Lyapunov Exponents and Kaplan-yorke Dimension for Five-dimensional System

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Abstract: This work introduces a new high-dimensional five-dimensional system with chaotic and periodic solutions. For special values of parameters, we calculate the Kaplan-Yorke dimension and we show the dynamics of Lyapunov exponents. Some definitions and propositions are given. The main intent is to use the 2D and 3D projections of the 5D trajectories on different subspaces, to construct the graphs of solutions for understanding and managing the system. Visualizations where possible, are provided.

Keywords: chaos, KaplanYorke dimension, Lyapunov exponents, chaotic solution, periodic solution

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1 Introduction

Chaos is a phenomenon that is not easily classified. There is no universally accepted definition for chaos, [1]. The authors interpret chaos in their way and give their own definitions for the concept of chaos. But the authors mention three main features of chaos: 1) long-term aperiodic (nonperiodic) behavior; sensitivity to initial conditions; fractal structure, [1]. One of the features of irregular regimes is the instability trajectories belonging to a chaotic or strange attractor. The quantitative measure of this instability is the characteristic call exponents, originally introduced by Lyapunov to study the behavior of one trajectory. Positive Lyapunov exponents (LE) and a high KaplanYorke dimension D_{KY} guarantee a chaotic behavior for long-times, [2]. LE quantifies the average increment of an infinitely small error at the initial point. $LE > 0$ indicates that the dynamic system is sensitive to the initial condition; $LE = 0$ means the system is stable; and $LE < 0$ reflects that the system tends to stabilize, [3]. The dissipative dynamical system has at least one negative Lyapunov exponent, the sum of all Lyapunov exponents is the negative, [4]. The same system with different parameters can have periodic or chaotic solutions, [5]. In this case, the system has a bifurcation. The bifurcation is a change in the dynamics system, accompanied by the disappearance of some and the appearance of other regimes. Firstly, the stable point goes into the periodic regime, then to the chaotic regime, [6].

2 Definitions and propositions

Definition 2.1. *A chaotic system is a deterministic system that exhibits irregular and unpredictable behavior, [7].*

Definition 2.2. *A strange attractor, (chaotic attractor, fractal attractor) is an attractor that exhibits sensitivity to initial conditions, [1].*

Definition 2.3. *A fractal is an object that displays self-similarity under magnification and can be constructed using a simple motif (an image repeated on ever-reduced scales), [1].*

Such strange objects were identified in nature. Sunflowers and broccoli (Figure 2), sea shells, fern, snowflakes (Figure 1), mountain chasms, coastlines, lightning bolts (Figure 3), tree branches, river beds, turbulent eddies, human vascular system. These fractal geometries play a significant role in the characterization of chaotic dynamical processes, fractal dimension is therefore an important attribute of such a process, [8].

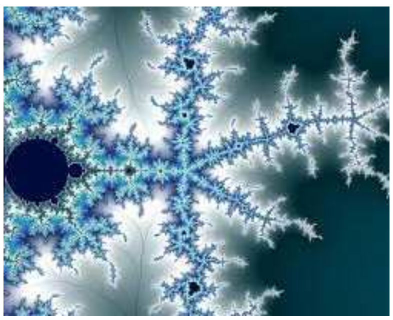

Figure 1: The picture from www.esa.org

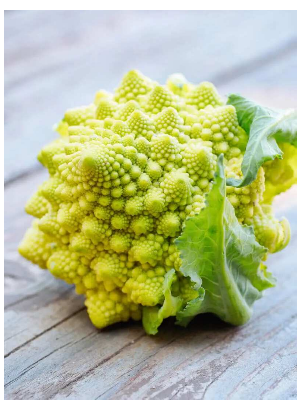

Figure 2: Remarkable Romanesco Broccoli. The picture from www.gardenbetty.com

Proposition 2.1. *Dissipative systems exhibit chaos for most initial conditions in a specified range of parameters. A conservative system exhibits periodic and quasi-periodic solutions for most values of parameters and initial conditions, and can exhibit chaos for special values only, [9].*

Proposition 2.2. *Only dissipative dynamical systems have attractors, [10].*

Figure 3: The picture from www.zmescience.com

3 Materials and methods

Our consideration is geometrical. The main intent is to use the 2D and 3D projections of the 5D trajectories on different subspaces, to construct the graphs of solutions for understanding and managing the system. Visualizations where possible, are provided. The dynamics of Lyapunov exponents are shown. Computations are performed using Wolfram Mathematics, [11]. In the article for Lyapunov exponents calculation the program Wolfram Mathematica “Lynch-DSAM.nb” was used, [1], [5].

3.1 The example of five-dimensional system

Consider the system

$$\begin{cases} x_1' = \tanh(x_4 - x_2) - bx_1, \\ x_2' = \tanh(x_1 + x_4) - bx_2, \\ x_3' = \tanh(x_1 + x_2 - x_4) - bx_3, \\ x_4' = \tanh(x_3 - x_2) - bx_4, \\ x_5' = \tanh(x_1 - x_2 - x_4 - x_5) - bx_5 \end{cases} \quad (1)$$

The initial conditions are

$$x_1(0) = 1.4;\ x_2(0) = 0.5;$$

$$x_3(0) = 1.4;\ x_4(0) = -1;\ x_5(0) = -1. \quad (2)$$

Table 1. Lyapunov exponents for the system (1), 2000 steps

b	LE_1	LE_2	LE_3	LE_4	LE_5
0.035	0	0	-0.06	-0.07	-0.08
0.036	0	0	-0.07	-0.07	-0.09
0.037	0	0	-0.06	-0.08	-0.09
0.038	0.02	0	-0.07	-0.09	-0.12
0.039	0.02	0	-0.07	-0.09	-0.11
0.04	0	0	-0.06	-0.09	-0.10
0.041	0.03	0	-0.07	-0.12	-0.15
0.042	0.04	0	-0.09	-0.12	-0.15
0.043	0.03	0	-0.07	-0.13	-0.15
0.044	0.03	0	-0.07	-0.13	-0.16
0.045	0.03	0	-0.07	-0.13	-0.16
0.046	0.01	0	-0.07	-0.13	-0.16
0.047	0.02	0	-0.07	-0.13	-0.16
0.048	0	0	-0.09	-0.11	-0.16
0.049	0	0	-0.06	-0.12	-0.17
0.05	0	0	-0.07	-0.12	-0.17
0.051	0	-0.4	-0.07	-0.09	-0.17

Let calculate the Kaplan-Yorke dimension using the following formula [12]

$$D_{KY} = j + \frac{1}{|L_{j+1}|}\sum_{j=1}^{j} L_j \qquad (3)$$

with j representing the index such that

$$\sum_{i=1}^{j} L_j > 0, \sum_{i=1}^{j+1} L_j < 0.$$

The formula (3) is called the Kaplan-Yorke formula after the names of the researchers who proposed this formula in 1979. The initial hypothesis was that this formula allows one to calculate the information dimension of attractors. For chaotic attractors of two-dimensional invertible mappings for which $\lambda_1 > 0$ and $\lambda_2 < 0$. This proposition has been proven rigorously. In the general case, the Kaplan-Yorke proposition cannot be proved.

Table 2. Kaplan-Yorke dimension for the system (1)

b	D_{KY}
0.035	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.38$
0.036	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.44$
0.037	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.44$
0.038	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.83$
0.039	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.72$
0.04	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.5$
0.041	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.93$
0.042	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.86$
0.043	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.87$
0.044	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.94$
0.045	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.94$
0.046	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.81$
0.047	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.88$
0.048	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.75$
0.049	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.94$
0.05	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 2.88$
0.051	$D_{KY} = 4 + \frac{\sum_i^4 LE_i}{\lvert LE_5\rvert} = 0.71$

Calculations showed the following:

- if $0.035 \leq b \leq 0.037$, then the system (1) has a quasi-periodic solution;
- if $0.038 \leq b \leq 0.039$, then the system (1) has a chaotic solution;
- if $b = 0.04$, then the system (1) has a quasi-periodic solution;
- $0.041 \leq b \leq 0.047$, then the system (1) has a chaotic solution;
- if $0.048 \leq b \leq 0.05$, then the system (1) has a quasi-periodic solution;
- if $b = 0.051$, then the system (1) has a periodic solution.

The projections of 5D trajectories on two-dimensional subspaces (x_1, x_5) and (x_1, x_4) are in the figures below.

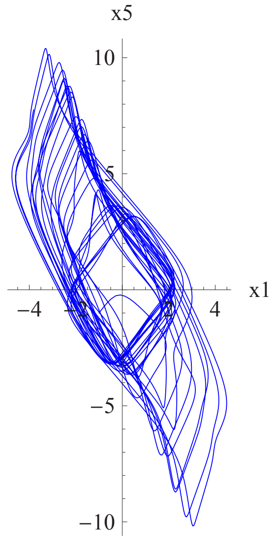

Figure 4: The projection of 5D trajectories to 2D subspace (x_1, x_5), $b = 0.042$.

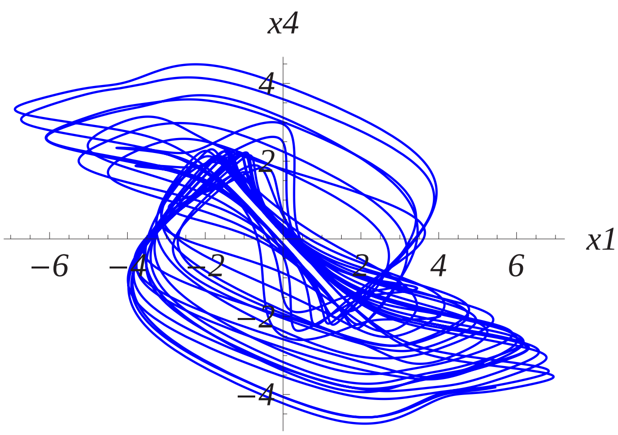

Figure 5: The projection of 5D trajectories to 2D subspace (x_1, x_4), $b = 0.042$.

The projections of 5D trajectories on three-dimensional subspaces (x_1, x_4, x_5), (x_1, x_2, x_4) and (x_1, x_3, x_4) are in the figures below.

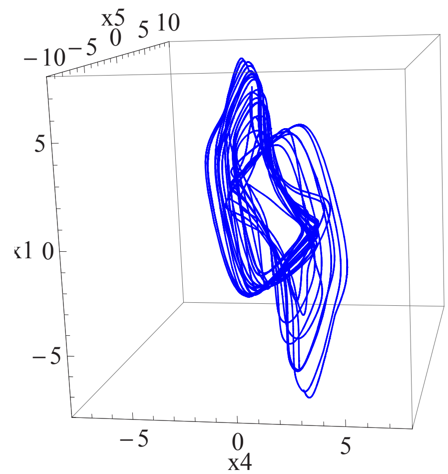

Figure 6: The projection of 5D trajectories to 3D subspace (x_1, x_4, x_5), $b = 0.042$.

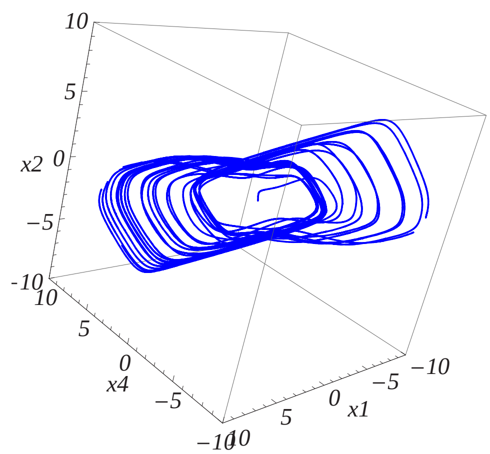

Figure 7: The projection of 5D trajectories to 3D subspace (x_1, x_2, x_4), $b = 0.042$.

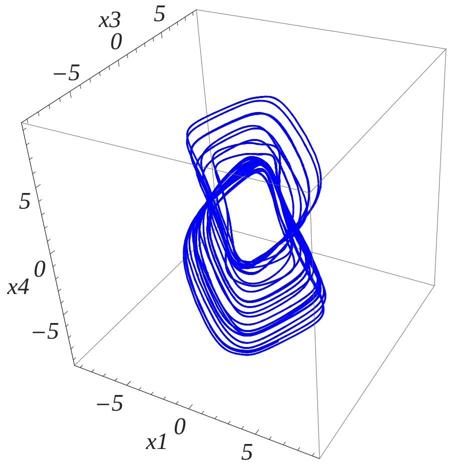

Figure 8: The projection of 5D trajectories to 3D subspace (x_1, x_3, x_4), $b = 0.042$.

The projections of 5D trajectories on two-dimensional subspaces (x_1, x_5) and (x_1, x_4) are shown in Figure 9 and Figure 10.

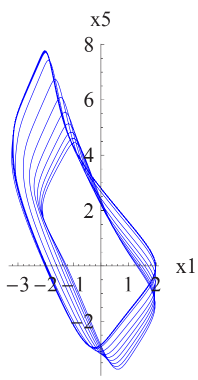

Figure 9: The projection of 5D trajectories to 2D subspace (x_1, x_5), $b = 0.051$.

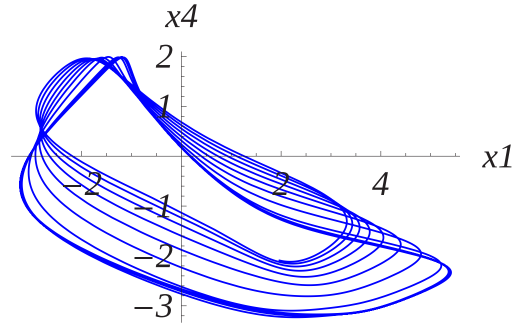

Figure 10: The projection of 5D trajectories to 2D subspace (x_1, x_4), $b = 0.051$.

The projections of 5D trajectories on three-dimensional subspaces (x_1, x_4, x_5), (x_1, x_2, x_4) and (x_1, x_3, x_4) are shown in Figure 11, Figure 12 and Figure 13.

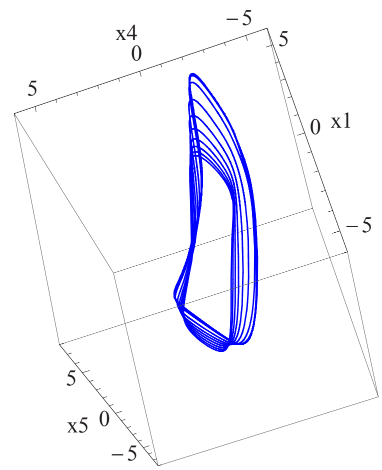

Figure 11: The projection of 5D trajectories to 3D subspace (x_1, x_4, x_5), $b = 0.051$.

Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ of the system (1), $b = 0.042$ are shown in Figure 14. Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ of the system (1), $b = 0.042$ are shown in Figure 15.

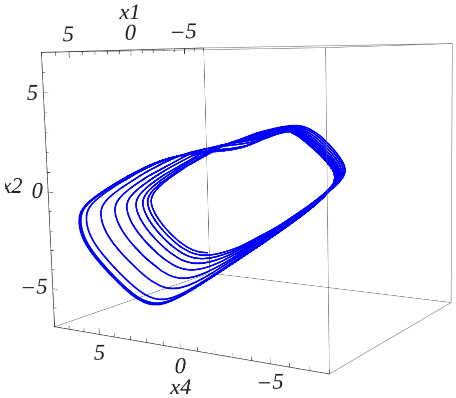

Figure 12: The projection of 5D trajectories to 3D subspace (x_1, x_2, x_4), $b = 0.051$.

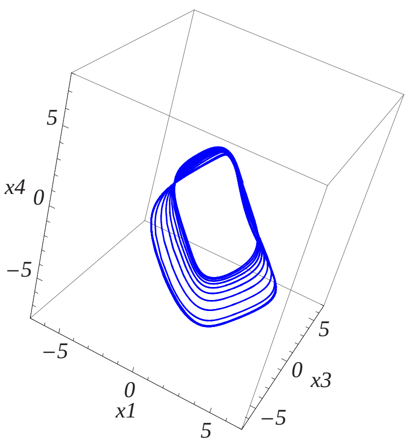

Figure 13: The projection of 5D trajectories to 3D subspace (x_1, x_3, x_4), $b = 0.051$.

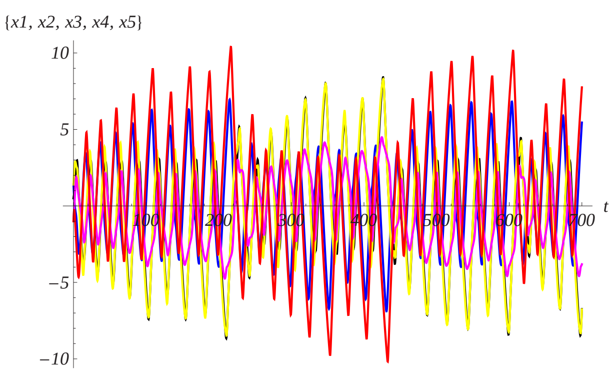

Figure 14: Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ of the system (1), $b = 0.042$.

The dynamics of Lyapunov exponents are shown in Figure 16 and Figure 17.

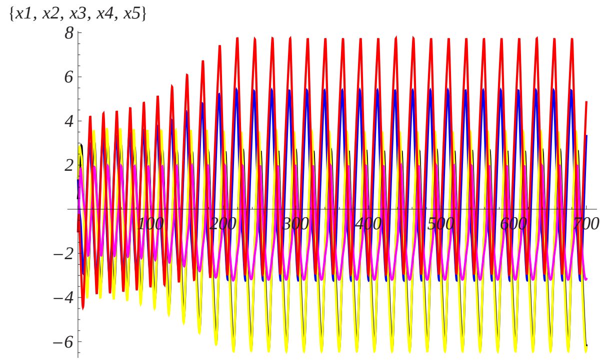

Figure 15: Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ of the system (1), $b = 0.051$.

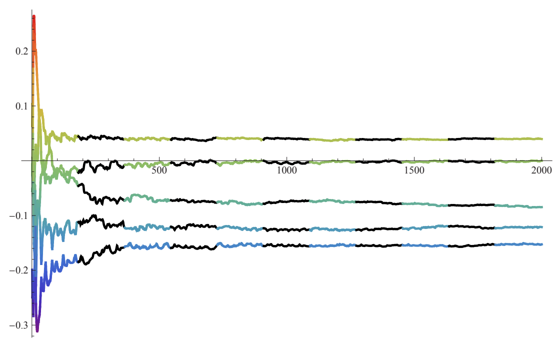

Figure 16: $b = 0.042$.

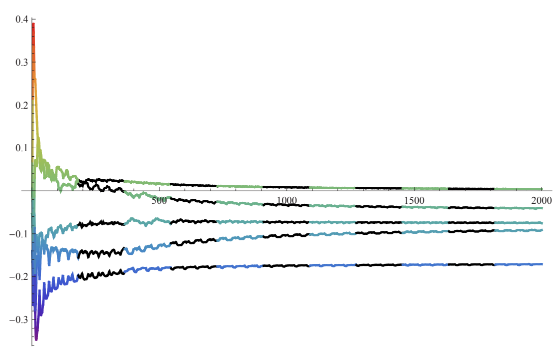

Figure 17: $b = 0.051$.

4 Conclusions

Mathematical systems with chaotic behavior are deterministic, that is, they obey some strict law and, in a sense, are ordered. This use of the word "chaos" differs from its usual meaning. Chaos theory says that complex systems are extremely dependent on initial conditions and small changes in the environment lead to unpredictable consequences. To study chaos, general mathematical principles and computer modeling are used.

This article showed that changing one parameter of the system (1) changes the solution of the system (a quasi-periodic solution, a chaotic solution, a periodic solution).

Lyapunov exponents are calculated using Wolfram Mathematica. Lyapunov exponents are one of the most useful diagnostic tools available for analyzing dynamical systems.

Visualizations of solutions of the system (1) are shown, projections onto two-dimensional and three-dimensional subspaces are provided.

Kaplan-Yorke dimension is calculated for the system (1) with different parameters.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Author Contributions:
All authors have contributed equally to creation of this article.

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