k_1 < y - x < k_2$.

Suppose that the function $f[X(t), Y(t)]$ in Eq. (1) is given by

$$f[X(t), Y(t)] = \alpha Y(t). \tag{14}$$

Moreover, we choose

$$K[X(\tau), Y(\tau)] = \begin{cases} 1 & \text{if } Y(\tau) - X(\tau) = k_1, \\ 0 & \text{if } Y(\tau) - X(\tau) = k_2. \end{cases} \tag{15}$$

That is, the aim is to make $Y(t) - X(t)$ leave the interval (k_1, k_2) at k_2. Since $Y(0) = y$ is assumed to be positive, $Y(t)$ is always positive in the continuation region $C := \{(x,y) \in \mathbb{R}^2 : 0 \leq k_1 < y - x < k_2\}$. It follows that $X(t)$ will be strictly increasing in C.

To obtain the function $\Phi(x,y)$ defined in Eq. (8), we must solve the PDE

$$\frac{1}{2}\sigma^2\,\Phi_{yy} - \alpha\, y\,\Phi_y + \alpha\, y\,\Phi_x = 0, \tag{16}$$

subject to the boundary conditions

$$\Phi(x,y) = \begin{cases} e^{-1/\theta} & \text{if } y - x = k_1, \\ 1 & \text{if } y - x = k_2. \end{cases} \tag{17}$$

Let $z := y - x$. We shall try to find a solution of the form

$$\Phi(x,y) = \Psi(z). \tag{18}$$

This is an application of the *method of similarity solutions*, and z is called the *similarity variable*.

Equation (16) simplifies to the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2\,\Psi''(z) = 0. \tag{19}$$

Hence, we may write that

$$\Psi(z) = c_1\, z + c_0. \tag{20}$$

The solution that satisfies the boundary conditions $\Psi(k_1) = e^{-1/\theta}$ and $\Psi(k_2) = 1$ is

$$\Psi(z) = \frac{(z - k_2)\,e^{-1/\theta} - z + k_1}{k_1 - k_2} \quad \text{for } k_1 \leq z \leq k_2. \tag{21}$$

We can now state the following proposition.

Proposition 2.1. *The value function in the problem considered above is given by*

$$F(x,y) = -\theta \ln\left(\frac{(y-x)\left(e^{-1/\theta} - 1\right) - k_2\,e^{-1/\theta} + k_1}{k_1 - k_2}\right) \tag{22}$$

for $k_1 \leq y - x \leq k_2$. Furthermore, from Eq. (6), the optimal control is

$$u^*(x,y) = -\frac{b_0}{q_0}\,\frac{\theta\left(e^{-1/\theta} - 1\right)}{k_2\,e^{-1/\theta} - k_1 - (y-x)\left(e^{-1/\theta} - 1\right)} \tag{23}$$

for $k_1 < y - x < k_2$, where $\theta = \sigma^2 q_0/b_0^2$.

In the special case when $k_1 = 0$ and $\sigma = q_0 = b_0 = k_2 = 1$, the value function is given by

$$F(x,y) = -\ln\left[\left(1-e^{-1}\right)(y-x) + e^{-1}\right] \quad (24)$$

and the optimal control becomes

$$u^*(x,y) = \frac{\left(1-e^{-1}\right)}{e^{-1} + (y-x)\left(1-e^{-1}\right)} \quad (25)$$

for $0 < y - x < 1$. This function is shown in Figure 1 when $x = 0$. We see that the optimal control decreases as y increases from 0 to 1, which is logical since the optimizer wants the controlled process to leave the continuation region C through the straight line $y - x = 1$.

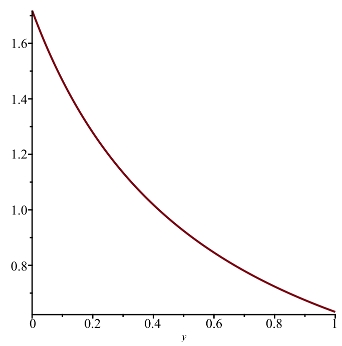

Figure 1: Function $u^*(0,y)$ in Case 1 for $y \in [0,1]$ when $k_1 = 0$ and $\sigma = q_0 = b_0 = k_2 = 1$.

Case 2. Suppose next that

$$\mathrm{d}X(t) = f_0\, X(t)/Y(t)\,\mathrm{d}t, \quad (26)$$
$$\mathrm{d}Y(t) = b_0\, Y^{1/2}(t)\, u(t)\,\mathrm{d}t + \sigma\, Y^{1/2}(t)\,\mathrm{d}W(t), \quad (27)$$

where $f_0 \neq 0$, and b_0 and σ are positive constants. This time the uncontrolled process $\{Y(t), t \geq 0\}$ is a limit case of a Cox-Ingersoll-Ross process, which is used in financial mathematics as a model for the evolution of interest rates, or a particular squared Bessel process (if $\sigma = 2$). Equation (7) is satisfied with $\theta = \sigma^2 q_0/b_0^2$, as in Case 1.

We define the first-passage time

$$\tau(x,y) = \inf\{t > 0 : X(t)/Y(t) = k_1 \text{ or } k_2\}, \quad (28)$$

where $x > 0$, $y > 0$ and $0 < k_1 < x/y < k_2$. Then $X(t)$ will increase with time if $f_0 > 0$, as it should in the case of a wear process.

Remark. If $X(t)$ represents the wear of a device at time t, then f_0 must be positive, whereas f_0 is negative if $X(t)$ is rather the remaining lifetime of the device.

The terminal cost function is given by

$$K[X(\tau), Y(\tau)] = \begin{cases} 1 & \text{if } X(\tau)/Y(\tau) = k_1, \\ 0 & \text{if } X(\tau)/Y(\tau) = k_2. \end{cases} \quad (29)$$

The function $\Phi(x,y)$ is a solution of the PDE

$$\frac{1}{2}\sigma^2\, y\, \Phi_{yy} + f_0\, \frac{x}{y}\, \Phi_x = 0, \quad (30)$$

subject to the boundary condition

$$\Phi(x,y) = \begin{cases} e^{-1/\theta} & \text{if } x/y = k_1, \\ 1 & \text{if } x/y = k_2. \end{cases} \quad (31)$$

We assume that

$$\Phi(x,y) = \Psi(z), \quad (32)$$

where the similarity variable is $z := x/y$. Equation (30) is then reduced to the ODE

$$\frac{1}{2}\sigma^2 z^2\, \Psi''(z) + \sigma^2 z\, \Psi'(z) + f_0\, z\, \Psi'(z) = 0 \quad (33)$$

and the boundary conditions are

$$\Psi(z) = \begin{cases} e^{-1/\theta} & \text{if } z = k_1, \\ 1 & \text{if } z = k_2. \end{cases} \quad (34)$$

Since $z > 0$ in the continuation region C, we may write that

$$\frac{1}{2}\sigma^2\, z\, \Psi''(z) + (\sigma^2 + f_0)\, \Psi'(z) = 0. \quad (35)$$

If

$$\kappa := \sigma^2 + f_0 = 0, \quad (36)$$

then the solution is the same as in Case 1. When κ is different from zero, we find that the general solution of Eq. (35) is

$$\Psi(z) = c_1 + c_2\, z^{\Delta}, \quad (37)$$

where

$$\Delta := -\left(2\,\frac{f_0}{\sigma^2} + 1\right). \quad (38)$$

Proposition 2.2. *The value function in Case 2 is given by*

$$F(x,y) = -\theta \ln[\Psi(x/y)], \quad (39)$$

where the function $\Psi(\cdot)$ is defined in Eq. (37) and the constants c_1 and c_2 are determined by using the boundary conditions in Eq. (34).

Moreover, the optimal control is

$$u^*(x,y) = -\frac{b_0\sqrt{y}}{q_0}\, F_y(x,y) \quad (40)$$

for $k_1 < x/y < k_2$.

Assume that $b_0 = q_0 = \sigma = 1$, so that $\theta = 1$, and $f_0 = 1$ as well. Then, $\kappa = 2$ and $\Delta = -3$. If $k_1 = 1$ and $k_2 = 2$, we find that

$$F(x,y) = \ln(7) - \ln\left[8 - e^{-1} + \frac{8(e^{-1}-1)y^3}{x^3}\right] \tag{41}$$

for $1 \leq x/y \leq 2$. Furthermore, the optimal control is

$$u^*(x,y) = \frac{24y^{5/2}(1-e^{-1})}{(x^3-8y^3)e^{-1}+8(y^3-x^3)}. \tag{42}$$

The function $F(x,y)$ is presented in Figure 2 in terms of $x/y \in [1,2]$, and the optimal control $u^*(1,y)$ is displayed in Figure 3 for $y \in [0.5, 1]$.

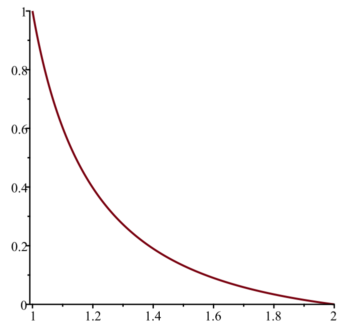

Figure 2: Value function $F(x,y)$ in Case 2 for $x/y \in [1,2]$ when $q_0 = b_0 = f_0 = \sigma = k_1 = 1$ and $k_2 = 2$.

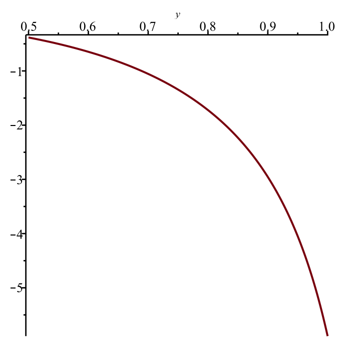

Figure 3: Optimal control $u^*(1,y)$ in Case 2 for $y \in [0.5,1]$ when $q_0 = b_0 = f_0 = \sigma = k_1 = 1$ and $k_2 = 2$.

Case 3. Finally, we consider the process defined by

$$\mathrm{d}X(t) = f_0 Y^2(t)\,\mathrm{d}t, \tag{43}$$
$$\mathrm{d}Y(t) = b_0 Y(t) u(t)\,\mathrm{d}t + \mu Y(t)\,\mathrm{d}t + \sigma Y(t)\,\mathrm{d}W(t), \tag{44}$$

where $f_0 \neq 0$, b_0 and σ are positive constants, and $\mu \in \mathbb{R}$. We assume that $(X(0), Y(0)) = (x,y)$, with x and y positive. The relation in Eq. (7) is satisfied if $\theta = \sigma^2 q_0/b_0^2$, as in the previous cases.

In the above case, the uncontrolled process $\{Y(t), t \geq 0\}$ is a geometric Brownian motion, which is very important in financial mathematics. Since this process is always positive (when $Y(0) > 0$), the variable $X(t)$ will increase with t if $u(t) \equiv 0$.

The first-passage time $\tau(x,y)$ is defined by

$$\tau(x,y) = \inf\{t > 0 : X(t)/Y^2(t) = k_1 \text{ or } k_2\}, \tag{45}$$

where $0 < k_1 < x/y^2 < k_2$, and the terminal cost function is

$$K[X(\tau), Y(\tau)] = \begin{cases} 1 & \text{if } X(\tau)/Y^2(\tau) = k_1, \\ 0 & \text{if } X(\tau)/Y^2(\tau) = k_2. \end{cases} \tag{46}$$

For this model, we can generalize the cost function defined in Eq. (4) to

$$\begin{aligned} J(x,y) \;&=\; \int_0^{\tau(x,y)} \left\{\frac{1}{2} q_0 u^2(t) + \lambda\right\} \mathrm{d}t \\ &\quad + K[X(\tau), Y(\tau)], \end{aligned} \tag{47}$$

where $\lambda \in \mathbb{R}$. Then, the aim is also to make the controlled process leave the continuation region C as rapidly as possible (if $\lambda > 0$) or remain in C as long as possible (if $\lambda < 0$).

The function $\Phi(x,y)$ defined in Eq. (8) is now given by

$$\Phi(x,y) = E\left[\exp\left(-\frac{\lambda T + K[X(T), Y(T)]}{\theta}\right)\right]. \tag{48}$$

It is a solution of the PDE

$$\frac{1}{2}\sigma^2 y^2 \Phi_{yy} + \mu y \Phi_y + f_0 y^2 \Phi_x = \beta \Phi, \tag{49}$$

where $\beta := \lambda/\theta$, and the boundary conditions are

$$\Phi(x,y) = \begin{cases} e^{-1/\theta} & \text{if } x/y^2 = k_1, \\ 1 & \text{if } x/y^2 = k_2. \end{cases} \tag{50}$$

Let $z := x/y^2$ and define $\Psi(z) = \Phi(x,y)$, as above. The function Ψ satisfies the ODE

$$2\sigma^2 z^2 \Psi''(z) + [(3\sigma^2 - 2\mu)z + f_0]\Psi'(z) = \beta \Psi(z). \tag{51}$$

The boundary conditions are the same as in the previous cases:

$$\Psi(z) = \begin{cases} e^{-1/\theta} & \text{if } z = k_1, \\ 1 & \text{if } z = k_2. \end{cases} \tag{52}$$

The general solution of Eq. (51) is in terms of the Kummer functions $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ [9],

$$\Psi(z) = z^{-\frac{\gamma-4\mu}{8\sigma^2}+\frac{1}{4}} \times \left[c_1 M\left(\frac{\gamma-4\mu}{8\sigma^2}+\frac{1}{4}, \frac{\gamma}{4\sigma^2}+1, \frac{f_0}{2\sigma^2 z}\right) + c_2 U\left(\frac{\gamma-4\mu}{8\sigma^2}+\frac{1}{4}, \frac{\gamma}{4\sigma^2}+1, \frac{f_0}{2\sigma^2 z}\right)\right], \quad (53)$$

where

$$\gamma := 2\sqrt{\sigma^4 + 4(2\beta-\mu)\sigma^2 + 4\mu^2}. \quad (54)$$

Proposition 2.3. *We may write that the value function in Case 3 is*

$$F(x,y) = -\theta \ln[\Psi(x/y^2)], \quad (55)$$

where $\Psi(\cdot)$ is defined in Eq. (53). The constants c_1 and c_2 are deduced from the boundary conditions in Eq. (52). Furthermore, the optimal control is given by

$$u^*(x,y) = -\frac{b_0\, y}{q_0} F_y(x,y) \quad (56)$$

for $k_1 < x/y^2 < k_2$.

Let us take $b_0 = q_0 = f_0 = \mu = 1$, $\sigma = \sqrt{2}$ and $\lambda = 8$. Then $\theta = 2$, $\beta = 4$ and the value function $F(x,y)$ can be expressed as elementary functions:

$$\Psi(z) = c_1(1+4z) + c_2\, z\, e^{1/(4z)}. \quad (57)$$

Let $k_1 = 1$ and $k_2 = 2$. We find that

$$c_1 = \frac{2e^{-5/8}-1}{10e^{-1/8}-9} \quad (58)$$

and

$$c_2 = \frac{e^{-1/4}(5-9e^{-1/2})}{10e^{-1/8}-9}. \quad (59)$$

Proceeding as above, we can obtain the value function $F(x,y)$ and the optimal control $u^*(x,y)$ explicitly.

Now, when $\lambda = 0$ (as in Cases 1 and 2), the function $\Psi(z)$ (denoted by $\Psi_0(z)$) becomes

$$\Psi_0(z) = d_1 + d_2\,\mathrm{Ei}_1\left(-\frac{1}{4z}\right), \quad (60)$$

where $\mathrm{Ei}_1(z)$ is an exponential integral function defined by

$$\mathrm{Ei}_1(z) = \int_1^\infty \frac{e^{-wz}}{w}\,\mathrm{d}w. \quad (61)$$

The constants d_1 and d_2 for which the boundary conditions are satisfied are

$$d_1 = \frac{\mathrm{Ei}_1(-1/8)\,e^{-1/2} - \mathrm{Ei}_1(-1/4)}{\mathrm{Ei}_1(-1/8) - \mathrm{Ei}_1(-1/4)} \quad (62)$$

and

$$d_2 = \frac{1-e^{-1/2}}{\mathrm{Ei}_1(-1/8) - \mathrm{Ei}_1(-1/4)}. \quad (63)$$

From $\Psi_0(z)$, we can now compute the corresponding value function $F_0(x,y)$ and the optimal control $u_0^*(x,y)$.

The value functions $F(x,y)$ and $F_0(x,y)$ and the optimal controls $u^*(x,y)$ and $u_0^*(x,y)$ are shown respectively in Figure 4 and Figure 5.

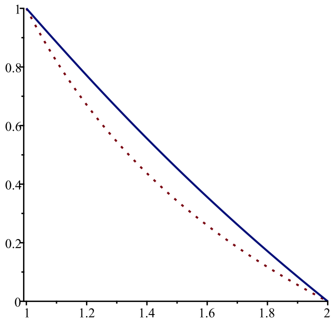

Figure 4: Functions $F(x,y)$ (solid line) and $F_0(x,y)$ in Case 3 for $x/y^2 \in [1,2]$ when $b_0 = q_0 = f_0 = \mu = 1$, $\sigma = \sqrt{2}$ and $\lambda = 8$.

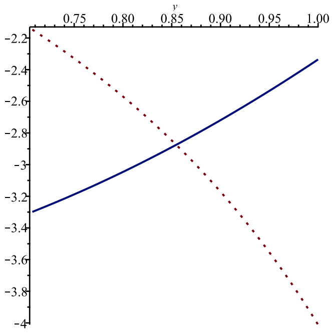

Figure 5: Functions $u^*(1,y)$ (solid line) and $u_0^*(1,y)$ in Case 3 for $y \in [(\sqrt{2})^{-1}, 1]$ when $b_0 = q_0 = f_0 = \mu = 1$, $\sigma = \sqrt{2}$ and $\lambda = 8$.

If we replace $\mu = 1$ by $\mu = -1$, we find that the general solution of Eq. (51) is given in terms of modified Bessel functions:

$$\Psi(z) = e^{1/(8z)}\left\{c_1 I_{\sqrt{5}/2}[1/(8z)] + c_2 K_{\sqrt{5}/2}[1/(8z)]\right\}. \quad (64)$$

The value function $F(x,y)$ and the optimal control $u^*(x,y)$ are presented respectively in Fig.6 & Fig.7, together with the corresponding functions when $\mu = 1$. We see that although the value functions are rather similar, the optimal controls are quite different.

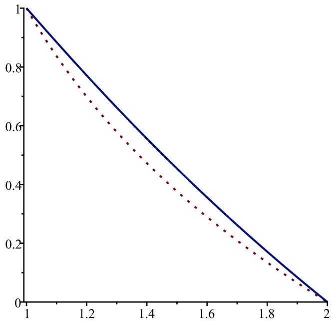

Figure 6: Functions $F(x, y)$ when $\mu = 1$ (solid line) and $\mu = -1$ in Case 3 for $x/y^2 \in [1, 2]$ when $b_0 = q_0 = f_0 = 1$, $\sigma = \sqrt{2}$ and $\lambda = 8$.

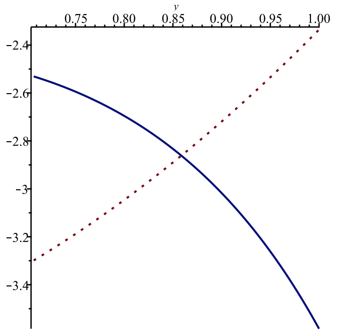

Figure 7: Functions $u^*(1, y)$ when $\mu = 1$ (solid line) and $\mu = -1$ in Case 3 for $y \in [(\sqrt{2})^{-1}, 1]$ when $b_0 = q_0 = f_0 = 1$, $\sigma = \sqrt{2}$ and $\lambda = 8$.

3 Conclusion

In this paper, we obtained explicit and exact solutions to optimal control problems for two-dimensional diffusion processes that could be used as models for the wear (or the remaining lifetime) of a device. These problems are particular LQG homing problems which are very difficult to solve, especially in two or more dimensions.

Using a result due to Whittle, it was possible to transform the control problems into purely probabilistic problems. Indeed, when the relation in Eq. (7) holds, it is possible to reduce the non-linear PDE satisfied by the value function to a linear PDE. This linear PDE is in fact the Kolmogorov backward equation satisfied by a certain mathematical expectation for the corresponding uncontrolled process.

Solving the Kolmogorov backward equation, subject to the appropriate boundary conditions, is in itself a difficult problem. Here, the linear PDE was solved explicitly in three important cases by making use of the method of similarity solutions.

When the relation in Eq. (7) does not hold, we can try other methods to obtain at least an approximate expression for the value function. We can also compute a numerical solution in any particular case. Another possibility is to calculate bounds for the value function and the corresponding optimal control, as was done in [10].

We could try to solve particular problems when there is more than one explanatory variable $Y(t)$. Finally, we could consider discrete-time versions of the problem treated in this paper.

Acknowledgements. The author would like to express his gratitude to the reviewers of this paper for their constructive comments.

References:

[1] R. Rishel, Controlled wear process: modeling optimal control, *IEEE Transactions on Automatic Control*, Vol. 36, No. 9, 1991, pp. 1100–1102. https://doi.org/10.1109/9.83548

[2] M. Lefebvre and P. Pazhoheshfar, An optimal control problem for the maintenance of a machine, *International Journal of Systems Science*, Vol. 53, No. 15, 2022, pp. 3364–3373. https://doi.org/10.1080/00207721.2022.2083258

[3] M. Lefebvre, First-passage times and optimal control of integrated jump-diffusion processes, *Fractal and Fractional*, Vol. 7, No. 2, 2023, Article 152. https://doi.org/10.3390/fractalfract7020152

[4] J. Kuhn, The risk-sensitive homing problem, *Journal of Applied Probability*, Vol. 22, No. 4, 1985, pp. 796–803. https://doi.org/10.2307/3213947

[5] C. Makasu, Risk-sensitive control for a class of homing problems, *Automatica*, Vol. 45, No. 10, 2009, pp. 2454–2455. https://doi.org/10.1016/j.automatica.2009.06.015

[6] M. Kounta and N.J. Dawson, Linear quadratic Gaussian homing for Markov processes with regime switching and applications to controlled population growth/decay, *Methodology and Computing in Applied Probability*, Vol. 23, 2021, pp. 1155–1172. https://doi.org/10.1007/s11009-020-09800-2

[7] P. Whittle, *Optimization over Time*, Vol. I, Wiley, Chichester, 1982.

[8] P. Whittle, *Risk-Sensitive Optimal Control*, Wiley, Chichester, 1990.

[9] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.

[10] C. Makasu, Homing problems with control in the diffusion coefficient, *IEEE Transactions on Automatic Control*, Vol. 67, No. 7, 2022, pp. 3770–3772. https://doi.org/10.1109/TAC.2022.3157077

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US