A sufficient condition for extinction and stability of a stochastic SIS model with random perturbation

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Abstract: The system dynamics of the randomly perturbed SIS depend on a certain threshold $\mathcal{R}_S$. If $\mathcal{R}_S < 1$, the disease is removed from our community, whereas an epidemic will occur if $\mathcal{R}_S > 1$. However, what happens when $\mathcal{R}_S = 1$? In this paper, we give a solution to this problem. Furthermore, we make some improvements to the free disease equilibrium state E_0 when $\mathcal{R}_S < 1$. Last, we give some computational simulations to explain our results.

Key-Words: Stochastic epidemic models, SIS models, Stability of disease, Extinction of disease, Threshold, Lyapunov function

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1 Introduction

The standard SIS epidemic model is defined as the following system

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt, \\ dI = (-(\mu+\gamma)I + \beta SI)\, dt, \end{cases} \tag{1}$$

where S and I are the numbers of susceptible and infected individuals, respectively. This model assumes a vital dynamic with a mortality rate that corresponds to the birth rate, implying that $S + I = 1$. Besides, β is the rate of infection, and γ is the rate of recovery. A deterministic form of system (1) given by the threshold $\mathcal{R}_0 = \dfrac{\beta}{\mu+\gamma}$ [3]. In other words, if $\mathcal{R}_0 \leq 1$, then the free disease equilibrium state $E_0\,(1,0)$ is globally asymptotically stable. While if $\mathcal{R}_0 > 1$, E_0 will become unstable, there is an endemic state of equilibrium $E_*\left(\dfrac{1}{\mathcal{R}_0}, \dfrac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\right)$ that is globally asymptotically stable. During the past few years, several mathematical programs for transmission dynamics of infectious diseases have been suggested [1, 2] such as (Susceptible-Infectious-Susceptible), SEIR (Susceptible-Exposed-Infectious-Recovered), SIRS (Susceptible-Infectious-Reduced-Susceptible). The purpose of building these models is to gain knowledge of the phenomenon of infectious diseases and forecast the consequences of applying public health actions to reduce the propagation of diseases, which helps us plan successful strategies for reducing the impact of infectious diseases. The SIS models provide adequate classifications of human population dynamics for particular bacterial diseases such as malaria, some protozoan diseases such as meningitis, and some sexually acquired diseases such as tuberculosis ("gonorrhea"), where individuals usually build up their immunity to the disease over 24 hours and do not develop any resistance to the disease when infected. There are various forms of model SIS diseases in continuous deterministic and stochastic settings in the literature (see, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11]). In [4], CAI. has considered the following stochastic form of model (1)

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt \\ \qquad -\sigma SI dB, \\ dI = (-(\mu+\gamma)I + \beta SI)\, dt \\ \qquad +\sigma SI dB. \end{cases} \tag{2}$$

According to the following initial conditions (S_0, I_0) in the set $\Delta = \left\{x \in \mathbb{R}^2_+; \quad x_1 + x_2 = 1\right\}$. Here, B is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and $\sigma > 0$ denotes the white noise intensity. CAI. [4] has shown that the set Δ is almost certainly positively invariant by the system (2). Next, the authors studied the dynamic behavior of $I(t)$ as a function of the new stochastic threshold $\mathcal{R}_S = \dfrac{\beta}{\mu+\gamma+\frac{1}{2}\sigma^2}$. They proved that if either $\mathcal{R}_S < 1$ and $\beta \geq \sigma^2$ or $\sigma^2 > \beta \vee \dfrac{\beta^2}{2(\mu+\gamma)}$, the disease will vanish. However, if $\mathcal{R}_S > 1$, then the disease will continue, and the model (2) will take a unique stationary distribution. CAI. [4] also suggested that if $\mathcal{R}_S < 1$ and $\beta < \sigma^2 \leq \dfrac{\beta^2}{2(\mu+\gamma)}$, then the disease disappears with the probability of 1. Now,

in our work, we consider the following deterministic system

$$\begin{cases} dS = \left(\mu - \mu S - \beta S^2 I + \gamma I\right) dt, \\ dI = \left(-(\mu+\gamma)I + \beta S^2 I\right) dt. \end{cases} \tag{3}$$

Using the technique of perturbation on the parameter β, we get the following stochastic form of the deterministic model (3)

$$\begin{cases} dS = \left(\mu - \mu S - \beta S^2 I + \gamma I\right) dt \\ \qquad -\sigma S^2 I dB, \\ dI = \left(-(\mu+\gamma)I + \beta S^2 I\right) dt \\ \qquad +\sigma S^2 I dB. \end{cases} \tag{4}$$

Therefore, it is enough to study the SDE for $I(t)$

$$\begin{aligned} dI &= \left[-(\mu+\gamma)I + \beta(1-I)^2 I\right] dt \\ &\quad +\sigma(1-I)^2 I dB, \\ &\triangleq f_1(I)dt + f_2(I)dB(t). \end{aligned} \tag{5}$$

For any twice continuously differentiable $V(.)$, the formula of Itô associated with (4) is defined by

$$dV(X) = \mathcal{L}V(X)dt + f_2(X)\frac{\partial V(X)}{\partial X}dB(t),$$

where

$$\mathcal{L}V(X) = f_1(X)\frac{\partial V(X)}{\partial X} + \frac{1}{2}f_2^2(X)\frac{\partial^2 V(X)}{\partial X^2},$$

is the generator of the process $X \in (0,1)$.
In this article, we assume that $\beta \geq \sigma^2$ is not exactly a limitation because it indicates that the estimation error σ^2 is smaller than the estimated value β. We investigated the case where $\mathcal{R}_S \leq 1$. More precisely, we show that if $\mathcal{R}_S < 1$, the equilibrium state without disease E_0 is κ-th exponentially stable moment. If $\mathcal{R}_S = 1$, E_0 is exponentially stable. Furthermore, the disease will be extinct on average.

2 Stability of disease

In this section, we will investigate the stability of the disease in the SDE system (4) to give the stochastic threshold condition for disease control or elimination.

Theorem 2.1 *Let $(S_0, I_0) \in \Delta$. If $\mathcal{R}_S < 1$, then for every κ such that*

$$0 < \kappa < \frac{2\beta}{\sigma^2}\left(\frac{1}{\mathcal{R}_S} - 1\right), \tag{6}$$

the solution $I(t)$ satisfies

$$\mathbb{E}\left(I^\kappa(t)\right) \leq I_0^\kappa e^{-\xi t},$$

where

$$\xi = -\kappa\left[\beta\left(1 - \frac{1}{\mathcal{R}_S}\right) + \frac{1}{2}\kappa\sigma^2\right] > 0. \tag{7}$$

Thus, the disease-free equilibrium state E_0 is κ-th moment exponentially stable.

Proof 1 *Let the function of Lyapunov $V(I) = I^\kappa$, where $\kappa > 0$ is real constants check the condition (6). By the formula of Itô, we obtain*

$$dV(I) = \mathcal{L}V(I)dt + \kappa\sigma(1-I)^2 I^\kappa dB, \tag{8}$$

or

$$\begin{aligned} \mathcal{L}V &= \kappa I^\kappa\left[-(\mu+\gamma) + \beta(1-I)^2\right. \\ &\quad \left. +\frac{1}{2}\sigma^2(\kappa-1)(1-I)^4\right], \\ &\leq \kappa I^\kappa\left[\sup_{0<x\leq 1}\left(-(\mu+\gamma) + \beta x^2 - \frac{1}{2}\sigma^2 x^4\right)\right. \\ &\quad \left. +\frac{\kappa}{2}\sigma^2\right], \\ &\triangleq \kappa I^\kappa\left(h(x) + \frac{\kappa}{2}\sigma^2\right). \end{aligned} \tag{9}$$

We can show clearly that if $\beta \geq \sigma^2$ and $\mathcal{R}_S < 1$, then

$$\begin{aligned} h(x) &= \sup_{0<x\leq 1}\left(-(\mu+\gamma) + \beta x^2 - \frac{1}{2}\sigma^2 x^4\right), \\ &= \beta\left(1 - \frac{1}{\mathcal{R}_S}\right). \end{aligned} \tag{10}$$

Combining this with (9), we get

$$\mathcal{L}I^\kappa(t) \leq -\xi I^\kappa(t),$$

where ξ is given in (7). Injecting it into (8), then integrating the result and taking the expectations on both sides, we get

$$\mathbb{E}\left(I^\kappa(t)\right) \leq I_0^\kappa - \xi\int_0^t \mathbb{E}\left(I^\kappa(u)\right) du,$$

which implies with the Gronwall inequality that

$$\mathbb{E}\left(I^\kappa(t)\right) \leq I_0^\kappa e^{-\xi t}.$$

The proof is finished.

Now, we will study the extinction of disease.

3 Extinction of disease

The following theorems discuss the situation where the stochastic threshold $\mathcal{R}_S = 1$.

Theorem 3.1 *For any initial value $(S_0, I_0) \in \Delta$, if $\mathcal{R}_S = 1$, then the solution of equation (5) follows*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds = 0. \tag{11}$$

Proof 2 *Let $(S_0, I_0) \in \Delta$ and the Lyapunov function*

$$V(I) = \log(I).$$

Using the formula of Itô, the equation (5), $I \leq 1$, the binomial formula of Newton, and $\mathcal{R}_S = 1$, we have

$$\begin{aligned} dV &= \left[-(\mu+\gamma) + \beta(1-I)^2 - \frac{1}{2}\sigma^2(1-I)^4\right]dt \\ &\quad +\sigma(1-I)^2 dB, \\ &\leq -\left(\beta - 4\sigma^2\right) I dt + \sigma(1-I)^2 dB. \end{aligned} \tag{12}$$

By integrating (12) from 0 to t, we obtain

$$\begin{aligned} \log I(t) &\leq \log I(0) - \left(\beta - 4\sigma^2\right)\int_0^t I(s)ds \\ &\quad +\sigma\int_0^t (1-I(s))^2 dB_s. \end{aligned} \tag{13}$$

According to the theorem of large numbers for martingales, there is a $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$, so that for every $\omega \in \Omega_1$ and $\epsilon > 0$, there is $T(w, \epsilon)$ such that for all $t \geq T$, we obtain

$$\log I(0) + \sigma\int_0^t (1-I(s))^2 dB_s \leq \epsilon t,$$

which implies with (13) that

$$\begin{aligned} e^{\epsilon t} &\geq I(t) e^{(\beta-4\sigma^2)\int_0^t I(s)ds}, \\ &\triangleq \frac{1}{(\beta-4\sigma^2)}\frac{d}{dt}\left[e^{(\beta-4\sigma^2)\int_0^t I(s)ds}\right]. \end{aligned} \tag{14}$$

By integrating (14) from T to t and multiplying both sides by $\frac{1}{t}$, one obtains

$$\begin{aligned} \frac{1}{t}\int_0^t I(s)ds &\leq \frac{1}{(\beta-4\sigma^2)\,t}\log\left[e^{(\beta-4\sigma^2)\int_0^T I(s)ds}\right. \\ &\quad \left. +\frac{\beta-4\sigma^2}{\epsilon}\left(e^{\epsilon T} - e^{\epsilon t}\right)\right]. \end{aligned} \tag{15}$$

Hence, applying the rule of Hospital on (15), we get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t I_s(\omega)ds \leq \frac{\epsilon}{\beta-4\sigma^2}.$$

Letting $\varepsilon \to 0$, we obtain the requested result (11).

Theorem 3.2 *Let $(S_0, I_0) \in \Delta$. If $\mathcal{R}_S = 1$, then for all $n > 0$ and $\varepsilon > 0$, we obtain*

$$\lim_{I_0\to 0} \mathbb{P}\left(\sup_{0\leq t\leq n} I(t) > \varepsilon\right) = 0, \tag{16}$$

that is, the disease-free steady state E_0 is stable in probability.

Proof 3 *Let $\kappa \leq 1$, $(S_0, I_0) \in \Delta$ and the Lyapunov function*

$$V(I(t)) = I^\kappa(t).$$

Using the formula of Itô, (8), (9), (10), and $\mathcal{R}_S = 1$, we obtain

$$dV(I(t)) \leq \frac{\kappa^2}{2}\sigma^2 I^\kappa + \kappa\sigma(1-I)^2 I^\kappa dB.$$

Integrating this inequality between $(0, t)$, and using $I \leq 1$, we can have easily for $\kappa \leq 1$

$$\begin{aligned} I^\kappa(t) - I^\kappa(0) &\leq \frac{\kappa^2}{2}\sigma^2 t \\ &\quad +\kappa\sigma\int_0^t (1-I(s))^2 I^\kappa(s) dB_s, \end{aligned}$$

thus

$$\begin{aligned} \sup_{0\leq t\leq n} I^\kappa(t) &\leq I^\kappa(0) + \frac{\kappa^2}{2}\sigma^2 n \\ &\quad +\kappa\sigma\sup_{0\leq t\leq n}\int_0^t (1-I(s))^2 I^\kappa dB_s. \end{aligned}$$

By $I < 1$, we obtain

$$\begin{aligned} \mathbb{P}\left(\sup_{0\leq t\leq n} I(t) > \varepsilon\right) &\leq \mathbb{I}_{I_0^\kappa \geq \frac{\varepsilon}{3}} + \mathbb{I}_{\frac{\kappa^2}{2}\sigma^2 n \geq \frac{\varepsilon}{3}} \\ &\quad +\mathbb{P}\left(\kappa\sigma\sup_{0\leq t\leq n} M_t > \frac{\varepsilon}{3}\right), \end{aligned}$$

where $\mathbb{I}_A$ denotes the characteristic function of A and

$$M_t = \int_0^t (1-I(s))^2 I^\kappa(s) dB_s,$$

which implies that

$$\begin{aligned} l &= \lim_{I_0\to 0}\mathbb{P}\left(\sup_{0\leq t\leq n} I(t) > \varepsilon\right), \\ &\leq \mathbb{I}_{\frac{\kappa^2}{2}\sigma^2 n \geq \frac{\varepsilon}{3}} \\ &\quad +\lim_{I_0\to 0}\mathbb{P}\left(\kappa\sigma\sup_{0\leq t\leq n} M_t > \frac{\varepsilon}{3}\right). \end{aligned} \tag{17}$$

Or, M_t is a continuous real-valued martingale, hence by the inequality of Doob, we obtain

$$\begin{aligned} P &= \mathbb{P}\left(\kappa\sigma \sup_{0\le t\le n} M_t > \frac{\varepsilon}{3}\right), \\ &\le \frac{9\kappa^2\sigma^2}{\varepsilon^2}\mathbb{E}\left(\left(\int_0^\eta (1-I(s))^2 I^\kappa(s) dB_s\right)^2\right), \\ &\triangleq \frac{9\kappa^2\sigma^2}{\varepsilon^2}\mathbb{E}\left(\int_0^\eta \left((1-I(s))^2 I^\kappa(s)\right)^2 ds\right), \\ &\le \frac{9\kappa^2\sigma^2}{\varepsilon^2}\eta. \end{aligned}$$

Using it in combination with (17), we get

$$\lim_{I_0\to 0}\mathbb{P}\left(\sup_{0\le t\le n} I(t) > \varepsilon\right) \le \mathbb{I}_{\frac{\kappa^2\sigma^2 n}{2}\ge\frac{\varepsilon}{3}} + \frac{9\kappa^2\sigma^2\eta}{\varepsilon^2}.$$

By letting $\kappa \to 0$, we get the required statement (16).

4 Simulation

The following simulation illustrates that if $\mathcal{R}_S = 1$, the stochastic disease will die when the deterministic illness occurs.

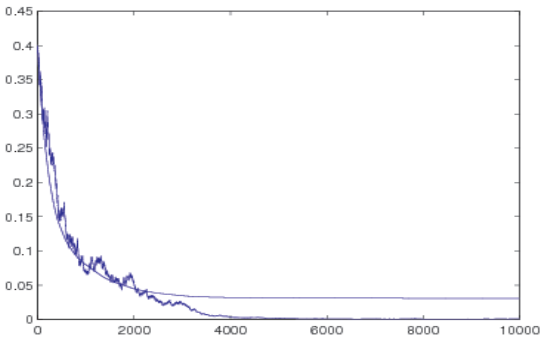

Figure 1: Single path computer simulation of $I(t)$ for the SDE model (5) with initial condition $I_0 = 0.4$ and its related deterministic model for the parameters: $\mu = 0.5$, $\beta = 0.902$, $\gamma = 0.4$, $\sigma = 0.2$, then $\mathcal{R}_0 > 1$ and $\mathcal{R}_S = 1$.

5 Conclusion

This article studied a stochastic SIS epidemiological model with a constant population size under white noise control. We discussed the behavior of the stochastic epidemiological SIS model over the long term. We show sufficient conditions for the extinction and stability of the disease. The stochastic population model provides one of several possible stochastic forms of the deterministic model. This model is generalizable. The argument is that the population can experience sudden changes in its parameters [5].

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Mourad El idrissi and Adel Settati: carried out the conceptualization, validation, formal analysis, writing - original draft, methodology, writing - review & editing.

Bilal Harchaoui and Aadil Lahrouz: have implemented the software, formal analysis, writing - original draft, writing - review & editing.

Abdeladim Nait Brahim and Ibrahim Bouzalmat: were responsible for validation, investigation, conceptualization, writing - review & editing.

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