Robust Recursive Least-Squares Fixed-Point Smoother and Filter using Covariance Information in Linear Continuous-Time Stochastic Systems with Uncertainties

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Abstract: - This study develops robust recursive least-squares (RLS) fixed-point smoothing and filtering algorithms for signals in linear continuous-time stochastic systems with uncertainties. The algorithms use covariance information, such as the cross-covariance function of the signal with the observed value and the autocovariance function of the degraded signal. A finite Fourier cosine series expansion approximates these functions. Additive white Gaussian noise is present in the observation of the degraded signal. A numerical simulation compares the estimation accuracy of the proposed robust RLS filter with the robust RLS Wiener filter, showing similar mean square values (MSVs) of the filtering errors. The MSVs of the proposed robust RLS fixed-point smoother are also compared to those of the proposed robust RLS filter.

Key-Words: - Robust RLS fixed-point smoother, robust RLS filter, degraded signal, stochastic systems with uncertainties, continuous-time stochastic systems, Fourier series expansion.

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1 Introduction

Over the past two decades, researchers have extensively studied robust estimation in continuous-time stochastic systems with uncertainties, covering both linear and nonlinear scenarios. The following is one classification for robust estimation problems. (1) Norm-bounded parameter uncertainty, [1], [2], [3], [4], [5], [6]. (2) Polytope uncertainty, [7], [8], [9], [10], [11], [12]. (3) Markovian jumps in the parameters [13], [14]. (4) In the presence of both parameter uncertainty and a known input signal, [2]. (5) Systems with finite frequency specifications, [15]. (6) Uncertain nonlinear systems with multiplicative observation noise, [16]. (7) Nonlinear systems via Takagi–Sugeno (T–S) fuzzy affine dynamic models, [17], [18]. (8) Robust finite impulse response (FIR) estimators, [5], [6]. (9) Recursive least-squares (RLS) Wiener filter, [19].

The book, [20], mainly discusses identification techniques for linear discrete-time stochastic systems. It is presented a method for estimating parameters of continuous-time linear systems by using differential equations to define the input-output relationship of the system. Recently, the author developed a robust recursive least-squares RLS Wiener filter for linear continuous-time uncertain stochastic systems by estimating the system matrix for the degraded signal, [19]. The estimated system matrix elements in [19] are unreliable because of negative values of the third and seventh powers of 10 in the third and fourth-order matrices, respectively, caused by large values in higher derivatives of the autocovariance function. To address this issue, it is recommended to use an alternative approach that does not involve estimating the system matrix.

Based on the preceding discussion, this paper suggests a novel robust estimation method for continuous-time uncertain stochastic systems. The observation of the degraded signal includes additive white Gaussian noise. Instead of estimating the system matrix for the degraded signal in [19], the robust RLS fixed-point smoothing and filtering algorithms of Theorem 1 are characteristic in that they use covariance information. The estimation algorithms described in Theorem 1 utilize the cross-covariance function of the signal with the observed value, along with the autocovariance function of the degraded signal. The finite Fourier cosine series expansion approximates the cross-covariance function between the signal and the observed value, as well as the autocovariance function of the degraded signal.

Section 2 introduces the state-space model for the signal and its degraded counterpart. In the degraded state-space model, uncertain parameters are present in both the observation vector and the system matrix. Section 3 presents a robust fixed-point smoothing problem in linear least-squares estimation. Theorem 1 in Section 4 presents the robust RLS fixed-point smoothing and filtering algorithms. Section 5 explains the finite Fourier cosine series approximation of the cross-covariance function between the signal and the observed value, as well as the autocovariance function of the degraded signal. In Section 6, we compare the estimation accuracy of the proposed robust RLS filter with the robust RLS Wiener filter, [19], in the first simulation example. The mean square value (MSV) of the robust RLS filter in Theorem 1 is smaller than that of the robust RLS Wiener filter, [19], when the observation noise is white Gaussian with variance 0.1^2. The proposed robust RLS fixed-point smoother is compared with the proposed robust RLS filter in terms of estimation properties.

2 State-Space Model and its Degraded State-Space Model with Uncertainties

Consider a state-space model (1) that satisfies the observability condition in linear continuous-time stochastic systems.

$$\begin{aligned} &y(t) = z(t) + v(t), z(t) = Hx(t),\\ &\frac{dx(t)}{dt} = Ax(t) + \Gamma w(t), x(0) = c,\\ &E[v(t)v(s)] = R\delta(t-s),\\ &E[w(t)w^T(s)] = Q\delta(t-s),\\ &E[v(t)w^T(s)] = 0, E[x(0)w^T(t)] = 0,\\ &E[x(0)v(t)] = 0, 0 \le \mathrm{s},\mathrm{t} \end{aligned} \tag{1}$$

$x(t) \in R^n$ is a state vector, and $z(t)$ is a scalar signal that needs to be estimated. The input noise $w(t) \in R^l$ and the observation noise $v(t)$ are mutually uncorrelated white Gaussian noises with zero means. Γ is an $n \times l$ input matrix, and H is a $1 \times n$ observation vector. The autocovariance functions of the input noise $w(t)$ and the observation noise $v(t)$ are expressed in (1) using the Dirac delta function. This paper examines the state and observation equations with uncertain parameters in the state-space model.

$$\begin{aligned} &\breve{y}(t) = \breve{z}(t) + v(t),\\ &\breve{z}(t) = \overleftrightarrow{H}(t)\overleftrightarrow{x}(t), \overleftrightarrow{H}(t) = H + \Delta H(t),\\ &\frac{d\overleftrightarrow{x}(t)}{dt} = \overleftrightarrow{A}(t)\overleftrightarrow{x}(t) + \Gamma w(t),\\ &\overleftrightarrow{A}(t) = A + \Delta \mathrm{A}(t),\\ &\mathrm{E}[\Delta A(\mathrm{t})w^T(\mathrm{s})] = 0,\\ &E[\Delta \mathrm{H}(t)v(s)] = 0, E[\overleftrightarrow{x}(0)w^T(t)] = 0,\\ &E[\overleftrightarrow{x}(0)v(t)] = 0,\ 0 \le \mathrm{s},\mathrm{t} \end{aligned} \tag{2}$$

In equation (2), the system matrix A and observation vector H from equation (1) are substituted with the degraded versions $\overleftrightarrow{A}(t)$ and $\overleftrightarrow{H}(t)$, respectively. The matrix elements of $\Delta \mathrm{A}(t)$ and the vector components of $\Delta H(t)$ contain uncertain variables. The initial state vector $\overleftrightarrow{x}(0)$ is randomly generated and independent of input or measurement noise.

The robust RLS Wiener filter, [19], utilizes the estimates of the degraded system and observation matrices. Estimating the matrices in linear continuous-time stochastic systems is more challenging than in linear discrete-time stochastic systems. Section 3 introduces linear least-squares estimation using covariance information without explicitly identifying the degraded system matrix and measurement vector.

3 Robust Least-Squares Fixed-Point Smoothing Problem

Let the fixed-point smoothing estimate $\hat{z}(t,T)$ of the signal $z(t)$ be given by

$$\hat{z}(t,T) = \int_0^T h(t,s,T)\breve{y}(s)ds \tag{3}$$

as a linear transformation of the observed value $\breve{y}(s), 0 \le s \le T$. Here, $h(t,s,T)$ represents an impulse response function. Let us consider minimizing the mean-square value:

$$J = E[(z(t) - \hat{z}(t,T))^2] \tag{4}$$

of the fixed-point smoothing error $z(t) - \hat{z}(t,T)$. The fixed-point smoothing estimate $\hat{z}(t,T)$ that minimizes the cost function J satisfies the relationship:

$$z(t) - \hat{z}(t,T) \perp \breve{y}(s), 0 \le s,t \le T, \tag{5}$$

from the orthogonal projection lemma, [21]. The optimal impulse response function satisfies the Wiener-Hopf integral equation:

$$\begin{aligned} &E[z(t)\breve{y}^T(s)]\\ &= \int_0^T h(t,\tau,T)E[\breve{y}(\tau)\breve{y}^T(s)]ds,\\ &0 \le s,t \le T. \end{aligned} \tag{6}$$

Substituting the degraded observation equation in (2) into (6), (6) is transformed into:

$$h(t,s,T)R = K_{z\breve{y}}(t,s) - \int_0^T h(t,\tau,T)K_{\breve{z}}(\tau,s)d\tau, \quad (7)$$
$$K_{z\breve{y}}(t,s) = E[z(t)\breve{y}^T(s)],$$
$$K_{\breve{z}}(t,s) = E[\breve{z}(t)\breve{z}^T(s)].$$

$K_{z\breve{y}}(t,s)$ is the cross-covariance function between the signal $z(t)$ and the observed value $\breve{y}(s)$. Assume that the cross-covariance function $K_{z\breve{y}}(t,s)$ is expressed as:

$$K_{z\breve{y}}(t,s) = \alpha(t)\beta^T(s), 0 \le s \le t. \quad (8)$$

$K_{\breve{z}}(t,s)$ is the autocovariance function of the degraded signal $\breve{z}(t)$, expressed by:

$$K_{\breve{z}}(t,s) = \begin{cases} \breve{A}(t)\breve{B}^T(s), 0 \le s \le t, \\ \breve{B}(t)\breve{A}^T(s), 0 \le t \le s. \end{cases} \quad (9)$$

In wide-sense stationary stochastic systems, $K_{z\breve{y}}(t,s)$ and $K_{\breve{z}}(t,s)$ are represented as $K_{z\breve{y}}(\tau)$ and $K_{\breve{z}}(\tau)$ respectively, with $\tau = t - s$. $K_{\breve{z}}(\tau)$ is an even function for every τ in its domain. From (7), Section 4 introduces Theorem 1 and proposes the robust RLS fixed-point smoothing and filtering algorithms using the covariance information provided by (8) and (9).

4 Robust RLS Fixed-Point Smoothing and Filtering Algorithms

Theorem 1 proposes the robust RLS fixed-point smoothing and filtering algorithms for the signal $z(t)$ using the covariance information $K_{z\breve{y}}(t,s)$ and $K_{\breve{z}}(t,s)$ defined by (8) and (9).

Theorem 1 Let the state-space model for the signal $z(t)$ be given by (1). Let the state-space model for the degraded signal $\breve{z}(t)$ be given by (2). Let the cross-covariance function $K_{z\breve{y}}(t,s)$ of the signal $z(t)$ with the observed value $\breve{y}(s)$ be represented as (8). Let the autocovariance function $K_{\breve{z}}(t,s)$ of the degraded signal $\breve{z}(t)$ be expressed as (9). Then, the robust RLS fixed-point smoothing and filtering algorithms for the signal $z(t)$ from the degraded observation $\breve{y}(t)$ in (2) using the covariance information consist of (10)-(19).

Fixed-point smoothing estimate of the signal $z(t)$ at the fixed point t: $\hat{z}(t,T)$

$$\frac{\partial \hat{z}(t,T)}{\partial T} = h(t,T,T)(\breve{y}(T) - \breve{A}(T)f(T)),$$
$$\hat{z}(t,T)|_{T=t} = \hat{z}(t,t) \quad (10)$$

Smoother gain: $h(t,T,T)$

$$h(t,T,T) = (K_{z\breve{y}}(t,T) - P(t,T)\breve{A}^T(T))/R \quad (11)$$

$$\frac{\partial P(t,T)}{\partial T} = h(t,T,T)(\breve{B}(T) - \breve{A}(T)S(T)),$$
$$P(t,t) = \alpha(t)r(t) \quad (12)$$

Filtering estimate of the signal $z(t)$: $\hat{z}(t,t)$

$$\hat{z}(t,t) = \alpha(t)e(t) \quad (13)$$

$$\frac{de(t)}{dt} = J(t,t)(\breve{y}(t) - \breve{A}(t)f(t)), \ e(0)=0 \quad (14)$$

$$\frac{df(t)}{dt} = L(t,t)(\breve{y}(t) - \breve{A}(t)f(t)),$$
$$f(0) = 0 \quad (15)$$

$$J(t,t) = (\beta^T(t) - r(t)\breve{A}^T(t))/R \quad (16)$$

$$L(t,t) = (\breve{B}^T(t) - S(t)\breve{A}^T(t))/R \quad (17)$$

$$\frac{dr(t)}{dT} = J(t,t)(\breve{B}(t) - \breve{A}(t)S(t)), \ r(0) = 0 \quad (18)$$

Autovariance function of the filtering estimate of the degraded signal $\breve{z}(t)$: $S(t)$

$$\frac{dS(t)}{dt} = L(t,t)(\breve{B}(t) - \breve{A}(t)S(t)),$$
$$S(0) = 0 \quad (19)$$

In (11), $K_{z\breve{y}}(t,T)$ represents the cross-covariance function of the signal $z(t)$ with the observed value $\breve{y}(T)$, $0 \le t \le T$.

Theorem 1 is derived based on the invariant imbedding method for integral equations, [22], [23]. Proof of Theorem 1 is deferred to the Appendix.

The robust RLS fixed-point smoother and filter are designed by minimizing the cost function (4) in the linear least-squares sense. In the combined Kalman filter and neural network estimation method, [24], [25], [26], [27], [28], the neural network weights are computed iteratively using a large amount of high-quality training data.

5 Finite Fourier Series Approximation of Autocovariance Function of Degraded Signal and Cross-Covariance Function of Signal with Observed Value

The autocovariance function $K_{\breve{z}}(t,s)$ of the degraded signal $\breve{z}(t)$ is represented as $K_{\breve{z}}(\tau)$ in wide-sense stationary stochastic systems, with $\tau = t - s$. $K_{\breve{z}}(\tau)$ is an even function for every τ in its domain. Let $K_{\breve{z}}(\tau)$ be approximated by the finite

Fourier cosine series expansion given in (20). $\widehat{K}_{\breve{z}}(\tau)$ represents a function that approximates $K_{\breve{z}}(\tau)$ using $N+1$ terms.

$$\widehat{K}_{\breve{z}}(\tau) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\omega_0\tau), \quad \omega_0 = \frac{2\pi}{T}, -\frac{T}{2} \le \tau \le \frac{T}{2} \qquad (20)$$

Here T represents the fundamental period of $K_{\breve{z}}(\tau)$. The finite Fourier cosine coefficients are calculated by:

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} K_{\breve{z}}(\tau)\cos(n\omega_0\tau)\,d\tau, \quad n = 0, 1, 2, \cdots, N. \qquad (21)$$

After comparing (9) and (20), we can represent the vector components of $\breve{A}(t)$ and $\breve{B}(t)$ as follows:

$\breve{A}(t) = [\breve{A}_1(t) \quad \breve{A}_2(t) \quad \breve{A}_3(t) \quad \cdots \quad \breve{A}_{2N+1}(t)]$,
$\breve{B}(t) = [\breve{B}_1(t) \quad \breve{B}_2(t) \quad \breve{B}_3(t) \quad \cdots \quad \breve{B}_{2N+1}(t)]$,
$\breve{A}_1(t) = \frac{a_0}{2}$,
$\breve{A}_i(t) = a_{i-1}\cos(\frac{2\pi(i-1)t}{T})$, $i = 2, 3, \cdots, N+1$,
$\breve{A}_i(t) = a_{i-(N+1)}\sin(\frac{2\pi(i-(N+1))t}{T})$, $i = N+2, N+3, \cdots, 2N+1$,
$\breve{B}_1(t) = 1$,
$\breve{B}_i(t) = \cos(\frac{2\pi(i-1)t}{T})$, $i = 2, 3, \cdots, N+1$,
$\breve{B}_i(t) = \sin(\frac{2\pi(i-(N+1))t}{T})$,
$i = N+2, N+3, \cdots, 2N+1$.

The cross-covariance function $K_{z\breve{y}}(t,s)$ of $z(t)$ with $\breve{y}(s)$ is given by (8). Let $K_{z\breve{y}}(\tau)$ be approximated by the finite Fourier cosine series expansion given in (22). $\widehat{K}_{z\breve{y}}(\tau)$ represents a function that approximates $K_{z\breve{y}}(\tau)$ using $N+1$ terms.

$$\widehat{K}_{z\breve{y}}(\tau) \approx \frac{\Xi_0}{2} + \sum_{n=1}^{N} \Xi_n \cos(n\omega_0\tau), \quad \omega_0 = \frac{2\pi}{T}, 0 \le \tau \le \frac{T}{2}. \qquad (22)$$

Here, T represents the fundamental period of $K_{z\breve{y}}(\tau)$. The finite Fourier cosine coefficients are calculated by:

$$\Xi_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} K_{z\breve{y}}(\tau)\cos(n\omega_0\tau)\,d\tau, \quad n = 0,1,2,\cdots,N. \qquad (23)$$

After comparing (8) and (22), we can represent the vector components of $\alpha(t)$ and $\beta(t)$ as follows:

$\alpha(t) = [\alpha_1(t) \quad \alpha_2(t) \quad \alpha_3(t) \quad \cdots \quad \alpha_{2N+1}(t)]$,
$\beta(t) = [\beta_1(t) \quad \beta_2(t) \quad \beta_3(t) \quad \cdots \quad \beta_{2N+1}(t)]$,
$\alpha_1(t) = \frac{\Xi_0}{2}$,
$\alpha_i(t) = \Xi_{i-1}\cos(\frac{2\pi(i-1)t}{T})$, $i = 2, 3, \cdots, N+1$,
$\alpha_i(t) = \Xi_{i-(N+1)}\sin(\frac{2\pi(i-(N+1))t}{T})$,
$i = N+2, N+3, \cdots, 2N+1$,
$\beta_1(t) = 1$,
$\beta_i(t) = \cos(\frac{2\pi(i-1)t}{T})$, $i = 2, 3, \cdots, N+1$,
$\beta_i(t) = \sin\left(\frac{2\pi(i-(N+1))t}{T}\right)$,
$i = N+2, N+3, \cdots, 2N+1$.

By substituting the functions $\breve{A}(t)$, $\breve{B}(t)$, $\alpha(t)$, $\beta(t)$, and the values of $K_{z\breve{y}}(t,T)$ into the robust RLS fixed-point smoothing and filtering algorithms of Theorem 1, we can recursively compute the fixed-point and filtering estimates of the signal $z(t)$.

6 Numerical Simulation Examples

Example 1

Let the observation equation for the signal $z(t)$ and the state differential equations for $x(t)$ be given by

$$\begin{aligned}
&y(t) = z(t) + v(t), z(t) = Hx(t),\\
&H = [1 \quad 0],\\
&\frac{dx(t)}{dt} = Ax(t) + \Gamma w(t),\\
&x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix},\\
&A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, a_1 = 4, a_2 = 3,\\
&\Gamma = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\\
&E[v(t)v(s)] = R\delta(t-s),\\
&E[w(t)w(s)] = Q\delta(t-s), Q = 1,\\
&E[v(t)w(s)] = 0, E[x(0)w(t)] = 0,\\
&E[x(0)v(t)] = 0.
\end{aligned} \qquad (24)$$

Let the observation equation for the degraded signal $\breve{z}(t)$, and the state differential equations for the degraded state $\overleftrightarrow{x}(t)$ be given by:

$$\begin{aligned}
&\breve{y}(t) = \breve{z}(t) + v(t), \breve{z}(t) = \overleftrightarrow{H}(t)\overleftrightarrow{x}(t),\\
&\frac{d\overleftrightarrow{x}(t)}{dt} = \overleftrightarrow{A}(t)\overleftrightarrow{x}(t) + \Gamma w(t),\\
&\overleftrightarrow{A}(t) = A + \Delta A(t), \overleftrightarrow{H}(t) = H + \Delta H(t),\\
&\Delta A(t) = \begin{bmatrix} 0 & 0 \\ -0.1 * rand & -0.1 * rand \end{bmatrix},\\
&\Delta H(t) = [0.1 \quad 0],\\
&E[\overleftrightarrow{x}(0)w(t)] = 0, E[\overleftrightarrow{x}(0)v(t)] = 0.
\end{aligned} \qquad (25)$$

$\Delta A(t)$ represents an uncertain matrix that is additional to the system matrix A. "$rand$" represents a scalar random number from a uniform distribution in (0,1). The value of N for the finite Fourier cosine series approximations in (20) and (22) is 15 in this simulation. Figure 1 illustrates the autocovariance function $K_{\breve{z}}(\tau)$ of the degraded signal $\breve{z}(t)$ vs. τ, $0 \le \tau \le \frac{T}{2}$, $T = 8$. The MSV of the finite Fourier cosine series approximation errors for $K_{\breve{z}}(i\Delta)$ is evaluated as $\frac{1}{4001}\sum_{i=0}^{4000}(K_{\breve{z}}(i\Delta) - \widehat{K}_{\breve{z}}(i\Delta))^2$ $= 1.038245914754162 \times 10^{-7}$, $\Delta = 0.001$.

Figure 2 illustrates the cross-covariance function $K_{z\breve{y}}(\tau)$ of the signal $z(t)$ with the observed value $\breve{y}(t)$ vs. τ, $0 \le \tau \le \frac{T}{2}$. Here, the fundamental period of $K_{\breve{z}}(\tau)$ and $K_{z\breve{y}}(\tau)$ is $T = 8$. The MSV of the finite Fourier cosine series approximation errors for $K_{z\breve{y}}(i\Delta)$ is evaluated as $\frac{1}{4001}\sum_{i=0}^{4000}(K_{z\breve{y}}(i\Delta) - \widehat{K}_{z\breve{y}}(i\Delta))^2$ $=8.103093306645037 \times 10^{-8}$. From the MSVs, the finite Fourier cosine series expansions accurately approximate $K_{\breve{z}}(i\Delta)$ and $K_{z\breve{y}}(i\Delta)$, $i = 0, \cdots, 4000$. Here, the Midpoint Rule calculates the numerical integration of (21) and (23) for the finite Fourier cosine series coefficients a_n and Ξ_n, $n = 0, 1, 2, \cdots, N$, with subintervals $[0 + kh, 0 + (k+1)h] \subset [0,4], h = \frac{4}{4000}, k = 0, 1, \cdots 3999$. By substituting the autocovariance information $\breve{A}(t)$ and $\breve{B}(t)$, the cross-covariance information $\alpha(t)$ and $\beta(t)$, and the values of $K_{z\breve{y}}(t, T)$ into the robust RLS fixed-point smoothing and filtering algorithms of Theorem 1, the fixed-point smoothing and filtering estimates are computed recursively.

Figure 3 illustrates the signal $z(t)$ and its filtering estimate $\hat{z}(t, t)$ vs t for the white Gaussian observation noise$N(0, 0.1^2)$.

Figure 4 illustrates the signal $z(t)$ and its filtering estimate $\hat{z}(t, t)$ vs t for the white Gaussian observation noise $N(0, 0.3^2)$. From Figure 3 and Figure 4, the filtering estimate for $N(0, 0.1^2)$ is closer to the signal process than for $N(0, 0.3^2)$.

Figure 5 illustrates the signal $z(t)$ and its fixed-point smoothing estimate $\hat{z}(t, t + 0.005)$ vs t for the white Gaussian observation noise $N(0, 0.1^2)$. Figure 3 and Figure 5 show that the fixed-point smoothing and filtering estimates have nearly identical waveforms. Table 1 shows the MSVs of filtering errors $z(t) - \hat{z}(t, t)$ by the robust RLS filter in Theorem 1 and the robust RLS Wiener filter [19], and those of fixed-point smoothing errors $z(t) - \hat{z}(t, t + 0.005)$ by the robust RLS fixed-point smoother in Theorem 1 for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. The MSV by the robust RLS filter in Theorem 1 is smaller than that by the robust RLS Wiener filter, [19], for the white Gaussian observation noise $N(0, 0.1^2)$. The MSV of the filtering errors by the robust RLS filter in Theorem 1 is almost the same as that of the fixed-point smoothing errors by the robust RLS fixed-point smoother in Theorem 1 for each white Gaussian observation noise.

Figure 6 illustrates the MSVs of the filtering and fixed-point smoothing errors by the robust RLS filter and the robust RLS fixed-point smoother in Theorem 1 vs. Lag for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. Here, the MSV for the filtering errors is evaluated by $\frac{1}{2500}\sum_{i=1}^{2500}(z(i\Delta) - \hat{z}(i\Delta, i\Delta))^2$. The evaluation of the MSV for the fixed-point smoothing errors is carried out by $\frac{1}{2500}\sum_{i=1}^{2500}(z(i\Delta) - \hat{z}(i\Delta, i\Delta + Lag))^2, 0.001 \le Lag \le 0.005$. From Figure 6, for the white Gaussian observation noise $N(0, 0.5^2)$, the MSV of the fixed-point smoothing errors $z(t) - \hat{z}(t, t + 0.002)$ is slightly smaller than that of the filtering errors. For the white Gaussian observation noises $N(0, 0.1^2)$ and $N(0, 0.3^2)$, the MSVs of the filtering errors are almost the same as those of the fixed-point smoothing errors $z(t) - \hat{z}(t, t + Lag)$ for $0.001 \le Lag \le 0.005$.

In the simulation example, 1,984 differential equations run simultaneously for each filtering estimation update. In updating the fixed-point smoothing estimate, 2,016 differential equations are computed recursively.

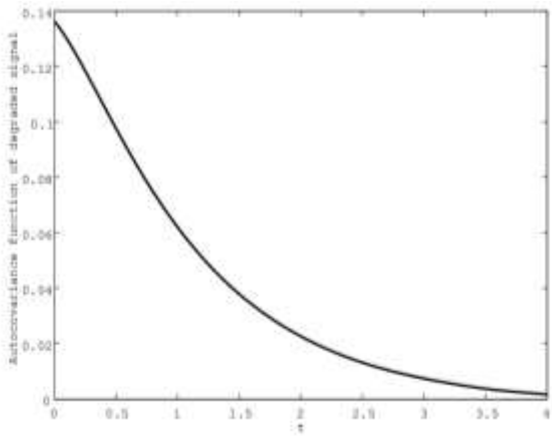

Fig. 1: Autocovariance function $K_{\breve{z}}(\tau)$ of the degraded signal $\breve{z}(t)$ vs. τ

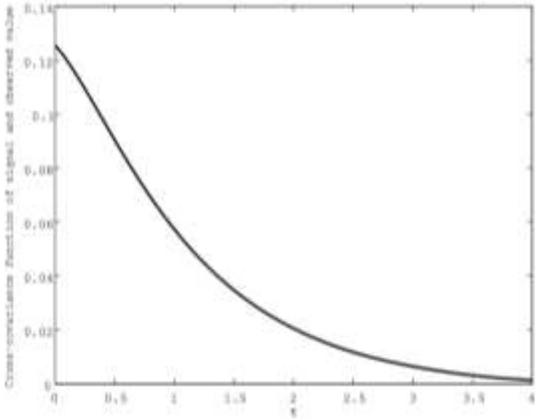

Fig. 2: Cross-covariance function $K_{z\tilde{y}}(\tau)$ of the signal $z(t)$ with the observed value $\tilde{y}(t)$ vs. τ

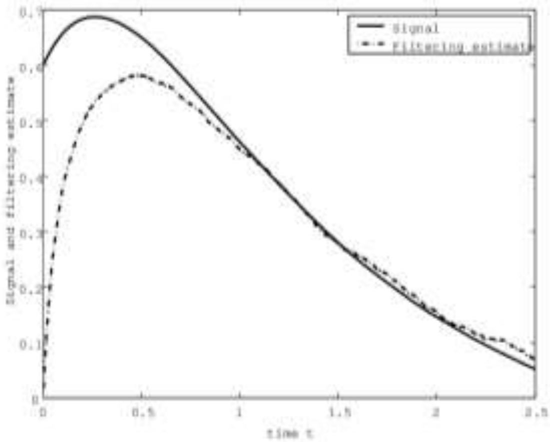

Fig. 3: Signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the white Gaussian observation noise $N(0,0.1^2)$

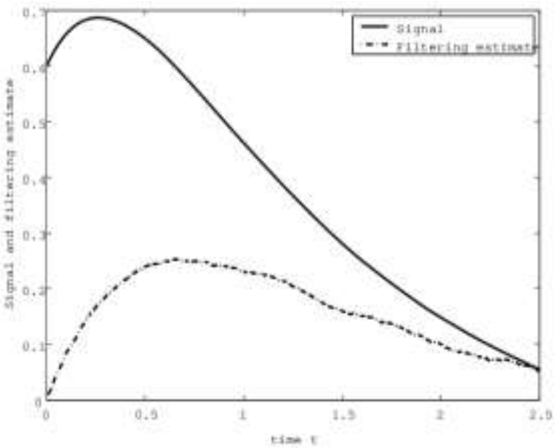

Fig. 4: Signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the white Gaussian observation noise $N(0,0.3^2)$

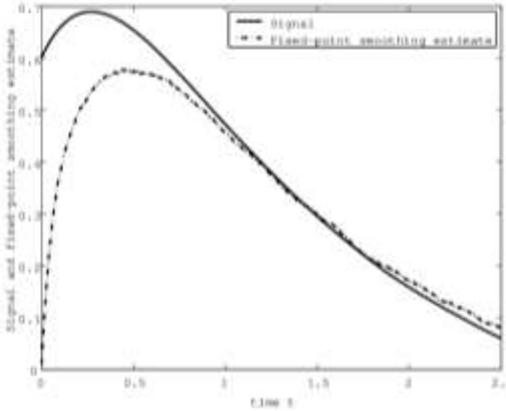

Fig. 5: Signal $z(t)$ and its fixed-point smoothing estimate $\hat{z}(t,t+0.005)$ vs. t for the white Gaussian observation noise $N(0,0.1^2)$

Table 1. MSVs of filtering errors $z(t)-\hat{z}(t,t)$ by the robust RLS filter in Theorem 1 and the robust RLS Wiener filter [19], and those of fixed-point smoothing errors $z(t)-\hat{z}(t,t+0.005)$ by the robust RLS fixed-point smoother in Theorem 1 for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, and $N(0,0.5^2)$.

White Gaussian observation noise	MSV of $z(t)-\hat{z}(t,t)$ by filter in [19]	MSV of $z(t)-\hat{z}(t,t)$ by filter in Theorem 1	MSV of $z(t)-\hat{z}(t,t+0.005)$ by fixed-point smoother in Theorem 1
$N(0,0.1^2)$	6.707662×10^{-2}	1.131457×10^{-2}	1.035210×10^{-2}
$N(0,0.3^2)$	4.526449×10^{-2}	8.441589×10^{-2}	8.337765×10^{-2}
$N(0,5^2)$	8.558077×10^{-2}	1.350774×10^{-1}	1.344245×10^{-1}

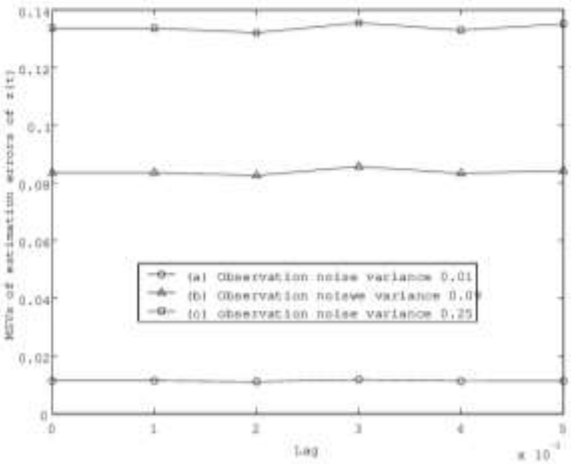

Fig. 6: MSVs of the filtering and fixed-point smoothing errors by the robust RLS filter and the robust RLS fixed-point smoother in Theorem 1 vs. Lag for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, and $N(0,0.5^2)$

Example 2

Let us consider the second-order mass-spring system driven by zero-mean white Gaussian noise [29], [30].

$$\begin{aligned}
&y(t) = z(t) + v(t), z(t) = Hx(t),\\
&H = [1 \quad 0],\\
&\frac{dx(t)}{dt} = Ax(t) + \Gamma w(t),\\
&x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix},\\
&A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \omega_n = \sqrt{3}, \qquad (26)\\
&\zeta = \frac{2}{\omega_n}, \Gamma = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix},\\
&\qquad E[v(t)v(s)] = R\delta(t-s),\\
&E[w(t)w(s)] = Q\delta(t-s), Q = 1,\\
&E[v(t)w(s)] = 0, E[x(0)w(t)] = 0,\\
&E[x(0)v(t)] = 0.
\end{aligned}$$

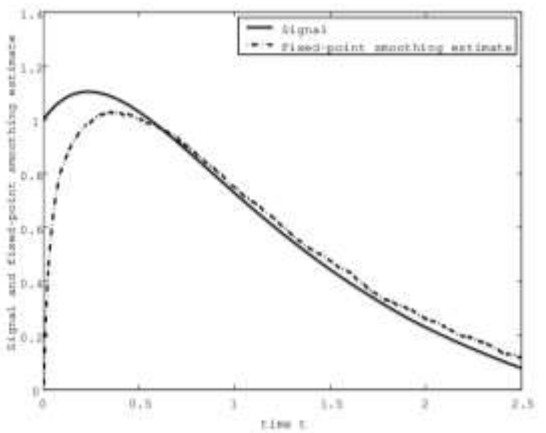

Fig. 7: Signal $z(t)$ and its fixed-point smoothing estimate $\hat{z}(t, t + 0.005)$ vs. t for the white Gaussian observation noise $N(0,0.1^2)$

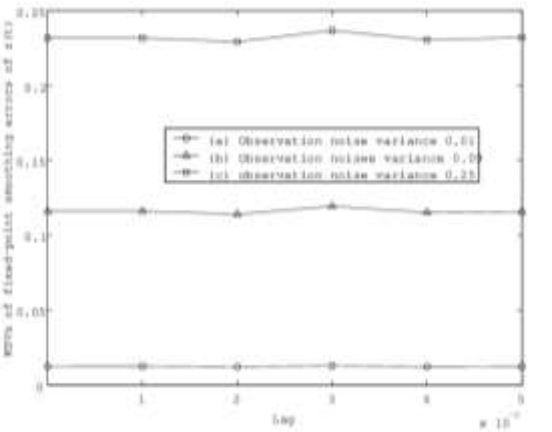

Fig. 8: MSVs of the filtering and fixed-point smoothing errors by the robust RLS filter and the robust RLS fixed-point smoother in Theorem 1 vs. Lag for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$

The state-space model is equivalently expressed by series RLC circuit, [29].

Figure 7 illustrates the signal $z(t)$ and its fixed-point smoothing estimate $\hat{z}(t, t + 0.005)$ vs. t for the white Gaussian observation noise $N(0,0.1^2)$. Figure 7 shows that $\hat{z}(t, t + 0.005)$ estimates $z(t)$ feasibly.

Figure 8 illustrates the MSVs of the filtering and fixed-point smoothing errors by the robust RLS filter and the robust RLS fixed-point smoother in Theorem 1 vs. Lag for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. From Figure 8, for the white Gaussian observation noise $N(0, 0.5^2)$, the MSV of the fixed-point smoothing errors $\mathrm{z(t)} - \hat{z}(t, t + 0.002)$ is slightly smaller than that of the filtering errors. For the white Gaussian observation noises $N(0, 0.1^2)$ and $N(0, 0.3^2)$, the MSVs of the filtering errors are almost the same as those of the fixed-point smoothing errors $\mathrm{z(t)} - \hat{z}(t, t + Lag)$ for $0.001 \leq Lag \leq 0.005$.

7 Conclusion

This paper has proposed a novel robust estimation technique for continuous-time uncertain stochastic systems. In the degraded state-space model, the observation vector and the system matrix include uncertain parameters. Additive white Gaussian noise is present in the observation of the degraded signal. The feature of utilizing covariance information is present in the robust RLS fixed-point smoothing and filtering algorithms in Theorem 1. The finite Fourier cosine series expansion approximates the cross-covariance function of the signal with the observed value, as well as the autocovariance function of the degraded signal.

In the first simulation example, the MSV by the robust RLS filter in Theorem 1 is smaller than that by the robust RLS Wiener filter for the white Gaussian observation noise $N(0, 0.1^2)$. In the two simulation examples, by using the robust RLS fixed-point smoother and filter in Theorem 1, for the white Gaussian observation noise $N(0, 0.5^2)$, the MSV of the fixed-point smoothing errors $\mathrm{z(t)} - \hat{z}(t, t + 0.002)$ is slightly smaller than that of the filtering errors. For the white Gaussian observation noises $N(0, 0.1^2)$ and $N(0, 0.3^2)$, the MSVs of the filtering errors are nearly identical to those of the fixed-point smoothing errors $\mathrm{z(t)} - \hat{z}(t, t + Lag)$ for Lag values between 0.001 and 0.005. Based on these results, the proposed fixed-point smoothing and filtering method utilizing covariance information is valid.

The proposed robust estimation method using covariance information leads to the development of new robust estimators for continuous-time stochastic systems.

References:

[1] U. Shaked and C. E. de Souza, Robust minimum variance filtering, *IEEE Transactions on Signal Processing*, Vol. 43, No. 11, 1995, pp. 2474-2483, DOI: 10.1109/78.482099.

[2] C. E. de Souza, U. Shaked, M. Fu, Robust H_∞ filtering for continuous time varying uncertain systems with deterministic input signals, *IEEE Transactions on Signal Processing*, Vol. 43, No. 3, 1995, pp. 709–719, DOI: 10.1109/78.370625.

[3] F. L. Lewis, L. Xie, D. Popa, *Optimal and Robust Estimation With an Introduction to Stochastic Control Theory*, Second Edition, CRC Press, 2008.

[4] S. O. R. Moheimani, A. V. Savkin, I. R. Petersen, Robust filtering, prediction, smoothing and observability of uncertain systems, *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, Vol. 45, No.4, 1998, pp. 446–457, DOI: 10.1109/81.669068.

[5] Z. Quan, S. Han, J. H. Park, W. H. Kwon, Robust FIR Filters for Linear Continuous-Time State-Space Models With Uncertainties, *IEEE Signal Processing Letters*, Vol. 15, 2008, pp. 621–624, DOI: 10.1109/LSP.2008.2004515.

[6] Y. Shmaliy, S. Zhao, *Optimal and Robust State Estimation: Finite Impulse Response (FIR) and Kalman Approaches*, IEEE Press, Piscataway, NJ, 2022.

[7] Q. Cheng, B. Cui, Improved results on robust energy-to-peak filtering for continuous-time uncertain linear systems, *Circuits, Systems, and Signal Processing*, Vol. 38, No. 5, 2019, pp. 2335–2350, DOI: 10.1007/s00034-018-0965-7.

[8] D.-W. Ding, G.-H. Yang, Robust H_2 filtering for uncertain continuous-time switched linear systems, *Proceedings of the IEEE International Conference on Control Applications, CCA 2009 and of the International Symposium on Intelligent Control, ISIC 2009, Saint Petersburg, Russia, July 8-10, 2009*, IEEE, 2009, pp. 1110–1115, DOI: 10.1109/CCA.2009.5281161.

[9] E. Gershon, D. J. N. Limebeer, U. Shaked, I. Yaesh, Robust H_∞ filtering of stationary continuous-time linear systems with stochastic uncertainties, *IEEE Transactions on Automatic Control*, Vol. 46, No. 11, 2001, pp. 1788–1793, DOI: 10.1109/9.964692.

[10] K. H. Lee, B. Huang, Robust H_2 optimal filtering for continuous-time stochastic systems with polytopic parameter uncertainty, *Automatica*, Vol. 44, No. 10, 2008, pp. 2686–2690, DOI: 10.1016/j.automatica.2008.02.025.

[11] X. Li, H. Gao, A delay-dependent approach to robust generalized H_2 filtering for uncertain continuous-time systems with interval delay, *Signal Processing*, Vol. 91, No. 10, 2011, pp. 2371–2378, DOI: 10.1016/j.sigpro.2011.04.032.

[12] J. Qiu, G. Feng, J. Yang, Improved delay-dependent robust H_∞ filtering of continuous-time polytopic linear systems with time-varying delay, *10th International Conference on Control, Automation, Robotics and Vision, ICARCV 2008, Hanoi, Vietnam, 17-20 December 2008, Proceedings*, IEEE, 2008, pp. 53–58, DOI: 10.1109/ICARCV.2008.4795491.

[13] O. L. V. Costa, M. D. Fragoso, Robust linear filtering for continuous-time hybrid Markov linear systems, *Proceedings of the 47th IEEE Conference on Decision and Control, CDC 2008, December 9-11, 2008, Cancún, Mexico*, IEEE, 2008, pp. 5098–5103, DOI: 10.1109/CDC.2008.4739044.

[14] X. Xiao, H. Xi, J. Zhu, H. Ji, Robust Kalman filter of continuous-time Markov jump linear systems based on state estimation performance, *International Journal of Systems Science*, Vol. 39, No. 1, 2008, pp. 9–16, DOI: 10.1080/00207720701597456.

[15] A. El-Amrani, B. Boukili, A. E. Hajjaji, A. Hmamed, Robust H filter for uncertain continuous-time systems with finite frequency ranges, *26th Mediterranean Conference on Control and Automation, MED 2018, Zadar, Croatia, June 19-22, 2018*, IEEE, 2018, pp. 807–812, DOI: 10.1109/MED.2018.8443021.

[16] A. G. Kallapur, I. G. Vladimirov, I. R. Petersen, Robust filtering for continuous-time uncertain nonlinear systems with an integral quadratic constraint, *American Control Conference, ACC 2012, Montreal, QC, Canada, June 27-29, 2012*, IEEE, 2012, pp. 4807–4812, DOI: 10.1109/ACC.2012.6314612.

[17] J. Qiu, H. Tian, Q. Lu, H. Gao, Nonsynchronized Robust Filtering Design for Continuous-Time T-S Fuzzy Affine Dynamic Systems Based on Piecewise Lyapunov Functions, *IEEE Transactions on Cybernetics*, Vol. 43, No. 6, 2013, pp. 1755–1766, DOI: 10.1109/TSMCB.2012.2229389.

[18] H. Tian, J. Qiu, H. Gao, Q. Lu, New results on robust filtering design for continuous-time nonlinear systems via T-S fuzzy affine dynamic models, *2012 12th International Conference on Control Automation Robotics & Vision (ICARCV), Guangzhou, China, Dec. 5-7, 2012*, pp. 1220–1225, DOI: 10.1109/ICARCV.2012.6485349.

[19] S. Nakamori, Robust recursive least-squares Wiener filter for linear continuous-time uncertain stochastic systems, *WSEAS Transactions on Signal Processing*, Vol. 19, No. 12, 2023, pp. 108–117, 10.37394/232014.2023.19.12.

[20] T. C. Hsia, *System Identification: Least-Squares Methods*, Lexington Books, 1977.

[21] A. P. Sage, J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill, 1971.

[22] R. Bellman, G. M. Wing, *An Introduction to Invariant Imbedding: Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics, 1992.

[23] H. Kagiwada, R. Kalaba, Imbedding Methods for Integral Equations with Applications, *Solution Methods for Integral Equations: Theory and Applications*, M. A. Golberg (Ed.), Mathematical Concepts and Methods in Science and Engineering, Springer, Boston, 1979, pp. 195–223.

[24] B. Millidge, A. Tschantz, A. Seth, C. Buckley, Neural Kalman Filtering, arXiv: 2102.10021, 2021, pp. 1-12, DOI: 10.48550/arXiv.2102.10021.

[25] S. Kim, I. Petrunin and H. -S. Shin, A Review of Kalman Filter with Artificial Intelligence Techniques," *2022 Integrated Communication, Navigation and Surveillance Conference (ICNS)*, Dulles, VA, USA, 2022, pp. 1-12, DOI: 10.1109/ICNS54818.2022.9771520.

[26] A. Juárez-Lora, L. M. García-Sebastián, V. H. Ponce-Ponce, E. Rubio-Espino, H. Molina-Lozano, H. Sossa, Implementation of Kalman Filtering with Spiking Neural Networks, *Sensors*, Vol. 22, No. 22, 2022, pp. 1-16, DOI: 10.3390/s22228845.

[27] Z. Cui, J. Dai, J. Sun, D. Li, L. Wan, K. Wang, Hybrid Methods Using Neural Network and Kalman Filter for the State of Charge Estimation of Lithium-Ion Battery, *Mathematical Problems in Engineering*, Vol. 2022, Article ID 9616124, 2022, pp. 1-11, DOI: 10.1155/2022/9616124.

[28] Y. Bai, B. Yan, C. Zhou, T. Su, X. Jin, State of art on state estimation: Kalman filter driven by machine learning, *Annual Reviews in Control*, Vol. 56, 2023, p. 100909, DOI: 10.1016/j.arcontrol.2023.100909.

[29] S. Nakamori, Design of linear continuous-time stochastic estimators using covariance information in Krein spaces, *IEICE Transactions on Fundamentals*, Vol. E84A, No. 9, 2001, pp. 2261-2271.

[30] M. S. Grewal, A. P. Andrews, *Kalman Filtering: Theory and Practice Using Matlab*, Third Edition, John Wiley & Sons, Inc., 2008.

APPENDIX

Proof of Theorem 1

Differentiating (7) with respect to T, we have

$$\frac{\partial h(t,s,T)}{\partial T}R = -h(t,T,T)K_{\breve{z}}(T,s) - \int_0^T \frac{\partial h(t,\tau,T)}{\partial T}K_{\breve{z}}(\tau,s)d\tau. \tag{A-1}$$

Introducing a function $L(s,t)$ satisfying

$$L(s,T)R = \breve{B}^T(s) - \int_0^T L(\tau,T)K_{\breve{z}}(\tau,s)d\tau, \tag{A-2}$$

we obtain

$$\frac{\partial h(t,s,T)}{\partial T} = -h(t,T,T)\breve{A}(T)L(s,T). \tag{A-3}$$

From (7), $h(t,T,T)$ satisfies

$$h(t,T,T)R = K_{z\breve{y}}(t,T) - \int_0^T h(t,\tau,T)K_{\breve{z}}(\tau,T)d\tau. \tag{A-4}$$

From (9), (A-4) is transformed into

$$h(t,T,T)R = K_{z\breve{y}}(t,T) - \int_0^T h(t,\tau,T)\breve{B}(\tau)\breve{A}^T(T)d\tau. \tag{A-5}$$

Introducing a function

$$P(t,T) = \int_0^T h(t,\tau,T)\breve{B}(\tau)d\tau, \tag{A-6}$$

$h(t,T,T)$ is given by

$$h(t,T,T) = (K_{z\breve{y}}(t,T) - P(t,T)\breve{A}^T(T))/R. \tag{A-7}$$

Differentiating (A-6) with respect to T, we have

$$\frac{\partial P(t,T)}{\partial T} = h(t,T,T)\breve{B}(T) + \int_0^T \frac{\partial h(t,\tau,T)}{\partial T}\breve{B}(\tau)d\tau. \tag{A-8}$$

Substituting (A-3) into (A-8) and introducing a function

$$S(T)=\int_0^T L(\tau,T)\breve{B}(\tau)d\tau, \quad \text{(A-9)}$$

we have

$$\frac{\partial P(t,T)}{\partial T}=h(t,T,T)(\breve{B}(T)-\breve{A}(T)S(T)). \quad \text{(A-10)}$$

The fixed-point smoothing estimate $\hat{z}(t,T)$ of the signal $z(t)$ is given by (3). Differentiating (3) with respect to T, we have

$$\frac{\partial \hat{z}(t,T)}{\partial T}=h(t,T,T)\breve{y}(T)+\int_0^T \frac{\partial h(t,s,T)}{\partial T}\breve{y}(s)ds. \quad \text{(A-11)}$$

Substituting (A-3) into (A-11), we have

$$\frac{\partial \hat{z}(t,T)}{\partial T}=h(t,T,T)\breve{y}(T)-h(t,T,T)\breve{A}(T)\int_0^T L(s,T)\breve{y}(s)ds. \quad \text{(A-12)}$$

Introducing $f(T)$ given by

$$f(T)=\int_0^T L(s,T)\breve{y}(s)ds, \quad \text{(A-13)}$$

(A-12) is transformed into

$$\frac{\partial \hat{z}(t,T)}{\partial T}=h(t,T,T)(\breve{y}(T)-\breve{A}(T)f(T)),\ \hat{z}(t,T)|_{T=t}=\hat{z}(t,t). \quad \text{(A-14)}$$

From (7), the impulse response function $h(t,s,t)$ for the filtering estimate $\hat{z}(t,t)$ of $z(t)$ satisfies

$$h(t,s,t)R=K_{z\breve{y}}(t,s)-\int_0^t h(t,\tau,t)K_{\breve{z}}(\tau,s)d\tau,\ 0\le s\le t. \quad \text{(A-15)}$$

Introducing a function $J(s,t)$ satisfying

$$J(s,t)R=\beta^T(s)-\int_0^t J(\tau,t)K_{\breve{z}}(\tau,s)d\tau. \quad \text{(A-16)}$$

$h(t,s,t)$ is given by

$$h(t,s,t)=\alpha(t)J(s,t). \quad \text{(A-17)}$$

Differentiating (A-16) with respect to t, we have

$$\frac{\partial J(s,t)}{\partial t}R=-J(t,t)K_{\breve{z}}(t,s)-\int_0^t \frac{\partial J(\tau,t)}{\partial t}K_{\breve{z}}(\tau,s)d\tau. \quad \text{(A-18)}$$

From (A-2) and (A-18), $\frac{\partial J(s,t)}{\partial t}$ satisfies

$$\frac{\partial J(s,t)}{\partial t}=-J(t,t)\breve{A}(t)L(s,t). \quad \text{(A-19)}$$

From (A-16), $J(t,t)$ satisfies

$$J(t,t)R=\beta^T(t)-\int_0^t J(\tau,t)K_{\breve{z}}(\tau,t)d\tau. \quad \text{(A-20)}$$

From (9), (A-20) is rewritten as

$$J(t,t)R=\beta^T(t)-\int_0^t J(\tau,t)\breve{B}(\tau)\breve{A}^T(t)d\tau. \quad \text{(A-21)}$$

Introducing a function

$$r(t)=\int_0^t J(\tau,t)\breve{B}(\tau)d\tau, \quad \text{(A-22)}$$

$J(t,t)$ is given by

$$J(t,t)=(\beta^T(t)-r(t)\breve{A}^T(t))/R. \quad \text{(A-23)}$$

Differentiating (A-22) with respect to t, we have

$$\frac{dr(t)}{dt}=J(t,t)\breve{B}(t)+\int_0^t \frac{\partial J(\tau,t)}{\partial t}\breve{B}(\tau)d\tau. \quad \text{(A-24)}$$

Substituting (A-19) into (A-24), we have

$$\frac{dr(t)}{dT}=J(t,t)\breve{B}(t)-J(t,t)\breve{A}(t)\int_0^t L(\tau,t)\breve{B}(\tau)d\tau. \quad \text{(A-25)}$$

Introducing a function

$$S(t)=\int_0^t L(\tau,t)\breve{B}(\tau)d\tau, \quad \text{(A-26)}$$

we obtain

$$\frac{dr(t)}{dT}=J(t,t)(\breve{B}(t)-\breve{A}(t)S(t)),\ \mathrm{r}(0)=0. \quad \text{(A-27)}$$

From (A-2), $L(s,t)$ satisfies

$$L(s,t)R=\breve{B}^T(s)-\int_0^t L(\tau,t)K_{\breve{z}}(\tau,s)d\tau. \quad \text{(A-28)}$$

Differentiating (A-28) with respect to t, we have

$$\frac{\partial L(s,t)}{\partial t}R=-L(t,t)K_{\breve{z}}(t,s)-\int_0^t \frac{\partial L(\tau,t)}{\partial t}K_{\breve{z}}(\tau,s)d\tau. \quad \text{(A-29)}$$

From (9), (A-29) is transformed into

$$\frac{\partial L(s,t)}{\partial t}R=-L(t,t)\breve{A}(t)\breve{B}^T(s)-\int_0^t \frac{\partial L(\tau,t)}{\partial T}K_{\breve{z}}(\tau,s)d\tau. \quad \text{(A-30)}$$

From (A-28), we obtain

$$\frac{\partial L(s,t)}{\partial t}=-L(t,t)\breve{A}(t)L(s,t). \quad \text{(A-31)}$$

From (A-28), $L(t,t)$ satisfies

$$L(t,t)R=\breve{B}^T(t)-\int_0^t L(\tau,t)K_{\breve{z}}(\tau,t)d\tau. \quad \text{(A-32)}$$

From (9), (A-32) is rewritten as

$$L(t,t)R=\breve{B}^T(t)-\int_0^t L(\tau,t)\breve{B}(\tau)\breve{A}^T(t)d\tau. \quad \text{(A-33)}$$

From (A-26), $L(t,t)$ is given by

$$L(t,t)=(\breve{B}^T(t)-S(t)\breve{A}^T(t))/R. \quad \text{(A-34)}$$

The filtering estimate $\hat{z}(t,t)$ of the signal $z(t)$ is given by

$$\hat{z}(t,t) = \int_0^t h(t,s,t)\breve{y}(s)ds \quad \text{(A-35)}$$

Substituting (A-17) into (A-35), we have

$$\hat{z}(t,t) = \alpha(t)\int_0^t J(s,t)\breve{y}(s)ds. \quad \text{(A-36)}$$

Introducing a function $e(T)$ given by

$$e(t) = \int_0^t J(s,t)\breve{y}(s)ds, \quad \text{(A-37)}$$

$\hat{z}(t,t)$ is given by

$$\hat{z}(t,t) = \alpha(t)e(t). \quad \text{(A-38)}$$

Differentiating (A-37) with respect to t, we have

$$\frac{de(t)}{dt} = J(t,t)\breve{y}(t) + \int_0^t \frac{\partial J(s,t)}{\partial t}\breve{y}(s)ds. \quad \text{(A-39)}$$

Substituting (A-19) into (A-39), we have

$$\frac{de(t)}{dt} = J(t,t)\breve{y}(t) - J(t,t)\breve{A}(t)\int_0^t L(s,t)\breve{y}(s)ds. \quad \text{(A-40)}$$

From (A-13), we obtain

$$\frac{de(t)}{dt} = J(t,t)(\breve{y}(t) - \breve{A}(t)f(t)), \quad e(0)=0. \quad \text{(A-41)}$$

From (A-13), differentiating $f(t)$ with respect to t, we have

$$\frac{df(t)}{dt} = L(t,t)\breve{y}(t) + \int_0^t \frac{\partial L(s,t)}{\partial t}\breve{y}(s)ds. \quad \text{(A-42)}$$

Substituting (A-31) into (A-42), we have

$$\frac{df(t)}{dt} = L(t,t)\breve{y}(t) - L(t,t)\breve{A}(t)\int_0^t L(s,t)\breve{y}(s)ds. \quad \text{(A-43)}$$

From (A-13), we obtain

$$\frac{df(t)}{dt} = L(t,t)(\breve{y}(t) - \breve{A}(t)f(t)), \quad f(0) = 0. \quad \text{(A-44)}$$

Differentiating (A-26) with respect to t, we have

$$\frac{dS(t)}{dt} = L(t,t)\breve{B}(t) + \int_0^t \frac{\partial L(\tau,t)}{\partial t}\breve{B}(\tau)d\tau. \quad \text{(A-45)}$$

Substituting (A-31) into (A-45), we have

$$\frac{dS(t)}{dt} = L(t,t)\breve{B}(t) - L(t,t)\breve{A}(t)\int_0^t L(\tau,t)\breve{B}(\tau)d\tau. \quad \text{(A-46)}$$

From (A-26), we obtain

$$\frac{dS(t)}{dt} = L(t,t)(\breve{B}(t) - \breve{A}(t)S(t)), \quad S(0) = 0. \quad \text{(A-47)}$$

From (A-6), $P(t,t)$ is given by

$$P(t,t) = \int_0^t h(t,\tau,t)\breve{B}(\tau)d\tau. \quad \text{(A-48)}$$

Substituting (A-17) into (A-48), we have

$$P(t,t) = \alpha(t)\int_0^t J(\tau,t)\breve{B}(\tau)d\tau. \quad \text{(A-49)}$$

From (A-22), $P(t,t)$ is given by

$$P(t,t) = \alpha(t)r(t). \quad \text{(A-50)}$$

(Q.E.D.)

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

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The author has no conflicts of interest to declare.