On Mathematical Models of a Finance System

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Abstract: - This paper opens new possibilities for application of chaos theory in the financial industry, namely analyzing solutions of systems of ordinary differential equations using the Lyapunov exponent and Kaplan-York dimensions. Using mathematical tools, including two-dimensional and three-dimensional attractor projections, a three-dimensional financial model constructed using ordinary differential equations is analyzed in detail, and conclusions are drawn about the chaotic behavior of the solutions of the system. This paper considers both a financial chaotic system proposed by Gao and Ma in 2009 and its modified analog. The 2D and 3D images of the attractor are provided.

Key-Words: - chaos, attractor, Lyapunov exponents, Kaplan-York dimension, chaotic solutions, 2D subspace, 3D image.

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1 Introduction

Chaos theory has established itself as an innovative approach to analyzing deterministic dynamical systems. Compared to neoclassical financial theory, which works under normal conditions and where reality is oversimplified, chaos theory can more successfully explain the complexity of the financial world. It deals with non-linear processes that are difficult to predict, control, and manage, [1]. As chaos is not yet fully understood, it is difficult to define it clearly. At the initial stage of its development, chaos theory was defined as irregular motion, which was described using deterministic equations, [2]. In turn, the modern concept of chaos theory is based on the butterfly effect, where complex systems are very dependent on initial conditions, and minimal changes can lead to unexpected consequences. One of the earliest practical implementations of chaos theory was presented in [3]. Based on efforts begun in 1961, this work attempted to predict the weather by using systems of equations with 12 variables. Nowadays, new opportunities for applying chaos theory are opening up, for example, in the financial industry. Much research involves modeling financial chaotic systems [4] and methods for managing them [5]. In 2009, a new financial chaotic system was developed, described using Hopf bifurcation, [6]. In our paper, we consider the system described in [6] and its modified version. We use Lyapunov exponents and Kaplan-York dimensionality to analyze the degree of chaoticity and the degree of complexity, respectively. The aim of this paper is to facilitate the understanding of the system solutions and Lyapunov dynamics by means of a geometric approach.

2 Materials and Methods

Our consideration is mostly geometrical. We offer projections onto suitable subspaces in both two-dimensional and three-dimensional formats. Our primary objective is to employ both 2D and 3D projections of the attractor within various subspaces. This approach enables the creation of solution graphs for the systems and facilitates the illustration of the Lyapunov dynamics inherent in the studied systems. In the article for Lyapunov exponents, Wolfram Mathematica program “Lynch-DSAM.nb” was used, [7].

3 Three-dimensional Systems

3.1 Gao-Ma System

Consider the system:

$$\begin{cases} \boldsymbol{x' = z + (y - a)x} \\ \boldsymbol{y' = 1 - by - x^2} \\ \boldsymbol{z' = -x - cz} \end{cases} \quad (1)$$

Within the three-dimensional financial model (1), the variable $\boldsymbol{x}$ signifies the interest rate, $\boldsymbol{y}$ corresponds to the degree of investment demand, and $\boldsymbol{z}$ denotes the exponential factor of prices, [4].

Additionally, the constant $a \geq 0$ signifies the household savings rate, while $b \geq 0$ represents the investment cost and and $c \geq 0$ is the elasticity of demand of commercial markets, [6]. The fluctuation of x isn't just shaped by the disparity between investment and savings; it's also structurally altered by the price, [8]. The rate of change of y depends on both the investment rate and inversely on the investment cost and interest rate. Meanwhile, the fluctuation of z is shaped by the supply-demand dynamics in commercial markets and is further influenced by inflation rates, [8].

The initial conditions are:
$\boldsymbol{x(0) = 0.5;\ y(0) = 3; z(0) = -0.4.}$

Consider $\boldsymbol{b = 0.1, a = 6, c = 1}$.

The projections of the attractor on 2D subspace are depicted in Figure 1, Figure 2 and Figure 3. The 3D image of the attractors depicted in Figure 4.

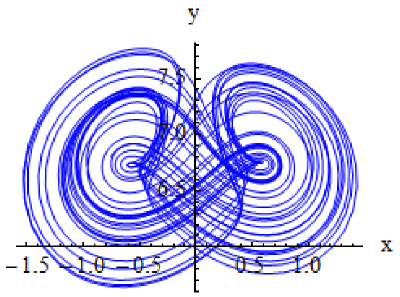

Fig. 1: The projection of the attractor on 2D subspace $(\mathrm{x}(t), y(t))$

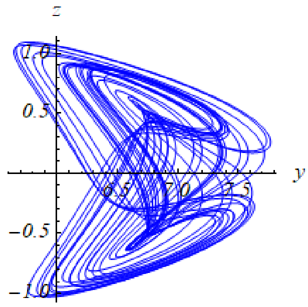

Fig. 2: The projection of the attractor on 2D subspace $(\mathbf{y}(\boldsymbol{t}), \mathbf{z}(\boldsymbol{t}))$

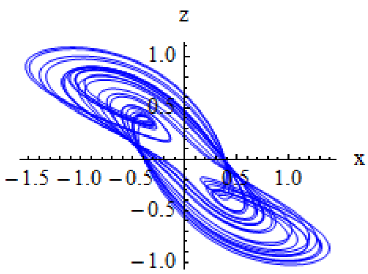

Fig. 3: The projection of the attractor on 2D subspace $(\mathrm{x}(t), \mathrm{z}(t))$

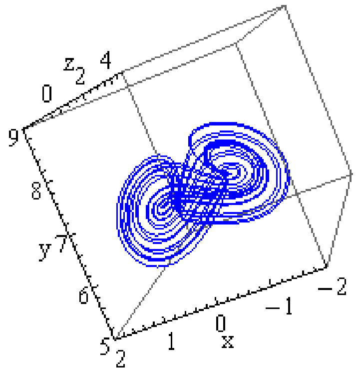

Fig. 4: The 3D image of the attractor

The graph of solutions is shown in Figure 5.

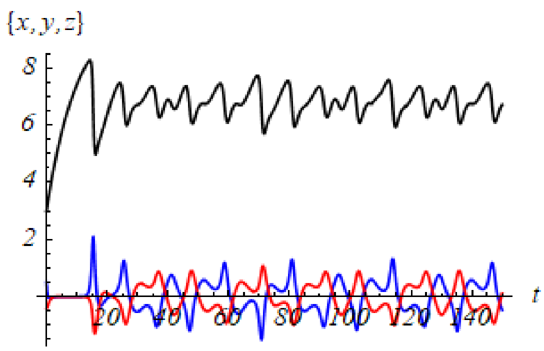

Fig. 5: Solutions $\left(\boldsymbol{x(t), y(t), z(t)}\right)$ of the system (1)

Lyapunov exponents after 3000 steps were $\mathbf{LE_1 = 0.097}; \boldsymbol{LE_2 = 0};\ \boldsymbol{LE_3 = -0.395}$. The largest Lyapunov exponent was greater than 0 as shown in Figure 6.

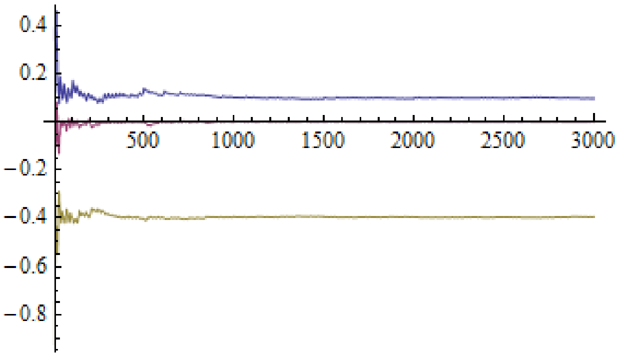

Fig. 6: $\mathbf{LE_1 = 0.097}; \boldsymbol{LE_2 = 0};\ \boldsymbol{LE_3 = -0.395}$

The identification of chaos can be accomplished through methods such as the Lyapunov exponent (LE), [9]. We have $(+; 0; -)$ then the system (1) has chaotic solutions, [10].

The Kaplan-Yorke dimension of the system (1) is:

$$D_{KY} = 2 + \frac{0.097 + 0}{|-0.395|} = 2.24$$

The value of $\boldsymbol{D_{KY} > 2}$ signifies the complexity of the model, [11].

3.2 The Modified Gao-Ma System

The parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the initial conditions are the same as in the system (1). In the second equation instead of $\boldsymbol{x^2}$, we consider $\boldsymbol{x^6}$. The new system is:

$$\begin{cases} \boldsymbol{x' = z + (y - a)x} \\ \boldsymbol{y' = 1 - by - x^6} \\ \boldsymbol{z' = -x - cz} \end{cases} \tag{2}$$

The projections of the attractor on 2D subspace are depicted in Figure 7, Figure 8 and Figure 9. The 3D image of the attractor is depicted in Figure 10. The graph of solutions is shown in Figure 11.

Lyapunov exponents after 3000 steps are shown in Figure 12.

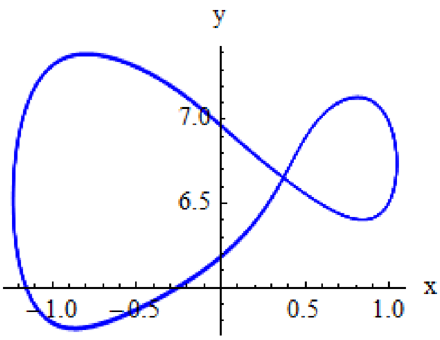

Fig. 7: The projection of the attractor on 2D subspace $(\mathbf{x}(\boldsymbol{t}), \mathbf{y}(\boldsymbol{t}))$

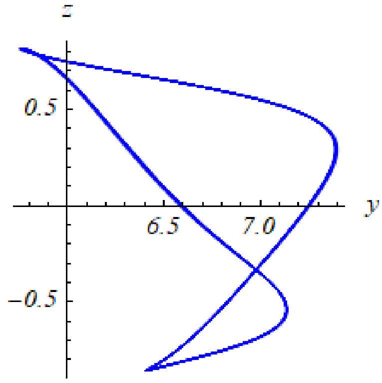

Fig. 8: The projection of the attractor on 2D subspace $(\mathbf{y}(\boldsymbol{t}), \mathbf{z}(\boldsymbol{t}))$

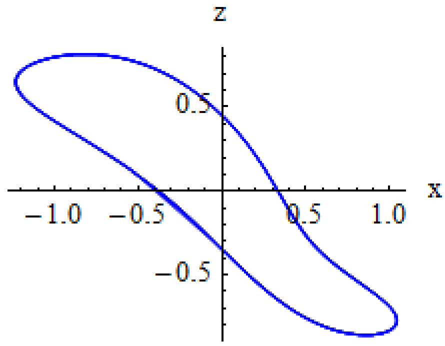

Fig. 9: The projection of the attractor on 2D subspace $(\mathrm{x}(t), \mathrm{z}(t))$

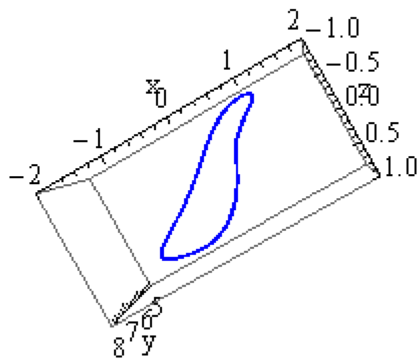

Fig. 10: The 3D image of the attractor

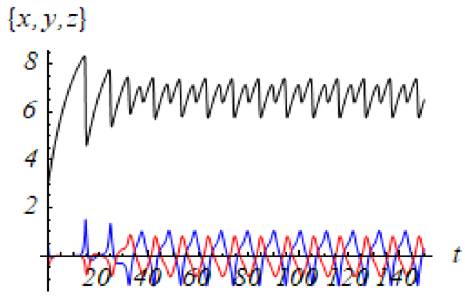

Fig. 11: Solutions $\left(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t}), \boldsymbol{z}(\boldsymbol{t})\right)$of the system (1)

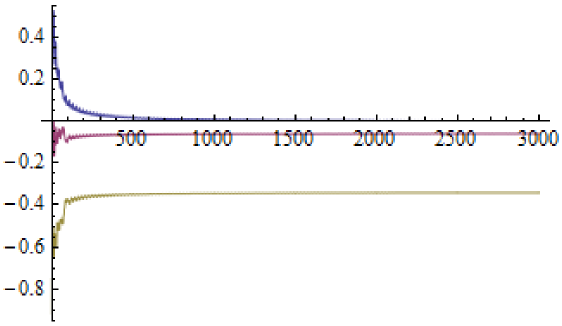

Fig. 12: $\boldsymbol{LE_1 = 0.030; LE_2 = -0.060; LE_3 = -0.340}$

$$D_{KY} = 2 + \frac{0.030 - 0.060}{|-0.340|} = 1.92.$$

The value of $D_{KY} < 2$ signifies the periodicity of the model.

Consider the system (2) and the new parameters are $\boldsymbol{a = 0.785, b = 0.013, c = 0.05}$. The initial conditions are the same.

The projections of the attractor on 2D subspace are depicted in Figure 13, Figure 15 and Figure 16. The 3D image of the attractor is depicted in Figure 14. The graph of solutions is shown in Figure 17.

Lyapunov exponents after 3000 steps were $\boldsymbol{LE_1 = 0.085; LE_2 = -0.001; LE_3 = -0.135}$. The largest Lyapunov exponent was greater than 0 as shown in Figure 18.

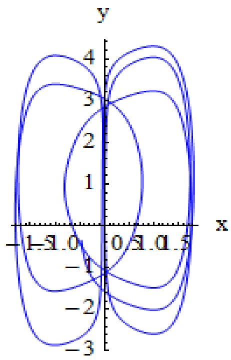

Fig. 13: The projection of the attractor on 2D subspace $\left(\mathrm{x}(t), \mathrm{y}(t)\right)$

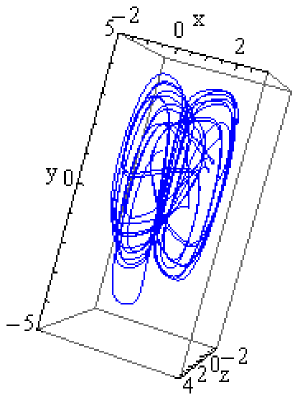

Fig. 14: The 3D image of the attractor

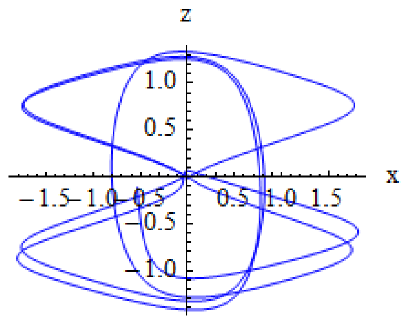

Fig. 15: The projection of the attractor on 2D subspace $(\mathrm{x}(t), \mathrm{z}(t))$

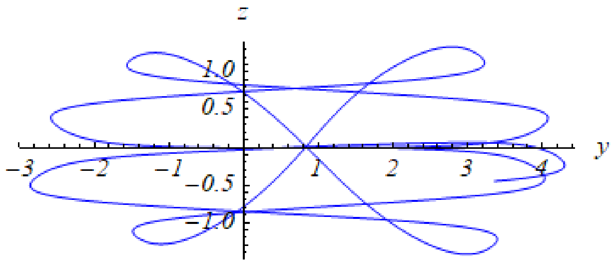

Fig. 16: The projection of the attractor on 2D subspace $(\mathrm{y}(t), \mathrm{z}(t))$

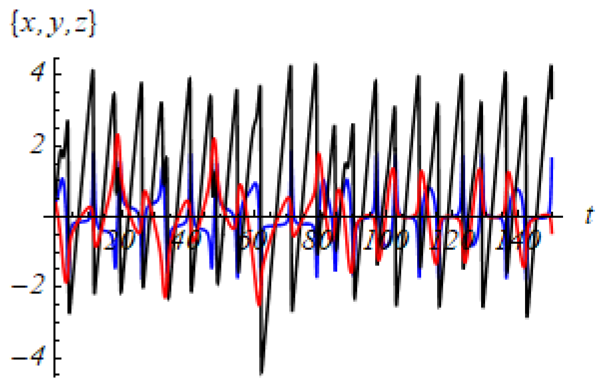

Fig. 17: Solutions $\left(x(t), y(t), z(t)\right)$of the system (1)

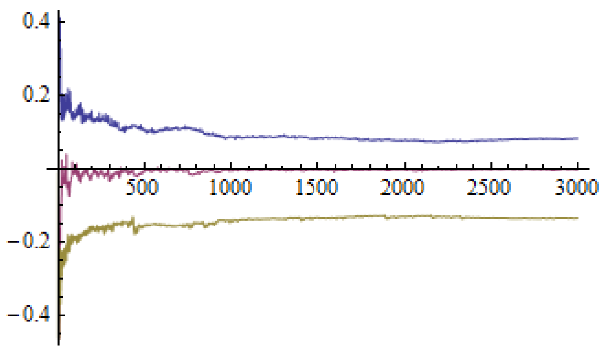

Fig. 18: $LE_1 = 0.085; LE_2 = -0.001; LE_3 = -0.135$

We have $(+; 0; -)$ then the system (1) has chaotic solutions, [10].

$$D_{KY} = 2 + \frac{0.085 - 0.001}{|-0.135|} = 2.59.$$

The value of $D_{KY} > 2$ signifies the complexity of the model, [10], [11], [12].

4 Conclusions

Applying chaos theory to finance presents new possibilities for comprehending, assessing, and envisioning the development prospects of financial markets. Conversely, efforts to appraise the future of the financial market through the lens of chaos theory extend the boundaries of its application, fostering the creation of novel tools and mathematical analysis methods.

Systems of ordinary differential equations were studied, which are interpreted as a finance model. These systems are dependent on three parameters. In this paper, three-dimensional systems were considered. We analyzed a financial chaotic system previously introduced in the literature and its modified version.

Identifying parameters, ensuring the intricate dynamics of trajectories, and navigating their responsive reliance on initial data pose significant challenges. The knowledge of the structure of the phase space and the ability to control this model is very important in the economy.

References:

[1] Igor Klioutchnikov, Mariia Sigova, Nikita Beizerov. Chaos Theory in Finance. *Procedia Computer Science*, Vol. 119, pp. 368–375, 2017. http://dx.doi.org/10.1016/j.procs.2017.11.196.

[2] Yu. G. Lavrikova, Olga Buchinskaia, Yu. G. Myslyakova, Chaos Theory: Expanding the Boundaries of Economic Research. *Alternatives*, Vol. 20, No. 1, pp. 85–100, 2023. http://dx.doi.org/10.31063/AlterEconomics/2023.20-1.5.

[3] Lorenz, E. N. Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences,* Vol. 20, No. 2, pp. 130–141, 1963. https://doi.org/10.1175/1520-0469(1963)020%3C0130:DNF%3E2.0.CO;2

[4] Sukono, Siti Hadiaty Yuningsih, Endang Rusyaman, Sundarapandian Vaidyanathan, Aceng Sambas. Investigation of chaos behavior and integral sliding mode control on financial risk model. *AIMS Mathematics*, Vol. 7, No. 10, pp. 18377–18392. https://doi.org/10.3934/math.20221012.

[5] Betty Subartini, Sukono, Sundarapandian Vaidyanathan, Aceng Sambas, Sen Zhang. Multistability in the Finance Chaotic System, Its Bifurcation Analysis and Global Chaos Synchronization via Integral Sliding Mode Control, *IAENG International Journal of Applied Mathematics*, Vol. 51, No. 4, pp. 1010-1018, 2021.

[6] Q. Gao, J. Ma. Chaos and Hopf bifurcation of a finance system. *Nonlinear Dynamics*, Vol. 58, pp. 209–216, 2009. DOI: 10.1007/s11071-009-9472-5.

[7] S. Lynch, Dynamical Systems with Applications Using Mathematica. *Springer*, 2017. DOI: 10.1007/978-0-8176-4586-1.

[8] Zhang Xuebing, Zhu Honglan, Hopf Bifurcation and Chaos of a Delayed Finance System. *Complexity*, Vol. 2019, Article ID 6715036, 2019. DOI: 10.1155/2019/6715036.

[9] Inna Samuilik, Felix Sadyrbaev, Svetlana Atslega, On mathematical models of artificial neural networks, *Proceedings of the 22nd International Scientific Conference Engineering for Rural Development*, Jelgava, Latvia. 2023. DOI: 10.22616/ERDev.2023.22.TF007.

[10] Aceng Sambas, Sundarapandian Vaidyanathan, Talal Bonny, Sen Zhang, Sukono, Yuyun Hidayat, Gugun Gundara, Mustafa Mamat, Mathematical Model and FPGA Realization of a Multi-Stable Chaotic Dynamical System with a Closed Butterfly-Like Curve of Equilibrium Points, *Applied Sciences*, Vol. 11, No. 2, Article 788, 2021. DOI: 10.3390/app11020788.

[11] Zeraoulia E., Yorke J. A. A flexible chaotic system with adjustable amplitude, largest Lyapunov exponent, and Kaplan–Yorke dimension. *Nonlinear Dynamics*, Vol. 92, pp. 1807–1818, 2018. DOI: 10.1007/s11071-018-4162-9.

[12] Ning Wang, Dan Xu, Ze Li, Quan Xu. A general configuration for nonlinear circuit employing current-controlled nonlinearity: Application in Chua's circuit. *Chaos, Solitons & Fractals*, Vol. 177, Article 114233, 2023. DOI: 10.1016/j.chaos.2023.114233.

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Conflict of Interest
The authors have no conflicts of interest to declare.