Comparative Analysis of Methods for Estimation of Rice Distribution Parameters

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Abstract: - This article discusses the methods and procedures for estimating the initial parameters of the Rice distribution known in the literature. The maximum likelihood method and the methods of different degrees' moments are selected. A series of experiments on estimating the parameters from the obtained samples are performed by mathematical modeling of the methods and samples with given values of the initial parameters of the distribution. The advantage of the maximum likelihood method over the others is shown. A procedure for estimating the distribution shape parameter for small signal-to-noise ratios is developed.

Key-Words: - Rice distribution, maximum likelihood method, method of moments, signal-to-noise ratio, coefficient of variation, empirical formula, estimation procedure.

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1 Introduction

The Rice distribution is used in various fields such as radio engineering, signal processing, magnetic resonance imaging (MRI) analysis and processing, and mobile communications, [1], [2], [3]. In MRI, this distribution is used to model the distribution of pixel intensities in images due to the presence of both coherent and incoherent noise sources. The Rice distribution is often used to model the received signal power in wireless and mobile communication systems, where the received signal is a combination of a dominant line-of-sight signal and scattered signals.

In [4], the Rice distribution is used as a model for improving medical diagnostics of heart diseases using ECG. There are many publications on the application of the Rice distribution dedicated to digital image processing tasks. In particular, effective algorithms for image similarity estimation have been developed in [5], [6], [7]. The central task in this context is the estimation of distribution parameters based on a sample of experimental data.

The analysis of Rice distribution data is related to problems where the output signal is the sum of a deterministic signal and random noise, and it is required to separately estimate the parameters of these components. Therefore, an important task is to estimate the initial parameters of the Rice distribution based on the available data sample obtained as a result of real process measurement or by mathematical modeling. The most used methods for estimating the parameters of the Rice distribution are the maximum likelihood (ML) method and the method of moments, [8], [9], [10], [11], [12], [13], [14], [15]. The ML method is based on maximizing the likelihood function based on the distribution parameters [9], [10], [11], [14], [15]. The method of moments is based on aligning the expressions of theoretical and empirical moments of the distribution and solving the corresponding equations. A review of the literature shows that to apply the method of moments, it is sufficient to use the first four initial moments of the sample, [12], [13], [15]. The method of moments is formally quite simple but can give less effective estimates, especially for small samples. The ML method, on the other hand, usually provides more efficient estimates, but its implementation, like certain variants of the method of moments, requires the use of successive approximation methods, which are computationally expensive. It is important to note that at large signal-to-noise ratios, the Rice distribution approaches the normal distribution. Therefore, all the good properties of the latter also apply to the Rice distribution.

One of the alternative approaches to estimating the parameters of the Rice distribution, which provides acceptable accuracy of estimates with minimal computational expenditure, is the development of empirical methods based on the approximation of certain characteristics by polynomials of low degrees. As an example, we point to the work [16], which studies development methods and many empirical formulas widely used in various fields of science and technology, in particular in the study of wind flow parameters.

In applications involving the Rice distribution, it is important to consider that as the signal-to-noise ratio increases, the distribution quickly converges towards a normal distribution. This leads to a significant simplification of the procedures for estimating the parameters. However, since cases with low signal-to-noise ratios are often of greater interest, there is a need for a thorough investigation of the accuracy of various methods for estimating the parameters of the Rice distribution, depending on the values and relationships of the distribution's initial parameters.

This article is dedicated to an experimental study of the aforementioned issues and the development of specific recommendations for applying the ML method and some procedures proposed in the scientific literature based on the method of moments.

2 Methods of Estimating Parameters

The density function of the Rice distribution has the form:

$$f(\nu,\sigma)=\frac{x}{\sigma^2}exp\left(\frac{-(x^2+\nu^2)}{2\sigma^2}\right)I_0\left(\frac{x\nu}{\sigma^2}\right), \quad (1)$$

where $I_0(z)$ is the modified Bessel function of the first kind of order zero.

To estimate the parameters of the Rice distribution and compare the accuracy of various methods, we selected the ML method and the method of moments, using combinations of the moments of the first four orders.

The expressions for the first four initial moments of the distribution have the form

$$\mu_1=\sigma\sqrt{\pi/2}L_{1/2}(-\nu^2/2\sigma^2), \quad (2)$$
$$\mu_2=2\sigma^2+\nu^2, \quad (3)$$
$$\mu_3=\sigma^3\sqrt{\frac{\pi}{2}}L_{3/2}(-\nu^2/2\sigma^2), \quad (4)$$
$$\mu_4=8\sigma^4+8\sigma^2\nu^2+\nu^4, \quad (5)$$

where

$$L_{1/2}(x)=e^{x/2}[(1-x)I_0(-x/2)xI_1(-x/2)]. \quad (6)$$

Here $L_{1/2}, L_{3/2}$ denote the Laguerre polynomials.

Let us denote the sample initial moments by $m_j, (j=1,2,3,4)$, and the variation coefficient by γ. The moments of the Rice distribution have been calculated and published in many educational and methodological publications, so the details of the proof are omitted. One can point, for example, to the well-known Abramovich reference book [17] and to [18]. It should be noted that a theoretical study of the statistical properties of the methods used, such as the consistency or bias of the ML estimate, as well as the sample moments of the Rice distribution, is not the objective of our study, so we will limit ourselves to references to the relevant works.

An experimental study of the accuracy of estimates obtained by the selected methods is of interest, comparing them with the ML method using samples with different values of the parameters υ and σ and sample size N from the Rice distribution. For the procedures based on the ML method, this paper utilizes the results obtained in [13], which also includes findings from numerical modeling.

We will consider the following three procedures for estimating the parameters of the Rice distribution: the ML estimation method (MLE) and the methods called MM12 and MM24, [13]. Next, we will consider the M_γ method for estimating parameters using the empirical formula proposed in [15]. At the end of the section, we will describe two estimation procedures for small values of the signal-to-noise ratio.

2.1 MLE Method

The properties of MLE estimates have been studied comprehensively in the scientific literature. We will not dwell on the details of these studies but limit ourselves only to referencing works, [11], [12], [13]. It should be noted that the referenced equations are fairly simple to implement.

A program based on simple equations given in [13] has been implemented to perform numerical calculations.

$$\nu=\frac{1}{N}\sum_{i=1}^{N}x_i\cdot\tilde{I}\left(\frac{2x_i\nu}{m_2-\nu^2}\right), \quad (7)$$
$$\sigma^2=\frac{1}{2}(m_2-\nu^2), \quad (8)$$

where $\tilde{I}(z)=I_1(z)/I_0(z)$.

First, equation (7) is solved with respect to the parameter ν using the method of simple iterations with a sufficiently small value of the relative approximation error $\varepsilon\geq 0$, then the σ parameter is estimated using formula (8).

2.2 Method of 1st and 2nd Moments MM12

It is proposed to estimate the parameters ν and σ in two stages [13].

Let us designate

$$\{K = \frac{\nu^2}{2\sigma^2},\ \Omega = \nu^2 + 2\sigma^2. \qquad (9)$$

In the literature, the K and Ω are called the parameters of the shape and scale of the Rice distribution, respectively. With the values of these quantities, one can obtain estimates of the initial parameters of the Rice distribution using the formulas:

$$\{\nu = \sqrt{\frac{K\Omega}{K+1}}, \sigma = \sqrt{\frac{\Omega}{2(K+1)}}. \qquad (10)$$

Moving on to the method of moments, we can obtain the following system of equations with respect to K and σ and using the first two sample moments:

$$\begin{cases} \mathrm{m}_1 = \sqrt{\frac{\pi}{4}\mathrm{m}_2}\ \sqrt{(1+\mathrm{K})}e^{\frac{\mathrm{K}}{2}}\, \mathrm{I}_0\left(\frac{\mathrm{K}}{2}\right)\left[1 + \frac{\mathrm{K}}{\mathrm{K}+1}\tilde{\mathrm{I}}\left(\frac{\mathrm{K}}{2}\right)\right], \\ \sigma^2 = \frac{\mathrm{m}_2}{2(1+\mathrm{K})}. \end{cases} \qquad (11)$$

As a result of solving the system of equations (11), we can estimate the parameters ν and σ using formulas (10).

2.3 Method of 2nd and 4th Moments MM24

Per the MM24 method, the parameters of the original signal are calculated using the following formulas:

$$\nu = \sqrt[4]{2(m_2)^2 - m_4}\,, \qquad (12)$$

$$\sigma = \sqrt{m_2 - \sqrt{2(m_2)^2 - m_4}}. \qquad (13)$$

2.4 Method M_γ of Estimating Parameters Using Empirical Formulas [15]

In [15], a new approach to the parameterization of the Rice distribution is proposed, and the resulting models are thoroughly studied. A new parameter λ is introduced, which essentially coincides with K in the notations adopted in this article. As a result of numerical modeling, the following empirical formula for estimating K, depending on the coefficient of variation $\hat{\gamma}$, is obtained:

$$\hat{K} \cong \frac{1 - 2.864928\hat{\gamma} - 3.193363\hat{\gamma}^2 + 15.715797\hat{\gamma}^3 - 11.713746\hat{\gamma}^4}{-0.017883 + 1.815109\hat{\gamma} + 7.318177\hat{\gamma}^2 + 8.717601\hat{\gamma}^3 - 2.362803\hat{\gamma}^4}. \qquad (14)$$

The scale parameter is determined by the formula $\Omega = m_2$. Comments on this formula are provided in the discussion of the modeling results for all the considered methods.

2.5 Estimation at Low Signal-To-Noise Ratio Values

In applications of the Rice distribution in signal processing tasks, it is often not the direct estimation of the distribution parameters from measurement results that is required, but rather the immediate determination of the signal-to-noise ratio, which is necessary for developing further signal processing strategies and making appropriate decisions. As seen from equations (9), the signal-to-noise ratio, denoted as $\xi = \upsilon/\sigma$, is related to the parameter K by the expression $\xi = \sqrt{2}K$. Consequently, the problem is reduced to estimating parameter K based on measurement results. This requires only knowledge of the coefficient of variation $\hat{\gamma}$. Moreover, simple procedures exist for estimating the parameter K at small values. Regarding high signal-to-noise ratio values, it is known in the fields of signal processing theory and technology that as $\xi \to \infty$, the Rice distribution approaches a normal distribution with parameters ν and σ. For $\xi \geq 3$, this approximation is quite acceptable, meaning that under this condition, we have $K \geq 4.5$, allowing the parameters of the Rice distribution to be estimated using traditional statistical methods.

In this article, we will consider two procedures. For what follows, we need to obtain the equation from (2) - (4)

$$\gamma^2 = \frac{2\sigma^2 + \upsilon^2}{\sigma^2(\pi/2)L^2_{1/2}(-\nu^2/2\sigma^2)} - 1 \qquad (15)$$

which, after a series of elementary transformations, is reduced to the form:

$$\gamma^2 = \frac{4(1+K)}{\pi L^2_{1/2}(-K)} - 1 \qquad (16)$$

Procedure one. In the above-mentioned work [15], the following formula is proposed, which relates γ to small values K

$$\gamma = 0.52272 - 0.15224K^4 \qquad (17)$$

Procedure two. We need to obtain an approximate expression (16) for small values of K, using only the terms from the Taylor series expansion of the Bessel function of the first kind of zero order. Then we obtain that

$$L^2_{1/2} = 1 + K + \frac{K^2}{8} + \square(K^3) \qquad (18)$$

Substituting (18) into (16) after some elementary transformations and denoting

$$\delta = \frac{4}{\pi(\gamma^2+1)} - 1, \qquad (19)$$

We obtain the following quadratic equation $K^2 - 8\delta K - 8\delta = 0$ relative to K, the solution of which is:

$$K = 4\delta + 2\sqrt{4\delta^2 + 2\delta}. \qquad (20)$$

Example. Let K = 0.1 (17) correspond to the value of the coefficient of variation $\gamma = 0.52138404$. Then $\delta = 0.0010994$ and the estimate per formula (20) is equal to $\widehat{K} = 0.09828$. The result is significantly better than the one obtained using the first procedure. Figure 1 shows a graph illustrating the accuracy of the parameter K estimate in the interval $[0, 1]$.

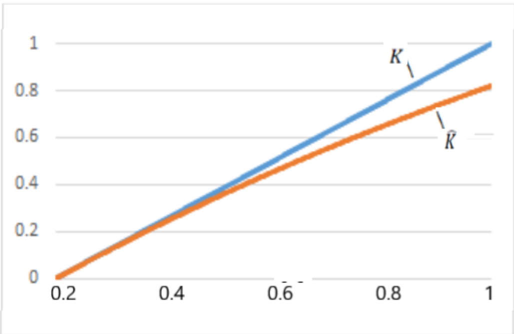

Fig. 1: Dependence of the parameter K estimate on the true value

2.6 Comparative Analysis of the Evaluation of the Parameters of the Above-Mentioned Methods

Methods 2.1-2.4. For a comparative analysis of the mentioned methods, modeling was performed in the Python environment by random generation of the Rice-distributed data with a length of n = 10,000 for different values of the Rice distribution parameters υ and σ. In this case, 100 samples were obtained for each pair of parameter values, and the estimates were averaged. Standard errors of the mean for the parameter ν were also calculated. The results are given in Table 1 (Appendix). Note that the degree of agreement of the obtained data with the Rice distribution was checked along the way, and only adequate samples were selected.

Based on the visual and numerical analysis of the data in Table 1 (Appendix), the following conclusions can be drawn:

1. The superior accuracy of the MLE method over other methods is clear, with a standard error of the mean less than 10^{-5}.

2. The MM12 and MM24 methods provide higher accuracy in estimating parameter ν compared to parameter σ. Additionally, the standard error for both methods is approximately the same.

3. The accuracy of parameter ν estimation using the MM24 method is nearly identical to that of MLE, but the estimation of parameter σ is significantly less accurate.

4. No dependency of the accuracy of parameter ν and σ estimates on SNR is observed with any of the methods considered.

The accuracy of the empirical and approximate formulas is shown in the text describing these formulas.

2.6.1 Comparison of the Procedures of Section 2.5 by Accuracy

As noted earlier, it is of interest to compare the procedures based on their accuracy in estimating the parameter K using only the value of the coefficient of variation γ. To do this, a test value of K is first selected, and then γ is determined from equation (17) using a numerical method. This value is then substituted into the corresponding formulas of the procedures. Table 2 (Appendix) presents the estimates of parameter K, calculated using formulas (14), (17), and (20).

The following observations should be made regarding the formulas used in the first procedure. An analysis of the numerical solution of equation (17) reveals that for sufficiently small values of the parameter K, the calculated values of the variation coefficient are very close to each other. For example, when $K = 0.001$ and $K = 0.002$, the variation coefficients are 0.522723165 and 0.52272271, respectively. Therefore, if these values are not rounded carefully, the estimates may deviate from the true values. This issue applies to both formulas in the first procedure. As a result, as shown in Table 2 (Appendix), these formulas do not provide the necessary accuracy for estimating K and, thus, cannot be recommended for use. In contrast, the second procedure offers sufficient accuracy.

3 Conclusion

The article examines and, for the first time, compares the accuracy of well-known methods for estimating the parameters of the Rice distribution found in the literature. Specifically, the MLE method and variants of the method of moments using moments of different degrees are selected. Samples generated for different distribution parameters serve as the source material. Through mathematical modeling of the selected methods, the unknown parameters are estimated for each sample, and the accuracy of the obtained estimates is compared. The MLE method proved to be the most accurate. Relevant comments are provided for the other methods. Additionally, a procedure for estimating the shape parameter of the Rice distribution at low signal-to-noise ratios is proposed. Its superior accuracy compared to other existing methods is demonstrated.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The author wrote, reviewed and edited the content as needed and the author has not utilised artificial intelligence (AI) tools. The author takes full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDIX

Table 1. Results of assessing the accuracy of different methods

Parameter			Estimates $(\hat{\nu},\ \hat{\sigma})$							
ν	σ	SNR	MLE $(\hat{\nu}, \hat{\sigma})$		MM12 $(\hat{\nu}, \hat{\sigma})$		MM24 $(\hat{\nu}, \hat{\sigma})$		Emp. M_γ $(\hat{\nu}, \hat{\sigma})$	
1.5	1.0	1.5	1.501	0.999	1.460	1.041	1.504	1.407	1.477	1.015
2.0	1.0	2.0	2.001	0.998	1.731	1.235	2.003	1.413	1.869	1.122
1.0	0.4	2.5	1.000	0.399	0.826	0.587	1.001	0.565	0.918	0.489
2.5	1.0	2.5	2.500	1.001	2.028	1.448	2.508	1.401	2.293	1.223
1.5	0.5	3.0	1.503	0.502	1.183	0.842	1.501	0.714	1.370	0.665
Std. Error of mean			≤ 0.0004		≤ 0.04		≤ 0.03		≤ 0.015	

Table 2. Values of parameter estimates

K	γ	Estimated values of K		
		Formula (14)	Formula (17)	Formula (20)
0.02	0.522664012	0.145821	0.140488304	0.01991
0.04	0.522492190	0.195760	0.196680272	0.03972
0.06	0.522216432	0.243474	0.239818334	0.05938
0.08	0.521844627	0.282825	0.275369698	0.07890
0.10	0.521384040	0.315803	0.306066754	0.09828