Overlapping Decentralized Dynamic Controller Design for Nonlinear Systems

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Abstract: - Inclusion and extension principles are defined for nonlinear time-invariant (NLTI) systems. These systems are assumed to be controlled by dynamic NLTI controllers and, thus, contractibility of such controllers is also discussed. It is shown that such a controller can always be contracted if the principle of extension is used. Design of overlappingly decentralized dynamic NLTI controllers within the framework of extension is then presented. It is shown that, when this approach is used, stability of the closed-loop system is assured.

Key-Words: - Nonlinear systems, Decentralized control, Large-scale systems, Overlapping decompositions, Stability

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1 Introduction

There are many examples of nonlinear large-scale systems, [1]. To design a controller for such a system, decomposition techniques are usually used, [2]. There are many examples of large-scale systems, however, which may not be decomposed into disjoint subsystems, due to the existence of *overlapping parts*. For such systems, the approach of *overlapping decompositions* has been introduced, [3]. Since then this approach has been used successfully for many different systems (e.g., [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]). In this approach, an overlappingly decomposed system is first *expanded* into a larger dimensional system in which overlapping subsystems appear as disjoint. A local controller is then designed for each disjoint subsystem such that the overall decentralized controller stabilizes the overall expanded system and satisfies desired performance criteria. This overall controller is then *contracted* to a controller (which appears as *overlappingly decentralized*) for application to the original system.

The overlapping decompositions approach is based on the principle of *inclusion*, [15]. However, when inclusion is used in controller design, there is no guarantee that the controller designed for the expanded system can be contracted for application to the original system, [16]. To overcome this difficulty, the principle of *extension*, [17], which is a special case of inclusion, has been introduced.

Although initial consideration of inclusion and extension principles and overlapping decompositions were restricted to finite-dimensional linear time-invariant (LTI) systems, since then, these have also been considered for linear time-varying, [18], and linear time-delay, [19], systems. Recently, these principles have also been considered for nonlinear time-invariant (NLTI) systems, [20]. However, only static controllers were considered in [20]. Furthermore, due to page limitations all the proofs were omitted in [20]. In the present work, we present all the necessary proofs and extend the results of [20], to the case of NLTI dynamic controllers. First, in the next section, we define the principle of inclusion for NLTI systems. We also show that when the original system is included by the expanded system, stability of the former is implied by the stability of the latter. Then, in Section 3, we define the principle of extension for NLTI systems. The necessary and sufficient conditions for extension are also presented in the same section and it is shown that it is a special case of inclusion. In Section 4, we consider contractibility of NLTI dynamic controllers. There, it is also shown that, if the expanded system is an extension of the original system, such a controller for the former system can always be contracted for application to the latter. In Section 5, we present overlapping decompositions of NLTI systems and show how overlappingly decentralized NLTI dynamic controllers can be designed within this framework.

Throughout, for positive integers μ and ν, $\mathbf{R}^{\mu}$ and $\mathbf{R}^{\mu\times\nu}$ denote the spaces of, respectively, μ-dimensional real vectors and $\mu\times\nu$-dimensional real matrices. $\mathbf{R}_{+}$ denotes the set of non-negative real numbers. $(\cdot)^T$ denotes the transpose of $(\cdot)$. I_{μ} denotes the $\mu\times\mu$-dimensional identity matrix. 0 may denote either the scalar zero, a zero vector, or a zero matrix. For $\xi\in\mathbf{R}^{\mu}$, $\|\xi\|$ denotes the 2-norm of ξ. Finally, for two matrices M and N, $\mathrm{bdiag}(M,N) := \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$.

2 Inclusion

Let us consider the NLTI systems:

$$\mathcal{S}: \begin{array}{l} \dot{\xi}(t)=\phi(\xi(t),\upsilon(t)) \\ \psi(t)=\gamma(\xi(t),\upsilon(t)) \end{array} \tag{1}$$

and

$$\hat{\mathcal{S}}: \begin{array}{l} \dot{\hat{\xi}}(t)=\hat{\phi}(\hat{\xi}(t),\hat{\upsilon}(t)) \\ \hat{\psi}(t)=\hat{\gamma}(\hat{\xi}(t),\hat{\upsilon}(t)) \end{array} . \tag{2}$$

The systems $\mathcal{S}$ and $\hat{\mathcal{S}}$ are to be referred to as the *original* and the *expanded* systems, respectively. Here, $\xi(t)\in\mathbf{R}^{\nu}$ and $\hat{\xi}(t)\in\mathbf{R}^{\hat{\nu}}$ are the state, $\upsilon(t)\in\mathbf{R}^{\mu}$ and $\hat{\upsilon}(t)\in\mathbf{R}^{\hat{\mu}}$ are the input, and $\psi(t)\in\mathbf{R}^{\eta}$ and $\hat{\psi}(t)\in\mathbf{R}^{\hat{\eta}}$ are the output vectors of, respectively, $\mathcal{S}$ and $\hat{\mathcal{S}}$ at time t. It is assumed that the dimensions of these vectors of $\hat{\mathcal{S}}$ are greater than or equal to the dimensions of the corresponding vectors of $\mathcal{S}$; i.e., $\hat{\nu}\geq\nu$, $\hat{\mu}\geq\mu$, and $\hat{\eta}\geq\eta$. The functions $\phi:\mathbf{R}^{\nu}\times\mathbf{R}^{\mu}\to\mathbf{R}^{\nu}$, $\gamma:\mathbf{R}^{\nu}\times\mathbf{R}^{\mu}\to\mathbf{R}^{\eta}$, $\hat{\phi}:\mathbf{R}^{\hat{\nu}}\times\mathbf{R}^{\hat{\mu}}\to\mathbf{R}^{\hat{\nu}}$, and $\hat{\gamma}:\mathbf{R}^{\hat{\nu}}\times\mathbf{R}^{\hat{\mu}}\to\mathbf{R}^{\hat{\eta}}$ are well-defined functions. Furthermore, $\frac{\partial\phi}{\partial\xi}$ and $\frac{\partial\hat{\phi}}{\partial\hat{\xi}}$ are also assumed to be well defined. This assumption guarantees that, for any initial conditions

$$\xi(0)=\xi_0\in\mathbf{R}^{\nu} \quad\text{and}\quad \hat{\xi}(0)=\hat{\xi}_0\in\mathbf{R}^{\hat{\nu}} , \tag{3}$$

the solutions to (1) and (2) uniquely exist, [21].

Furthermore, to assure the existence and uniqueness of solutions for the closed-loop systems, we also assume that $\frac{\partial\phi}{\partial\upsilon}$, $\frac{\partial\hat{\phi}}{\partial\hat{\upsilon}}$, $\frac{\partial\gamma}{\partial\xi}$, $\frac{\partial\hat{\gamma}}{\partial\hat{\xi}}$, $\frac{\partial\gamma}{\partial\upsilon}$, and $\frac{\partial\hat{\gamma}}{\partial\hat{\upsilon}}$ are all well defined. It is further assumed that $\phi(0,0)=0$, $\gamma(0,0)=0$, $\hat{\phi}(0,0)=0$, and $\hat{\gamma}(0,0)=0$. Thus, $0\in\mathbf{R}^{\nu}$ and $0\in\mathbf{R}^{\hat{\nu}}$ are equilibrium points of (1) and (2), respectively, and, together with zero input, they produce zero output.

Next, we define inclusion for NLTI systems:

Definition 1: $\hat{\mathcal{S}}$ is said to *include* $\mathcal{S}$ and $\mathcal{S}$ is said to be *included by* $\hat{\mathcal{S}}$ if there exist full-rank matrices $U\in\mathbf{R}^{\hat{\nu}\times\nu}$, $V\in\mathbf{R}^{\hat{\mu}\times\mu}$, $U^{\dagger}\in\mathbf{R}^{\nu\times\hat{\nu}}$, and $R^{\dagger}\in\mathbf{R}^{\eta\times\hat{\eta}}$, with $U^{\dagger}U=I_{\nu}$, such that for any initial condition $\xi_0\in\mathbf{R}^{\nu}$ and any input $\upsilon:\mathbf{R}_{+}\to\mathbf{R}^{\mu}$ of $\mathcal{S}$, the choice

$$\hat{\xi}(0)=U\xi_0 \tag{4}$$

for the initial condition and

$$\hat{\upsilon}(t)=V\upsilon(t) , \quad t\geq 0 \tag{5}$$

for the input of $\hat{\mathcal{S}}$ implies

$$\xi(t)=U^{\dagger}\hat{\xi}(t) , \quad t\geq 0 \tag{6}$$

and

$$\psi(t)=R^{\dagger}\hat{\psi}(t) , \quad t\geq 0 . \tag{7}$$

It is important for the expanded system to include the original system so that certain properties, such as stability, are retained between the two systems. Although other definitions of stability can also be considered, here, for brevity, we only consider *global asymptotic stability of the zero state*. Thus, we present the following definition for the sake of completeness:

Definition 2: $\mathcal{S}$ is said to be *stable*, if for any initial state $\xi_0\in\mathbf{R}^{\nu}$, there exists a bound $M<\infty$ such that with $\upsilon(t)=0$, $t\geq 0$, the solution $\xi(t)$ to (1) satisfies $\|\xi(t)\|\leq M$, $\forall t\geq 0$, and $\lim_{t\to\infty}\xi(t)=0$. Furthermore, a controller is said to *stabilize* a system if the closed-loop system under that controller is stable.

Our first result shows that stability of $\hat{\mathcal{S}}$ implies stability of $\mathcal{S}$ when $\hat{\mathcal{S}}$ includes $\mathcal{S}$:

Theorem 1: Let $\hat{\mathcal{S}}$ include $\mathcal{S}$ and $\hat{\mathcal{S}}$ be stable. Then $\mathcal{S}$ is also stable.

Proof: Let $\xi_0\in\mathbf{R}^{\nu}$ be arbitrary and $\upsilon(t)=0$, for $t\geq 0$. Let $\hat{\xi}(0)$ and $\hat{\upsilon}(t)$ be given by (4) and (5), respectively (thus $\hat{\upsilon}(t)=0$, for $t\geq 0$). Then, since $\hat{\mathcal{S}}$

includes $\mathcal{S}$, (6) holds. Furthermore, since $\hat{\mathcal{S}}$ is stable, $\|\hat{\xi}(t)\|$ is bounded $\forall t \geq 0$ and $\lim_{t\to\infty} \hat{\xi}(t) = 0$. By (6), this implies that $\|\xi(t)\|$ is bounded $\forall t \geq 0$ and $\lim_{t\to\infty} \xi(t) = 0$, which implies stability of $\mathcal{S}$. □

3 Extension

As it will be shown in Section 5, in the overlapping decompositions approach, a controller is first designed for the expanded system and then contracted to a controller for the original system. However, if the former simply includes the latter, there is no guarantee that such a controller can be contracted, [16]. For this guarantee, a special case of inclusion, called *extension*, is needed.

We define extension for NLTI systems as follows:

Definition 3: $\hat{\mathcal{S}}$ is said to be an *extension* of $\mathcal{S}$ and $\mathcal{S}$ is said to be a *disextension* of $\hat{\mathcal{S}}$ if there exist full-rank matrices

$$U \in \mathbf{R}^{\hat{\nu}\times\nu}\ , \quad V^\dagger \in \mathbf{R}^{\mu\times\hat{\mu}}\ , \quad \text{and} \quad R \in \mathbf{R}^{\hat{\eta}\times\eta} \tag{8}$$

such that for any initial condition $\xi_0 \in \mathbf{R}^\nu$ of $\mathcal{S}$ and any input $\hat{\upsilon} : \mathbf{R}_+ \to \mathbf{R}^{\hat{\mu}}$ of $\hat{\mathcal{S}}$, the choice (4) for the initial condition of $\hat{\mathcal{S}}$ and

$$\upsilon(t) = V^\dagger \hat{\upsilon}(t)\ , \quad t \geq 0 \tag{9}$$

for the input of $\mathcal{S}$ implies

$$\hat{\xi}(t) = U\xi(t)\ , \quad t \geq 0 \tag{10}$$

and

$$\hat{\psi}(t) = R\psi(t)\ , \quad t \geq 0\ . \tag{11}$$

Next, we give the necessary and sufficient conditions for extension:

Theorem 2: Necessary and sufficient condition for $\hat{\mathcal{S}}$ being an *extension* of $\mathcal{S}$ is the existence of full-rank matrices as in (8) such that

$$\hat{\phi}(U\xi, \hat{\upsilon}) = U\phi(\xi, V^\dagger\hat{\upsilon}) \tag{12}$$

and

$$\hat{\gamma}(U\xi, \hat{\upsilon}) = R\gamma(\xi, V^\dagger\hat{\upsilon}) \tag{13}$$

for all $\xi \in \mathbf{R}^\nu$ and $\hat{\upsilon} \in \mathbf{R}^{\hat{\mu}}$.

Proof: To prove sufficiency, premultiply both sides of the first equation in (1) by U and use (9) and (12) to obtain

$$\begin{aligned} U\dot{\xi}(t) &= U\phi(\xi(t), \upsilon(t)) = U\phi(\xi(t), V^\dagger\hat{\upsilon}(t)) \\ &= \hat{\phi}(U\xi(t), \hat{\upsilon}(t))\ . \end{aligned} \tag{14}$$

By (4) and uniqueness of solutions to (2), this implies (10). Next, premultiply both sides of the second equation in (1) by R and use (9), (13), and (10) to obtain

$$\begin{aligned} R\psi(t) &= R\gamma(\xi(t), \upsilon(t)) = R\gamma(\xi(t), V^\dagger\hat{\upsilon}(t)) \\ &= \hat{\gamma}(U\xi(t), \hat{\upsilon}(t)) = \hat{\gamma}(\hat{\xi}(t), \hat{\upsilon}(t))\ , \end{aligned} \tag{15}$$

which, by the second equation in (2), implies (11). This proves sufficiency.

To prove the necessity of (12), suppose that (12) does not hold for some $\xi_1 \in \mathbf{R}^\nu$ and/or for some $\hat{\upsilon}_1 \in \mathbf{R}^{\hat{\mu}}$. Let $\xi(0) = \xi_0 = \xi_1$ and $\hat{\upsilon}(0) = \hat{\upsilon}_1$. Let $\hat{\xi}(0)$ and $\upsilon(0)$ be given by (4) and (9), respectively. Then we will have $\dot{\hat{\xi}}(0) \neq U\dot{\xi}(0)$, which will imply $\hat{\xi}(0^+) \neq U\xi(0^+)$, which means that (10) will not hold. Thus, (12) is necessary.

Next. to prove the necessity of (13), suppose that (13) does not hold for some $\xi_1 \in \mathbf{R}^\nu$ and/or for some $\hat{\upsilon}_1 \in \mathbf{R}^{\hat{\mu}}$. Let $\xi(0) = \xi_0 = \xi_1$ and $\hat{\upsilon}(0) = \hat{\upsilon}_1$. Let $\hat{\xi}(0)$ and $\upsilon(0)$ be given by (4) and (9), respectively. Then we will have $\hat{\psi}(0) \neq R\psi(0)$, which means that (11) will not hold. Thus, (13) is also necessary. This concludes the proof. □

Next, we prove the following:

Theorem 3: If $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$, then, $\hat{\mathcal{S}}$ includes $\mathcal{S}$.

Proof: Since U, $V^\dagger$, and R are full-rank matrices, there exist full-rank matrices $U^\dagger$, V, and $R^\dagger$ such that $U^\dagger U = I_\nu$, $V^\dagger V = I_\mu$, and $R^\dagger R = I_\eta$. Then, (5) implies (9), (10) implies (6), and (11) implies (7). Thus, according to Definition 1, $\hat{\mathcal{S}}$ includes $\mathcal{S}$. □

The following result then follows:

Corollary 1: Let $\hat{\mathcal{S}}$ be an extension of $\mathcal{S}$ and $\hat{\mathcal{S}}$ be stable. Then $\mathcal{S}$ is also stable.

Proof: By Theorems 3 and 1. □

4 Contractibility

As mentioned in the beginning of the previous section, in the overlapping decompositions approach, first the design of a controller for the expanded system is undertaken. Then, this controller is contracted to a controller for the original system. However, for this to work, the designed controller must be *contractible*. Thus, contractibility of controllers for NLTI systems is discussed in this section. We will consider NLTI dynamic (which can be reduced to static controllers by choosing $\sigma = 0$ or $\hat{\sigma} = 0$ below) controllers of the form

$$\mathcal{G}: \begin{array}{l} \dot{\zeta}(t) = \chi(\zeta(t), \lambda(t)) \\ \omega(t) = \kappa(\zeta(t), \lambda(t)) \end{array} , \tag{16}$$

for $\mathcal{S}$, and

$$\hat{\mathcal{G}}: \begin{array}{l} \dot{\hat{\zeta}}(t) = \hat{\chi}(\hat{\zeta}(t), \hat{\lambda}(t)) \\ \hat{\omega}(t) = \hat{\kappa}(\hat{\zeta}(t), \hat{\lambda}(t)) \end{array} , \tag{17}$$

for $\hat{\mathcal{S}}$. Here, $\zeta(t) \in \mathbf{R}^{\sigma}$ and $\hat{\zeta}(t) \in \mathbf{R}^{\hat{\sigma}}$ are the state, $\lambda(t) \in \mathbf{R}^{\eta}$ and $\hat{\lambda}(t) \in \mathbf{R}^{\hat{\eta}}$ are the input, and $\omega(t) \in \mathbf{R}^{\mu}$ and $\hat{\omega}(t) \in \mathbf{R}^{\hat{\mu}}$ are the output vectors of, respectively, $\mathcal{G}$ and $\hat{\mathcal{G}}$ at time t. Furthermore, the functions $\chi : \mathbf{R}^{\sigma} \times \mathbf{R}^{\eta} \to \mathbf{R}^{\sigma}$, $\kappa : \mathbf{R}^{\sigma} \times \mathbf{R}^{\eta} \to \mathbf{R}^{\mu}$, $\hat{\chi} : \mathbf{R}^{\hat{\sigma}} \times \mathbf{R}^{\hat{\eta}} \to \mathbf{R}^{\hat{\sigma}}$, and $\hat{\kappa} : \mathbf{R}^{\hat{\sigma}} \times \mathbf{R}^{\hat{\eta}} \to \mathbf{R}^{\hat{\mu}}$ are well-defined functions. To assure the existence and uniqueness of solutions to (16) and (17) for any initial conditions

$$\zeta(0) = \zeta_0 \in \mathbf{R}^{\sigma} \quad \text{and} \quad \hat{\zeta}(0) = \hat{\zeta}_0 \in \mathbf{R}^{\hat{\sigma}} , \tag{18}$$

it is also assumed that $\dfrac{\partial \chi}{\partial \zeta}$ and $\dfrac{\partial \hat{\chi}}{\partial \hat{\zeta}}$ are also well defined. It is further assumed that $\chi(0, 0) = 0$, $\kappa(0, 0) = 0$, $\hat{\chi}(0, 0) = 0$, and $\hat{\kappa}(0, 0) = 0$.

The controllers $\mathcal{G}$ and $\hat{\mathcal{G}}$ are respectively implemented on $\mathcal{S}$ and $\hat{\mathcal{S}}$ by letting

$$\lambda(t) = \psi(t) - \rho(t) \quad \text{and} \quad \upsilon(t) = \omega(t) \tag{19}$$

and

$$\hat{\lambda}(t) = \hat{\psi}(t) - \hat{\rho}(t) \quad \text{and} \quad \hat{\upsilon}(t) = \hat{\omega}(t) \tag{20}$$

for $t \geq 0$, where $\rho(t) \in \mathbf{R}^{\eta}$ and $\hat{\rho}(t) \in \mathbf{R}^{\hat{\eta}}$ are *reference* inputs at time t, for $\mathcal{S}$ and $\hat{\mathcal{S}}$, respectively.

To guarantee solvability, here it is also assumed that $\kappa : \mathbf{R}^{\sigma} \times \mathbf{R}^{\eta} \to \mathbf{R}^{\mu}$ is such that

$$\upsilon = \kappa(\zeta, \gamma(\xi, \upsilon) - \rho) \tag{21}$$

has a unique solution

$$\upsilon = \theta(\zeta, \xi, \rho) \tag{22}$$

for all $\zeta \in \mathbf{R}^{\sigma}$, $\xi \in \mathbf{R}^{\nu}$, and $\rho \in \mathbf{R}^{\eta}$; and $\hat{\kappa} : \mathbf{R}^{\hat{\sigma}} \times \mathbf{R}^{\hat{\eta}} \to \mathbf{R}^{\hat{\mu}}$ is such that

$$\hat{\upsilon} = \hat{\kappa}(\hat{\zeta}, \hat{\gamma}(\hat{\xi}, \hat{\upsilon}) - \hat{\rho}) \tag{23}$$

has a unique solution

$$\hat{\upsilon} = \hat{\theta}(\hat{\zeta}, \hat{\xi}, \hat{\rho}) \tag{24}$$

for all $\hat{\zeta} \in \mathbf{R}^{\hat{\sigma}}$, $\hat{\xi} \in \mathbf{R}^{\hat{\nu}}$, and $\hat{\rho} \in \mathbf{R}^{\hat{\eta}}$. Note that, the above assumptions guarantee that $\theta(0, 0, 0) = 0$ and $\hat{\theta}(0, 0, 0) = 0$. It is also assumed that θ, $\hat{\theta}$, $\dfrac{\partial \theta}{\partial \xi}$, $\dfrac{\partial \theta}{\partial \zeta}$, $\dfrac{\partial \hat{\theta}}{\partial \hat{\xi}}$, $\dfrac{\partial \hat{\theta}}{\partial \hat{\zeta}}$, $\dfrac{\partial \chi}{\partial \lambda}$, and $\dfrac{\partial \hat{\chi}}{\partial \hat{\lambda}}$ are all well defined functions.

We can now define contractibility of NLTI controllers for NLTI systems:

Definition 4: Assume that only the first connection in (19) and only the first connection in (20) are made. $\hat{\mathcal{G}}$ for $\hat{\mathcal{S}}$ is said to be *contractible* to $\mathcal{G}$ for $\mathcal{S}$ if there exist full-rank matrices as in (8) and a full row-rank matrix $S^{\dagger} \in \mathbf{R}^{\sigma \times \hat{\sigma}}$ such that for any initial condition $\xi_0 \in \mathbf{R}^{\nu}$ of $\mathcal{S}$, for any input $\hat{\upsilon} : \mathbf{R}_+ \to \mathbf{R}^{\hat{\mu}}$ of $\hat{\mathcal{S}}$, for any reference $\rho : \mathbf{R}_+ \to \mathbf{R}^{\eta}$ for $\mathcal{S}$, and for any initial condition $\hat{\zeta}_0 \in \mathbf{R}^{\hat{\sigma}}$ of $\hat{\mathcal{G}}$, the choice (4), (9),

$$\zeta_0 = S^{\dagger} \hat{\zeta}_0 \tag{25}$$

and

$$\hat{\rho}(t) = R\rho(t) , \quad t \geq 0 \tag{26}$$

implies

$$\zeta(t) = S^{\dagger} \hat{\zeta}(t) , \quad t \geq 0 \tag{27}$$

and

$$\omega(t) = V^{\dagger} \hat{\omega}(t) , \quad t \geq 0 . \tag{28}$$

Note that, a necessary condition for $S^{\dagger} \in \mathbf{R}^{\sigma \times \hat{\sigma}}$ to be of full row-rank is $\hat{\sigma} \geq \sigma$. This condition, however, is not restrictive, since $\mathcal{S}$ is in general included in $\hat{\mathcal{S}}$ and thus should not require a larger dimensional controller. Contractibility should be satisfied so that condition (9) holds after the application of the controllers. We now present the conditions for contractibility of $\hat{\mathcal{G}}$ to $\mathcal{G}$:

Theorem 4: Let $\hat{\mathcal{S}}$ be an extension of $\mathcal{S}$ and $\hat{\mathcal{G}}$ be a controller for $\hat{\mathcal{S}}$. Then, $\hat{\mathcal{G}}$ is contractible to the controller $\mathcal{G}$ for $\mathcal{S}$ if there exists a full row-rank matrix $S^{\dagger} \in \mathbf{R}^{\sigma \times \hat{\sigma}}$ such that

$$\chi(S^{\dagger} \hat{\zeta}, \lambda) = S^{\dagger} \hat{\chi}(\hat{\zeta}, R\lambda) \tag{29}$$

and

$$\kappa(S^{\dagger} \hat{\zeta}, \lambda) = V^{\dagger} \hat{\kappa}(\hat{\zeta}, R\lambda) , \tag{30}$$

for all $\hat{\zeta} \in \mathbf{R}^{\hat{\sigma}}$ and $\lambda \in \mathbf{R}^{\eta}$, where $V^{\dagger}$ and R are as in (8).

Proof: Since $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$, (4) and (9) implies (11). (11) and (26), however, implies

$$\hat{\lambda}(t) = R\lambda(t) , \quad t \geq 0 . \tag{31}$$

Now, premultiply both sides of the first equation in (17) by $S^\dagger$ and use (31) and (29) to obtain:

$$\begin{aligned} S^\dagger\dot{\hat{\zeta}}(t) &= S^\dagger\hat{\chi}(\hat{\zeta}(t),\hat{\lambda}(t)) = S^\dagger\hat{\chi}(\hat{\zeta}(t),R\lambda(t)) \\ &= \chi(S^\dagger\hat{\zeta}(t),\lambda(t)) \; . \end{aligned} \tag{32}$$

By (25) and uniqueness of solutions to (16), this implies (27).

Next, premultiply both sides of the second equation in (17) by $V^\dagger$ and use (31), (30), and (27) to obtain

$$\begin{aligned} V^\dagger\hat{\omega}(t) &= V^\dagger\hat{\kappa}(\hat{\zeta}(t),\hat{\lambda}(t)) = V^\dagger\hat{\kappa}(\hat{\zeta}(t),R\lambda(t)) \\ &= \kappa(S^\dagger\hat{\zeta}(t),\lambda(t)) = \kappa(\zeta(t),\lambda(t)) \; , \end{aligned} \tag{33}$$

which, by (16), implies (28). This concludes the proof. □

Next we show that, when $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$, contractibility of any controller $\hat{\mathcal{G}}$ is guaranteed:

Corollary 2: Let $\hat{\mathcal{S}}$ be an extension of $\mathcal{S}$ and $\hat{\mathcal{G}}$ be a controller for $\hat{\mathcal{S}}$. Then, $\hat{\mathcal{G}}$ is contractible to a controller $\mathcal{G}$ for $\mathcal{S}$ with

$$\chi(\zeta,\lambda) := \hat{\chi}(\zeta,R\lambda) \tag{34}$$

and

$$\kappa(\zeta,\lambda) := V^\dagger\hat{\kappa}(\zeta,R\lambda) \; , \tag{35}$$

for all $\zeta \in \mathbf{R}^{\hat{\sigma}}$ and $\lambda \in \mathbf{R}^{\eta}$, where $V^\dagger$ and R are as in (8).

Proof: Follows from Theorem 4 with $S^\dagger = I_{\hat{\sigma}}$. □

When $\mathcal{G}$ is applied to $\mathcal{S}$ by making both connections in (19), the closed-loop system can be described as:

$$\mathcal{S}_c: \begin{array}{l} \dot{\bar{\xi}}(t) = \bar{\phi}(\bar{\xi}(t),\rho(t)) \\ \psi(t) = \bar{\gamma}(\bar{\xi}(t),\rho(t)) \end{array} \; , \tag{36}$$

where $\bar{\xi}(t) := \begin{bmatrix} \xi^T(t) & \zeta^T(t) \end{bmatrix}^T \in \mathbf{R}^{\nu+\sigma}$ is the state of $\mathcal{S}_c$,

$$\bar{\phi}(\begin{bmatrix} \xi \\ \zeta \end{bmatrix},\rho) := \begin{bmatrix} \phi(\xi,\theta(\zeta,\xi,\rho)) \\ \chi(\zeta,\gamma(\xi,\theta(\zeta,\xi,\rho))-\rho) \end{bmatrix} ,$$

and $\bar{\gamma}(\begin{bmatrix} \xi^T,\zeta^T \end{bmatrix}^T,\rho) := \gamma(\xi,\theta(\zeta,\xi,\rho))$. Similarly, when $\hat{\mathcal{G}}$ is applied to $\hat{\mathcal{S}}$ by making both connections in (20), the closed-loop system can be described as:

$$\hat{\mathcal{S}}_c: \begin{array}{l} \dot{\hat{\bar{\xi}}}(t) = \hat{\bar{\phi}}(\hat{\bar{\xi}}(t),\hat{\rho}(t)) \\ \hat{\psi}(t) = \hat{\bar{\gamma}}(\hat{\bar{\xi}}(t),\hat{\rho}(t)) \end{array} \; , \tag{37}$$

where $\hat{\bar{\xi}}(t) := \begin{bmatrix} \hat{\xi}^T(t) & \hat{\zeta}^T(t) \end{bmatrix}^T \in \mathbf{R}^{\hat{\nu}+\hat{\sigma}}$ is the state of $\hat{\mathcal{S}}_c$,

$$\hat{\bar{\phi}}(\begin{bmatrix} \hat{\xi} \\ \hat{\zeta} \end{bmatrix},\hat{\rho}) := \begin{bmatrix} \hat{\phi}(\hat{\xi},\hat{\theta}(\hat{\zeta},\hat{\xi},\hat{\rho})) \\ \hat{\chi}(\hat{\zeta},\hat{\gamma}(\hat{\xi},\hat{\theta}(\hat{\zeta},\hat{\xi},\hat{\rho}))-\hat{\rho}) \end{bmatrix} ,$$

and $\hat{\bar{\gamma}}(\begin{bmatrix} \hat{\xi}^T,\hat{\zeta}^T \end{bmatrix}^T,\hat{\rho}) := \hat{\gamma}(\hat{\xi},\hat{\theta}(\hat{\zeta},\hat{\xi},\hat{\rho}))$. Note that, under the foregoing assumptions, $\bar{\phi}$, $\bar{\gamma}$, $\hat{\bar{\phi}}$, $\hat{\bar{\gamma}}$, $\dfrac{\partial\bar{\phi}}{\partial\bar{\xi}}$, and $\dfrac{\partial\hat{\bar{\phi}}}{\partial\hat{\bar{\xi}}}$ are all well-defined functions and satisfy $\bar{\phi}(0,0)=0$, $\bar{\gamma}(0,0)=0$, $\hat{\bar{\phi}}(0,0)=0$, and $\hat{\bar{\gamma}}(0,0)=0$.

Now, we can show that when $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$ and $\hat{\mathcal{G}}$ is contractible to $\mathcal{G}$, $\hat{\mathcal{S}}_c$ includes $\mathcal{S}_c$:

Theorem 5: Let $\hat{\mathcal{S}}$ be an extension of $\mathcal{S}$ and $\hat{\mathcal{G}}$ be contractible to $\mathcal{G}$. Then, $\hat{\mathcal{S}}_c$ includes $\mathcal{S}_c$.

Proof: Let U, $V^\dagger$, and R be as in Definition 3 and $S^\dagger$ be as in Definition 4. Let $U^\dagger$, $R^\dagger$, and S be such that $U^\dagger U = I_\nu$, $R^\dagger R = I_\eta$, and $S^\dagger S = I_\sigma$ (there exist such matrices since U, R, and $S^\dagger$ are full-rank). Let $\bar{U} := \mathrm{bdiag}(U,S)$, $\bar{U}^\dagger := \mathrm{bdiag}(U^\dagger,S^\dagger)$ (note that $\bar{U}^\dagger\bar{U} = I_{\nu+\sigma}$), $\bar{V} := R$, and $\bar{R}^\dagger := R^\dagger$. Let $\bar{\xi}(0) = \bar{\xi}_0 := \begin{bmatrix} \xi_0^T & \zeta_0^T \end{bmatrix}^T \in \mathbf{R}^{\nu+\sigma}$ and $\rho : \mathbf{R}_+ \to \mathbf{R}^\eta$ be arbitrary. Let

$$\hat{\bar{\xi}}(0) = \bar{U}\bar{\xi}_0 \tag{38}$$

and

$$\hat{\rho}(t) = \bar{V}\rho(t) \; , \quad t \geq 0 \; . \tag{39}$$

Note that (38) implies $\hat{\xi}(0) = U\xi_0$ and $\hat{\zeta}(0) = S\zeta_0$. The former of which is simply (4) and the latter implies (by premultiplying by $S^\dagger$) (25). Furthermore, (39) is simply (26). Thus, since $\hat{\mathcal{G}}$ is contractible to $\mathcal{G}$, (27) and (28) hold. However, (28) implies (9). Thus, since $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$, (10) and (11) hold. However, (10) and (27) imply

$$\bar{\xi}(t) = \bar{U}^\dagger\hat{\bar{\xi}}(t) \; , \quad t \geq 0 \; , \tag{40}$$

and (11) implies

$$\psi(t) = \bar{R}^\dagger\hat{\psi}(t) \; , \quad t \geq 0 \; . \tag{41}$$

This, however, implies that $\hat{\mathcal{S}}_c$ includes $\mathcal{S}_c$. □

Now, we can prove the following:

Theorem 6: Let $\hat{\mathcal{S}}$ be an extension of $\mathcal{S}$ and $\hat{\mathcal{G}}$ be contractible to $\mathcal{G}$. Also suppose that $\hat{\mathcal{G}}$ stabilizes $\hat{\mathcal{S}}$. Then, $\mathcal{G}$ stabilizes $\mathcal{S}$.

Proof: Follows from Theorems 5 and 1. □

5 Overlapping Decompositions

Theorem 6 implies that, if we can obtain an extension $\hat{\mathcal{S}}$ of a given system $\mathcal{S}$, design a controller $\hat{\mathcal{G}}$ which stabilizes $\hat{\mathcal{S}}$, and contract $\hat{\mathcal{G}}$ to a controller $\mathcal{G}$, this controller can be applied to $\mathcal{S}$ to stabilize it. This approach can be used in particular if the given system is made up of *overlapping subsystems*. In such a case, using *overlapping decompositions and expansions*, an expanded system, made up of disjoint subsystems, which is an extension of the original system can be obtained. A local controller can then be designed for each disjoint subsystem. By combining these local controllers, a controller for the expanded system is obtained, which has a decentralized structure. This controller can then be contracted to obtain a controller for the original system, which has an overlappingly decentralized structure.

For brevity, we will only consider the simplest case of two overlapping subsystems. In such a case, the state, the input, and the output vectors of the original system $\mathcal{S}$, described by (1), can be decomposed as:

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_c \\ \xi_2 \end{bmatrix} , \quad \upsilon = \begin{bmatrix} \upsilon_1 \\ \upsilon_c \\ \upsilon_2 \end{bmatrix} , \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_c \\ \psi_2 \end{bmatrix} , \tag{42}$$

where, for $i = 1, 2$, $\xi_i \in \mathbf{R}^{\nu_i}$, $\upsilon_i \in \mathbf{R}^{\mu_i}$, and $\psi_i \in \mathbf{R}^{\eta_i}$ are the respective vectors for the i^{th} subsystem only, and $\xi_c \in \mathbf{R}^{\nu_c}$, $\upsilon_c \in \mathbf{R}^{\mu_c}$, and $\psi_c \in \mathbf{R}^{\eta_c}$ are the corresponding vectors for the overlapping part.

Here, we make the assumption that there are only dynamic interactions between the subsystems over the overlapping part. In this case, the functions ϕ and γ have the structure:

$$\phi(\xi, \upsilon) = \begin{bmatrix} \phi_1(\xi_1, \xi_c, \upsilon_1) \\ \phi_c(\xi_1, \xi_c, \xi_2, \upsilon_c) \\ \phi_2(\xi_c, \xi_2, \upsilon_2) \end{bmatrix} \tag{43}$$

and

$$\gamma(\xi, \upsilon) = \begin{bmatrix} \gamma_1(\xi_1, \upsilon_1) \\ \gamma_c(\xi_c, \upsilon_c) \\ \gamma_2(\xi_2, \upsilon_2) \end{bmatrix} \tag{44}$$

where the partitionings are compatible with those in (42).

To define the expansion, we define the vectors:

$$\hat{\xi} = \begin{bmatrix} \xi_1 \\ \xi_{c1} \\ \xi_{c2} \\ \xi_2 \end{bmatrix} , \quad \hat{\upsilon} = \begin{bmatrix} \upsilon_1 \\ \upsilon_{c1} \\ \upsilon_{c2} \\ \upsilon_2 \end{bmatrix} , \quad \hat{\psi} = \begin{bmatrix} \psi_1 \\ \psi_{c1} \\ \psi_{c2} \\ \psi_2 \end{bmatrix} , \tag{45}$$

where $\xi_{c1}, \xi_{c2} \in \mathbf{R}^{\nu_c}$, $\upsilon_{c1}, \upsilon_{c2} \in \mathbf{R}^{\mu_c}$, and $\psi_{c1}, \psi_{c2} \in \mathbf{R}^{\eta_c}$ (thus, $\hat{\xi} \in \mathbf{R}^{\nu+\nu_c}$, $\hat{\upsilon} \in \mathbf{R}^{\mu+\mu_c}$, and $\hat{\psi} \in \mathbf{R}^{\eta+\eta_c}$). The expanded system $\hat{\mathcal{S}}$ is then described by (2), where

$$\hat{\phi}(\hat{\xi}, \hat{\upsilon}) := \begin{bmatrix} \phi_1(\xi_1, \xi_{c1}, \upsilon_1) \\ \phi_c(\xi_1, \xi_{c1}, \xi_2, \frac{1}{2}(\upsilon_{c1} + \upsilon_{c2})) \\ \phi_c(\xi_1, \xi_{c2}, \xi_2, \frac{1}{2}(\upsilon_{c1} + \upsilon_{c2})) \\ \phi_2(\xi_{c2}, \xi_2, \upsilon_2) \end{bmatrix} \tag{46}$$

and

$$\hat{\gamma}(\hat{\xi}, \hat{\upsilon}) := \begin{bmatrix} \gamma_1(\xi_1, \upsilon_1) \\ \gamma_c(\xi_{c1}, \frac{1}{2}(\upsilon_{c1} + \upsilon_{c2})) \\ \gamma_c(\xi_{c2}, \frac{1}{2}(\upsilon_{c1} + \upsilon_{c2})) \\ \gamma_2(\xi_2, \upsilon_2) \end{bmatrix} . \tag{47}$$

Note that, with $\hat{\phi}$ and $\hat{\gamma}$ as above, (12) and (13) are satisfied with

$$U := \begin{bmatrix} I_{\nu_1} & 0 & 0 \\ 0 & I_{\nu_c} & 0 \\ 0 & I_{\nu_c} & 0 \\ 0 & 0 & I_{\nu_2} \end{bmatrix} , \tag{48}$$

$$V^{\dagger} := \begin{bmatrix} I_{\mu_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2} I_{\mu_c} & \frac{1}{2} I_{\mu_c} & 0 \\ 0 & 0 & 0 & I_{\mu_2} \end{bmatrix} , \tag{49}$$

and

$$R := \begin{bmatrix} I_{\eta_1} & 0 & 0 \\ 0 & I_{\eta_c} & 0 \\ 0 & I_{\eta_c} & 0 \\ 0 & 0 & I_{\eta_2} \end{bmatrix} . \tag{50}$$

Thus, Theorem 2 implies that $\hat{\mathcal{S}}$ is an extension of $\mathcal{S}$.

Furthermore, $\hat{\mathcal{S}}$ is composed of two disjoint subsystems: $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, where $\hat{\mathcal{S}}_1$ has the state, input, and output vectors

$$\hat{\xi}_1 = \begin{bmatrix} \xi_1 \\ \xi_{c1} \end{bmatrix} , \quad \hat{\upsilon}_1 = \begin{bmatrix} \upsilon_1 \\ \upsilon_{c1} \end{bmatrix} , \quad \hat{\psi}_1 = \begin{bmatrix} \psi_1 \\ \psi_{c1} \end{bmatrix} \tag{51}$$

and $\hat{\mathcal{S}}_2$, has the corresponding vectors

$$\hat{\xi}_2 = \begin{bmatrix} \xi_{c2} \\ \xi_2 \end{bmatrix} , \quad \hat{\upsilon}_2 = \begin{bmatrix} \upsilon_{c2} \\ \upsilon_2 \end{bmatrix} , \quad \hat{\psi}_2 = \begin{bmatrix} \psi_{c2} \\ \psi_2 \end{bmatrix} . \tag{52}$$

Note that there are only minimal interactions between these two subsystems. Thus, a local controller $\hat{\mathcal{G}}_i$ of the form

$$\dot{\hat{\zeta}}_i(t) = \hat{\chi}_i(\hat{\zeta}_i(t), \hat{\lambda}_i(t)) \tag{53}$$

$$\hat{\omega}_i(t) = \hat{\kappa}_i(\hat{\zeta}_i(t), \hat{\lambda}_i(t)) \tag{54}$$

can be designed for $\hat{\mathcal{S}}_i$, for each $i = 1, 2$. These controllers are designed such that the overall decentralized controller $\hat{\mathcal{G}}$, described by (17) with

$$\hat{\chi}(\hat{\zeta}, \hat{\lambda}) = \begin{bmatrix} \hat{\chi}_1(\hat{\zeta}_1, \hat{\lambda}_1) \\ \hat{\chi}_2(\hat{\zeta}_2, \hat{\lambda}_2) \end{bmatrix} \tag{55}$$

and

$$\hat{\kappa}(\hat{\zeta}, \hat{\lambda}) = \begin{bmatrix} \hat{\kappa}_1(\hat{\zeta}_1, \hat{\lambda}_1) \\ \hat{\kappa}_2(\hat{\zeta}_2, \hat{\lambda}_2) \end{bmatrix}, \tag{56}$$

where

$$\hat{\zeta} := \begin{bmatrix} \hat{\zeta}_1 \\ \hat{\zeta}_2 \end{bmatrix} \quad \text{and} \quad \hat{\lambda} := \begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{bmatrix}, \tag{57}$$

stabilizes the overall expanded system $\hat{\mathcal{S}}$. We can now contract this controller to a controller $\mathcal{G}$ of the form (16), where the functions χ and κ are obtained using (34) and (35) as

$$\chi(\hat{\zeta}, \lambda) = \begin{bmatrix} \hat{\chi}_1(\hat{\zeta}_1, \hat{\lambda}_1) \\ \hat{\chi}_2(\hat{\zeta}_2, \hat{\lambda}_2) \end{bmatrix} \tag{58}$$

and

$$\kappa(\hat{\zeta}, \lambda) = \begin{bmatrix} \hat{\kappa}_{11}(\hat{\zeta}_1, \hat{\lambda}_1) \\ \frac{1}{2}\hat{\kappa}_{1c}(\hat{\zeta}_1, \hat{\lambda}_1) + \frac{1}{2}\hat{\kappa}_{2c}(\hat{\zeta}_2, \hat{\lambda}_2) \\ \hat{\kappa}_{22}(\hat{\zeta}_2, \hat{\lambda}_2) \end{bmatrix} \tag{59}$$

where we decomposed $\hat{\kappa}_1$ and $\hat{\kappa}_2$ as

$$\hat{\kappa}_1(\hat{\zeta}_1, \hat{\lambda}_1) = \begin{bmatrix} \hat{\kappa}_{11}(\hat{\zeta}_1, \hat{\lambda}_1) \\ \hat{\kappa}_{1c}(\hat{\zeta}_1, \hat{\lambda}_1) \end{bmatrix} \tag{60}$$

and

$$\hat{\kappa}_2(\hat{\zeta}_2, \hat{\lambda}_2) = \begin{bmatrix} \hat{\kappa}_{2c}(\hat{\zeta}_2, \hat{\lambda}_2) \\ \hat{\kappa}_{22}(\hat{\zeta}_2, \hat{\lambda}_2) \end{bmatrix}. \tag{61}$$

Furthermore, here,

$$\hat{\lambda}_1 := \begin{bmatrix} \psi_1 - \rho_1 \\ \psi_c - \rho_c \end{bmatrix} \text{ and } \hat{\lambda}_2 := \begin{bmatrix} \psi_c - \rho_c \\ \psi_2 - \rho_2 \end{bmatrix}, \tag{62}$$

where $\rho := \begin{bmatrix} \rho_1^T & \rho_c^T & \rho_2^T \end{bmatrix}^T$ is the reference input for $\mathcal{S}$. Then, by Corollary 2, $\hat{\mathcal{G}}$ is contractible to $\mathcal{G}$, and hence, by Theorem 6, $\mathcal{G}$ stabilizes $\mathcal{S}$. The implementation of the contracted controller $\mathcal{G}$ is illustrated in Fig. 1, where the overlapping decentralized structure of the controller is apparent.

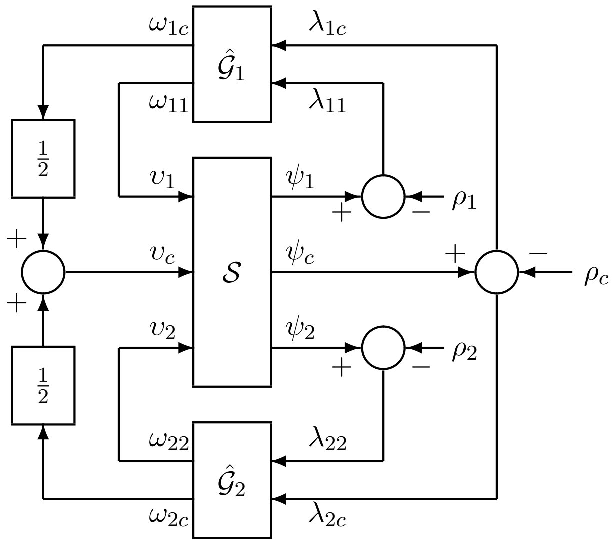

Fig. 1. Controller implementation.

6 Conclusions

Principles of inclusion and extension have been defined for NLTI systems. It has been shown that if an NLTI system is included by another, then stability of the latter implies stability of the former. Contractibility of NLTI dynamic controllers has also been discussed. Dynamic NLTI controller design using the principle of extension is then presented. This appraoch produces a controller with an overlapping decentralized structure, as shown in Fig. 1, which may be desirable in many applications. Furthermore, the proposed approach also assures the stability of the original closed-loop system.

Although, for brevity, only the case of two overlapping subsystems has been considered, the results can be extended to many other different cases. Such cases include a number of subsystems with a common overlapping part and a string of a number of overlapping subsystems. Furthermore, the present approach can also be extended to nonlinear time-varying systems and infinite-dimensional nonlinear systems, such as nonlinear time-delay systems.

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The author has no conflicts of interest to declare that are relevant to the content of this article.