A New Consideration of the Influence of Shelter on the Kinetic Behavior of the Leslie-Gower Predator Prey System with Fear Effect

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Abstract: - In this study, a Leslie-Gower predator-prey model that incorporates both fear effect and shelter is presented and investigated. It is assumed that predator species only capture and cause fear in prey species outside the refuge, but have no impact on prey species inside the refuge. We demonstrate that the fear effect and the refuge have no impact on the positive equilibrium's existence and local stability. Next, we explore the system's persistence characteristic. By applying the Bendixson-Dulac criterion, we demonstrate that the requirement assures the system's permanence is enough to guarantee the global attractivity of the positive equilibrium. According to our investigation, the birth rate of prey species and the refuge are two of the most critical factors in ensuring the sustainable development of the system.

Key-Words: Predator; Prey; Fear effect; Refuge

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1 Introduction

The study of the predator-prey system dominates the field of biomathematics due to its universal existence.

Leslie [1, 2] provided the following predator-prey model:

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H)H,$$

$$\frac{dP}{dt} = (r_2 - a_2 \frac{P}{H})P,$$
(1)

here H and P represent the density of prey species and the predator species at time t, respectively. The approach described above permits a unique coexisting fixed point

$$H^* = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}, \ P^* = \frac{r_1 r_2}{a_1 r_2 + a_2 b_1}.$$
 (2)

By developing an appropriate Lyapunov function, Korobeinikov [3] demonstrated that the positive equilibrium is globally stable.

It is well known that prey species may stay in the refuge to avoid the capture of predator species. This decreases the likelihood of extinction as a result of predation. Chen, Chen, and Xie [5] introduced the Leslie-Gower predator-prey model with a prey sanctuary as follows:

$$\frac{dH}{dt} = (r_1 - b_1 H)H - a_1(1 - m)HP,$$

$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1 - m)H}\right)P,$$
(3)

where $m \in (0,1)$ is constant, and the authors assumed that there is a refuge sheltering mH of the prey. This makes (1 - m)H of the prey accessible to the predator. The system, as was shown by the authors, permits a globally stable positive equilibrium. According to their research, the refuge has a complicated effect on the ultimate density of predator species.

Recent research has shown that fear of predators alters anti-predator defenses to such a degree that it drastically reduces prey reproduction. Wang, Zanette, and Zou [6] presented the following generic preypredator model, reflecting the cost of fear:

$$\begin{aligned} \frac{du}{dt} &= ur_0 f(k, v) - du - au^2 - g(u)v, \\ \frac{dv}{dt} &= v \Big(-m + cg(u) \Big). \end{aligned}$$

$$(4)$$

In this system, f(k, v) compensates for the cost of anti-predator defense owning to fear, with $f(k, v) = \frac{1}{1+kv}$ being one of the viable expression. Since the pioneering work of Wang, Zanette, and Zou [6], many scholars have studied the predator-prey system with the fear impact on the prey species, see [5]-[28] and the references cited therein. For more work on Leslie-Gower predator prey system, one could refer to [29]-[32] and the references cited therein.

Exploring the dynamic behaviors of a predatorprey system that contains both a fear impact and a refuge is natural. Indeed, several scholars [7, 13, 16, 20] had done works on this direction.

The following Holling type III prey-predator system with both fear effect and prey shelter was researched by Xie and Zhang [13]:

$$\frac{dx}{dt} = \frac{ax}{1+ny} - bx^2 - \frac{\alpha(1-m)^2 x^2 y}{\beta^2 + (1-m)^2 x^2},
\frac{dy}{dt} = -cy + \frac{k\alpha(1-m)^2 x^2 y}{\beta^2 + (1-m)^2 x^2}.$$
(5)

Their research has shown that an increase in fear levels might enhance system stability by removing periodic solutions and reducing predator species abundance at the coexist equilibrium.

Zhang, Cai, Fu, and Wang [16] conducted research on the Holling type II prey-predator system described below, which includes both the fear effect and the prey shelter:

$$\frac{dx}{dt} = \frac{\alpha x}{1+Ky} - bx^2 - \frac{\beta(1-m)xy}{\beta+(1-m)x},$$

$$\frac{dy}{dt} = -\gamma y + \frac{c\beta(1-m)xy}{\beta+(1-m)x}.$$
(6)

The authors performed comprehensive research of the aforementioned system and received comprehensive results.

With the aim of finding out the combined effect of prey refuge, fear effect, and Allee effect, the following predator-prey model was examined by Huang, Zhu, and Li [7]:

$$\frac{dv}{dt} = rv\left(1 - \frac{v}{k}\right)\left(v - \theta_0\right)\frac{1}{1 + ky}$$
$$-a(1 - \eta)uv, \tag{7}$$

$$\frac{du}{dt} = a\alpha(1-\eta)uv - m_0v.$$

They demonstrated how the system's dynamic behaviors might become more complex via boosting the prey shelter or Allee effect or the fear effect, which does not change the density of prey but might reduce the density of predator species.

Firdiansyah [20] employed a Leslie-Gower predator-prey model with Beddington-DeAngelis functional response to examine the influence of fear; the model had the following form:

$$\frac{dx}{dt} = \frac{r_1 x}{1 + Ky} - bx - px^2$$
$$-\frac{\alpha(1 - m)xy}{a + b(1 - m)x + cy}, \qquad (8)$$

$$\frac{dy}{dt} = y\Big(r_2 - \frac{\beta y}{(1-m)x + \gamma}\Big).$$

The author demonstrated that an increase in fear might reduce the population density of both species. However in the event of a constant fear rate, the prey refuge is beneficial to the survival of both species.

It brings to our attention that in the systems (5)-(8), they all assumed that the prey species staying in the refuge also has a fear effect to the predator species. However, we argued that for those in the refuge, since predators could not find them, predator species certainly have no influence on them, no matter the direct killing or the anti-predator behaviors to reduce the birth rate.

To this day, no academic disputes this assertion. We believe providing a more appropriate model and examining the system's dynamic behaviour is preferable. In fact, we shall investigate the dynamic characteristics of the subsequent model.

$$\frac{dH}{dt} = r_{11}mH + \frac{r_{11}(1-m)H}{1+kP} -r_{12}H - b_1H^2 - a_1(1-m)HP, \quad (9)$$
$$\frac{dP}{dt} = \left(r_2 - a_2\frac{P}{(1-m)H}\right)P,$$

now, The birth rate of the prey species is r_{11} , and the prey species' fear effect is $\frac{1}{1+kP}$. We assume that the predator only impacts prey outside the refuge and has no influence on those prey species that remain inside the refuge.

One could easily see that if m = 0, k = 0, i.e., without considering the influence of refuge and fear effect, system (1.9) will degenerate to the following system

$$\frac{dH}{dt} = (r_{11} - r_{12})H - b_1 H^2 - a_1 HP,
\frac{dP}{dt} = (r_2 - a_2 \frac{P}{H})P,$$
(10)

then, if $r_{11} > r_{12}$, system (10) is equivalent to system (1).

Also, if only restrict k = 0 in the system (9), then the model will degenerate to

$$\frac{dH}{dt} = (r_{11} - r_{12})H - b_1 H^2
-a_1(1 - m)HP,$$
(11)
$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1 - m)H}\right)P,$$

then, if $r_{11} > r_{12}$, system (11) is equivalent to system (3).

The purpose of this study is to thoroughly analyze the dynamical behavior of the system (9) and to give a positive answer on how m, k affects the dynamical behavior of the system. The essay's remaining sections are grouped as follows: In the next part, we will examine the presence of equilibrium states and their local stability properties. In Sections 3 and 4, respectively, the properties of permanence and global stability were examined. The impact of the fear effect and shelter is then covered in Section 5. We describe the related modeling and findings in Section 6 to demonstrate the key distinction between our model and the model with prey species experiencing the fear effect in all cases. The fundamental distinction between our model and the one with all prey suffering due to the fear effect is then shown. To demonstrate the plausibility of the key conclusions, numerical simulations are undertaken. Discussion followed the paper's conclusion.

2 Existence of equilibria and their local stability property

In this part, we will investigate the existence of equilibrium states and their local stability quality.

Theorem 2.1. If $r_{11} > r_{12}$ is true, then system (9) admits two nonnegative equilibria: $E_1\left(\frac{r_{11}-r_{12}}{b_1},0\right)$ and $E_2(H^*, P^*)$, where

$$H^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1},$$

$$P^* = \frac{r_2(1-m)H^*}{a_2},$$
(12)

here

$$A_{1} = k(a_{1}(-1+m)^{2}r_{2} + a_{2}b_{1})(1-m)r_{2} > 0,$$

$$A_{2} = \left(((mr_{11} - r_{12})k + a_{1}(-1+m))(-1+m)r_{2} + a_{2}b_{1} \right)a_{2},$$

$$A_{3} = -a_{2}^{2}(r_{11} - r_{12}) < 0.$$
(13)

Proof. The existence of nonnegative equilibria of the system (9) is determined by the following equations:

$$r_{11}mH + \frac{r_{11}(1-m)H}{1+kP} - r_{12}H$$

$$r - b_1H^2 - a_1(1-m)HP = 0, \qquad (14)$$

$$\left(r_2 - a_2\frac{P}{(1-m)H}\right)P = 0.$$

It follows directly from the second equation of (14) that

$$P = 0 \text{ or } P = \frac{r_2(1-m)H}{a_2}.$$
 (15)

Incorporating P = 0 into the first equation, if $r_{11} > r_{12}$, the system admits a nonnegative boundary equilibrium $E_1\left(\frac{r_{11} - r_{12}}{b_1}, 0\right)$. By substituting $P = \frac{r_2(1-m)H}{a_2}$ into the first equation and simplifying,

we get the equation

$$A_1H^2 + A_2H + A_3 = 0, (16)$$

where A_1, A_2 and A_3 are specified by (13). (16) allows a unique positive solution H^* , consequently, system (9) permits a single positive equilibrium $E_2(H^*, P^*)$ due to the presence of H^* .

Theorem 2.1 has now been proved.

Theorem 2.2. Assume that $r_{11} > r_{12}$ holds, then $E_2(H^*, P^*)$ is locally asymptotically stable and $E_1\left(\frac{r_{11}-r_{12}}{b_1}, 0\right)$ is unstable.

Proof. The system's Jacobian matrix (9) can be represented as

$$J(H,P) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (17)$$

where

$$A_{11} = r_{11}m + \frac{r_{11}(1-m)}{kP+1} - r_{12}$$

$$-2b_1H - a_1(1-m)P,$$

$$A_{12} = -\frac{r_{11}(1-m)Hk}{(kP+1)^2} - a_1(1-m)H,$$

$$A_{21} = \frac{P^2a_2}{(1-m)H^2},$$

$$A_{22} = r_2 - \frac{2a_2P}{(1-m)H}.$$

(18)

The variational matrix's characteristic equation is

$$\lambda^2 - tr(J)\lambda + det(J) = 0.$$
(19)

As long as tr(J) < 0 and det(J) > 0, which indicates that both eigenvalues have negative real components, the asymptotic stability of an equilibrium solution for a continuous-time system is satisfied.

At the equilibrium $E_1\left(\frac{r_{11}-r_{12}}{b_1},0\right)$, the Jacobian matrix is represented by

$$J\left(E_{1}\left(\frac{r_{11}-r_{12}}{b_{1}},0\right)\right) = \begin{pmatrix} r_{12}-r_{11} & B \\ 0 & r_{2} \end{pmatrix},$$

$$\begin{pmatrix} (1+m)(n-r_{12})(n-k+r_{12}) \end{pmatrix}$$
(20)

where $B = \frac{(-1+m)(r_{11}-r_{12})(r_{11}k+a_1)}{b_1}$. Hence, it has one positive characteristic root $\lambda_1 = r_2$, consequently, $E_1\left(\frac{r_{11}-r_{12}}{b_1},0\right)$ is unstable. Noting that $E_2(H^*, P^*)$ satisfies the equation

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP^*} - r_{12}$$

$$-b_1H^* - a_1(1-m)P^* = 0, \qquad (21)$$

$$r_2 - a_2\frac{P^*}{(1-m)H^*} = 0.$$

By applying (21), at the equilibrium $E_2(H^*, P^*)$, we have

$$J(E_{2}(H^{*}, P^{*}))$$

$$= \begin{pmatrix} -H^{*}b_{1} & C_{1} \\ C_{2} & C_{3} \end{pmatrix}.$$
(22)

where

$$C_{1} = -a_{1}H^{*}(1-m) - \frac{r_{11}(1-m)H^{*}k}{(kP^{*}+1)^{2}},$$

$$C_{2} = \frac{a_{2}(P^{*})^{2}}{(1-m)(H^{*})^{2}},$$

$$C_{3} = -\frac{a_{2}P^{*}}{(1-m)H^{*}}.$$

Then we have

$$Det J \Big(E_2(H^*, P^*) \Big)$$

$$= -\frac{H^* b_1 a_2 P^*}{(1-m)H^*}$$

$$+ \Big(a_1 H^* (1-m) + \frac{r_{11} (1-m) H^* k}{(kP^*+1)^2} \Big) \times$$

$$\frac{a_2(P^*)^2}{(1-m)(H^*)^2}$$

$$> 0,$$

and

$$TrJ(E_2(H^*, P^*)) = -H^*b_1 - \frac{a_2P^*}{(1-m)H^*} < 0.$$

Consequently, $E_2(H^*, P^*)$ is locally asymptotically stable.

The proof of Theorem 2.2 is now complete.

3 Permanence

By means of permanence, we imply that the positive solution of the system has a positive upper and lower bound after a sufficient amount of time, and that these bounds are independent of the solution. If the system (9) is permanent, predator and prey species will cohabit on a long-term basis.

Set

$$P_1 \stackrel{\text{def}}{=} \frac{r_2(1-m)\frac{r_{11}-r_{12}}{b_1}}{a_2}.$$
 (23)

Theorem 3.1. Assuming that

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP_1} > r_{12} + a_1(1-m)P_1 \quad (24)$$

holds, system (9) is permanent.

Proof. For any enough small positive constants $\varepsilon > 0$, set

$$P_1^{\varepsilon} \stackrel{\text{def}}{=} \frac{r_2(1-m)\frac{r_{11}-r_{12}}{b_1}}{a_2} + \varepsilon.$$
 (25)

Inequality (24) implies that for any enough small positive constants $\varepsilon > 0$, the following inequalities

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP_1^{\varepsilon}} > r_{12} + a_1(1-m)P_1^{\varepsilon},$$

$$\frac{r_{11}m + \frac{r_{11}(1-m)}{1+kP_1} - r_{12} - a_1(1-m)P_1}{b_1} - \varepsilon > 0$$
(26)

holds.

By using the first equation of system (9), we have

$$\frac{dH}{dt} = r_{11}mH + \frac{r_{11}(1-m)H}{1+kP}
-r_{12}H - b_1H^2 - a_1(1-m)HP
\leq r_{11}mH + \frac{r_{11}(1-m)H}{1+kP}
-r_{12}H - b_1H^2
\leq (r_{11} - r_{12} - b_1H)H,$$
(27)

By using Lemma 2.3 of [23] to (27), it follows that

$$\limsup_{t \to +\infty} H(t) \le \frac{r_{11} - r_{12}}{b_1}.$$
 (28)

Hence, for $\varepsilon > 0$ which satisfies inequality (26), there exists a $T_1 > 0$, such that

$$H(t) < \frac{r_{11} - r_{12}}{b_1} + \varepsilon$$
 for all $t \ge T_1$. (29)

For $t > T_1$, we have, according to the second equation of (9),

$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1-m)H}\right)P$$

$$\leq \left(r_2 - a_2 \frac{P}{(1-m)\left(\frac{r_{11} - r_{12}}{b_1} + \varepsilon\right)}\right)P,$$
(30)

Applying Lemma 2.3 of [23] to (30) leads to

$$\limsup_{t \to +\infty} P(t) \le \frac{r_2(1-m)\Big(\frac{r_{11}-r_{12}}{b_1}+\varepsilon\Big)}{a_2}.$$
 (31)

Since $\varepsilon > 0$ enough small, setting $\varepsilon \to 0$ in the above inequality, one has

$$\limsup_{t \to +\infty} P(t) \le \frac{r_2(1-m)\frac{r_{11}-r_{12}}{b_1}}{a_2}.$$
 (32)

For $\varepsilon > 0$ which satisfies inequality (25), there exists a $T_2 > T_1$, such that for all $t \ge T_2$,

$$P(t) < \frac{r_2(1-m)\frac{r_{11}-r_{12}}{b_1}}{a_2} + \varepsilon \stackrel{\text{def}}{=} P_1^{\varepsilon}.$$
 (33)

For $t > T_2$, the first equation of system (9) yields

$$\frac{dH}{dt} = r_{11}mH + \frac{r_{11}(1-m)H}{1+kP}
-r_{12}H - b_1H^2 - a_1(1-m)HP
\geq r_{11}mH + \frac{r_{11}(1-m)H}{1+kP_1^{\varepsilon}}
-r_{12}H - b_1H^2 - a_1(1-m)HP_1^{\varepsilon}
= \left(r_{11}m + \frac{r_{11}(1-m)}{1+kP_1^{\varepsilon}} - r_{12} - a_1(1-m)P_1^{\varepsilon} - b_1H\right)H,$$
(34)

Applying Lemma 2.3 of [23] to (34) results in

$$\liminf_{t \to +\infty} H(t) \ge \frac{D_{1\varepsilon}}{b_1}.$$
(35)

where

$$D_{1\varepsilon} = r_{11}m + \frac{r_{11}(1-m)}{1+kP_1^{\varepsilon}} - r_{12} - a_1(1-m)P_1^{\varepsilon}.$$

Setting $\varepsilon \to 0$ in the inequality shown above, we get

$$\liminf_{t \to +\infty} H(t) \ge \frac{D_1}{b_1},\tag{36}$$

where $D_1 = r_{11}m + \frac{r_{11}(1-m)}{1+kP_1} - r_{12} - a_1(1-m)P_1$.

Hence, for $\varepsilon > 0$ which satisfies inequality (26), there exists a $T_3 > T_2$, such that

$$H(t) > \frac{D_1}{b_1} - \varepsilon \quad \text{for all } t \ge T_3. \tag{37}$$

For $t > T_3$, the second equation of (9) provides us

$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1-m)H}\right)P$$

$$\geq \left(r_2 - a_2 \frac{P}{(1-m)\left(\frac{D_1}{b_1} - \varepsilon\right)}\right)P,$$
(38)

Applying Lemma 2.3 of [23] to (38) leads to

$$\liminf_{t \to +\infty} P(t) \ge \frac{r_2(1-m)\left(\frac{D_1}{b_1} - \varepsilon\right)}{a_2}.$$
 (39)

Since $\varepsilon>0$ enough small, setting $\varepsilon\to 0$ in the above inequality, one has

$$\liminf_{t \to +\infty} P(t) \ge \frac{r_2(1-m)\frac{D_1}{b_1}}{a_2}.$$
 (40)

The equations (28), (32), (36) and (40) demonstrate that if (24) holds, the system (9) is permanent. Theorem 3.1 is proved.

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4 Global stability

The subsequent Theorem addresses the global stability of the positive equilibrium E_2 .

Theorem 4.1. Assuming

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP_1} > r_{12} + a_1(1-m)P_1 \quad (41)$$

holds, the positive equilibrium $E_2(H^*, P^*)$ is globally stable.

Proof. Already, we had showed in Theorem 2.2 that under the assumption $r_{11} > r_{12}$, system (1) permits a locally asymptotically stable positive equilibrium $E_2(H^*, P^*)$. To demonstrate the attractivity of E_2 , we first claim that (9) permits no limit cycles inside the first quadrant. Set the Dulac function as $B = \frac{1}{TTP}$. Then

$$= HP^{-1} Her$$

$$= \frac{\partial (F_1B)}{\partial H} + \frac{\partial (F_2B)}{\partial P}$$

$$= \frac{E_{11}}{HP} - \frac{E_{12}}{H^2P} - \frac{a_2}{(1-m)H^2}$$

$$= -\frac{b_1}{P} - \frac{a_2}{(1-m)H^2} < 0.$$
(42)

where

$$E_{11} = r_{11}m + \frac{r_{11}(1-m)}{kP+1}$$

-r_{12} - 2b_1H - a_1(1-m)P,
$$E_{12} = r_{11}mH + \frac{r_{11}(1-m)H}{kP+1}$$

Now the claim follows from Dulac Theorem. Then the claim, combined with permanence guaranteed by Theorem 3.1 and the Bendixson-Dulac criterion, tells us that all solutions with positive initial conditions approach $E_2(H^*, P^*)$ as $t \to \infty$. The conclusion of Theorem 4.1 is now followed.

5 Influence of fear effect and refuge

Let us denote

$$F(H^*, P^*, m, k)$$

$$= r_{11}m + \frac{r_{11}(1-m)}{1+kP^*}$$

$$-r_{12} - b_1H^* - a_1(1-m)P^*, \quad (43)$$

$$G(H^*, P^*, m, k)$$

$$= r_2 - a_2 \frac{P^*}{(1-m)H^*}.$$

Then $E_2(H^*, P^*)$ satisfies the equation

$$\begin{cases} F(H^*, P^*, m, k) = 0, \\ G(H^*, P^*, m, k) = 0. \end{cases}$$
(44)

By straightforward calculation, we have

$$J = \frac{D(F,G)}{D(H^*,P^*)} = \begin{vmatrix} F_{H^*} & F_{H^*} \\ G_{H^*} & G_{P^*} \end{vmatrix}$$

$$= \begin{vmatrix} -b_1 & F_1 \\ F_2 & F_3 \end{vmatrix}$$

$$= \frac{b_1 a_2}{(1-m) H^*} + F_1 F_2$$
(45)

where

> 0.

$$F_{1} = -\frac{r_{11} (1 - m) k}{(kP^{*} + 1)^{2}} - a_{1} (1 - m) ,$$

$$F_{2} = \frac{a_{2}P^{*}}{(1 - m) (H^{*})^{2}},$$

$$F_{3} = -\frac{a_{2}}{(1 - m) H^{*}}.$$

Using implicit function set theorem, the equation (44) has an unique solution in the neighborhood of $E_2(H^*, P^*)$

 $H^* = H^*(m,k), \ P^* = P^*(m,k).$ (46)

and

$$\frac{\partial H^*}{\partial k} = -\frac{1}{J} \frac{D(F,G)}{D(k,P^*)},$$

$$\frac{\partial P^*}{\partial k} = -\frac{1}{J} \frac{D(F,G)}{D(H^*,k)},$$

$$\frac{\partial H^*}{\partial m} = -\frac{1}{J} \frac{D(F,G)}{D(m,P^*)},$$

$$\frac{\partial P^*}{\partial m} = -\frac{1}{J} \frac{D(F,G)}{D(H^*,m)}.$$
(47)

By computation, we have

$$\frac{\partial H^*}{\partial k} = -\frac{1}{J} \frac{r_{11} P^* a_2}{(kP^* + 1)^2 H^*} < 0, \qquad (48)$$

that is, the prey density H* decreases as k increases; (2)

$$\frac{\partial P^*}{\partial k} = -\frac{1}{J} \frac{r_{11}(P^*)^2 a_2}{(kP^*+1)^2 (H^*)^2} < 0, \tag{49}$$

that is, the predator density P^* decreases with increasing k; The reason may be relying on the fact that with the fear of predator species, the prey's density will decrease, and less of the food resource will finally reduce the density of predator species, too;

(3)

(1)

$$\frac{\partial H^*}{\partial m} = -\frac{1}{J} \frac{K_1}{H^* \left(-1+m\right) \left(kP^*+1\right)^2} > 0, \quad (50)$$

where

$$K_{1} = 2a_{2} \left((P^{*})^{2}a_{1}k^{2} + \left(\frac{1}{2}k^{2}r_{11} + 2ka_{1}\right)P^{*} + r_{11}k + a_{1} \right)P^{*}.$$

that is, the prey density, H^* , increase as m increases. (4)

$$\frac{\partial P^*}{\partial m} = -\frac{1}{J} \frac{a_2 P^* G}{(-1+m)^2 (H^*)^2 (kP^*+1)},$$
 (51)

where

$$G = b_1(kP^*+1)H^* + (-1+m)(ka_{11}P^*+r_{11}k+a_1)P^*$$
(52)

The sign of $\frac{\partial P^*}{\partial m}$ depends on the sign of the term G. (i)If

$$b_1(kP^*+1)H^* - (ka_1P^*+r_{11}k+a_1)P^* > 0,$$
 (53)

we obtain $\frac{\partial P^*}{\partial m} < 0$, Thus, predator density decreases with increasing refuge size. (ii)If

$$b_1(kP^*+1)H^* - (ka_1P^*+r_{11}k+a_1)P^* < 0,$$
 (54)

then there exists a

$$m^* = 1 - \frac{b_1(kP^* + 1)H^*}{(ka_1P^* + r_{11}k + a_1)P^*}$$
(55)

such that

$$\frac{\partial P^*}{\partial m} > 0 \quad \text{for all } 0 < m < m^*, \qquad (56)$$

$$\frac{\partial P^*}{\partial m} < 0 \quad \text{for all } 1 > m > m^*, \qquad (57)$$

That is, for $0 < m < m^*$, P^* is the increasing function of m, while for m sufficiently big, maybe as more prey species remain in the refuge, predators have insufficient food supplies, leading to a decrease in the ultimate density of predator species.

6 Prey species all suffer from fear effect

Note that in the system (9), we assume that prey species are divided into two classes: outside the refuge and inside the refuge. Only those outside the refuge have a fear effect. Hence, it is natural to compare our results with previously scholars' works. However, since (5)-(8) incorporating the functional response or Allee effect, We were unable to contrast our findings with their findings directly. In this section, we would like to study the following model.

$$\frac{dH}{dt} = \frac{r_{11}H}{1+kP} - r_{12}H -b_1H^2 - a_1(1-m)HP,$$
(58)

$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1-m)H}\right)P.$$

We will only state the results but omit the detail proof.

Theorem 6.1. Assume that $r_{11} > r_{12}$ holds, then system (58) admits a boundary equilibrium $F_1\left(\frac{r_{11}-r_{12}}{b_1},0\right)$ and a unique positive equilibrium $F_2(H_1^*, P_1^*)$, where

$$H^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1},$$

$$P^* = \frac{r_2(1-m)H^*}{a_2},$$
(59)

here

$$B_{1} = kr_{2}(1-m) \left(a_{1}m^{2}r_{2} - 2a_{1}mr_{2} + a_{1}r_{2} + a_{2}b_{1} \right) > 0,$$

$$B_{2} = a_{2} \left(a_{1}m^{2}r_{2} - kmr_{12}r_{2} - 2a_{1}mr_{2} + kr_{12}r_{2} + a_{1}r_{2} + a_{2}b_{1} \right),$$

$$B_{3} = -a_{2}^{2}(r_{11} - r_{12}) < 0.$$
(60)

Theorem 6.2. Considering the case $r_{11} > r_{12}$ holds, then $F_1\left(\frac{r_{11} - r_{12}}{b_1}, 0\right)$ is unstable, whereas $F_2(H_1^*, P_1^*)$ is locally asymptotically stable.

Theorem 6.3. Assume that

$$\frac{r_{11}}{1+kP_1} > r_{12} + a_1(1-m)P_1 \tag{61}$$

holds, where P_1 is defined in (23), then system (58) is permanent.

Theorem 6.4. Assume that

$$\frac{r_{11}}{1+kP_1} > r_{12} + a_1(1-m)P_1 \tag{62}$$

holds, then the positive equilibrium $F_2(H_1^*, P_1^*)$ is globally stable.

Remark 6.1. Compared with the system (9) and (58), we found that under the assumption $r_{11} > r_{12}$ holds, both system admits two equilibria, and the stability property of the equilibria is identical. However, the situation becomes very different regarding the persistent property or global stability property. Noting that for fixed r_{11} and P_1 , $\frac{r_{11}}{1+kP_1} \rightarrow 0$ as $k \rightarrow \infty$. Hence, in the system (58), with the increasing fear effect, the system may not be persistent. However, in the system (9), regardless the large of K, as long as m tends to 1, the inequality will still be maintained. The system could allow for a globally stable positive equilibrium. In other words, a refuge contributes significantly to the system's persistence and stability.

7 Numeric simulations

Example 7.1 In system (9), let's take the following parameter set

$$r_{11} = 3, r_{12} = 1, k = 10, b_1 = a_2 = r_1 = a_1 = r_2 = 1.$$
(63)

By computation, $P_1 = 2(1 - m)$, hence if

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP_1}$$

$$= 3m + \frac{3(1-m)}{1+20(1-m)}$$

$$> 1+2(1-m)^2$$

$$= r_{12} + a_1(1-m)P_1,$$
(64)

holds, the positive equilibrium $E_2(H^*, P^*)$ is stable on a global level. That is, for m > 0.4728930857, system (9) provides a single globally stable positive equilibrium $E_2(H^*, P^*)$. For m = 0.5, system (9) allows a single positive equilibrium $E_2(0.5873499784, 0.2936749892)$. Fig.1 shows that E_2 is globally asymptotically stable. Considering the coefficients mentioned above, then

$$\frac{r_{11}}{1+kP_1} = \frac{3}{1+10}$$
< 1 = r_{12}
< 1+2(1-m)^2 = r_{12} + a_1(1-m)P_1.
(65)

Therefore, the inequality (61) could not hold, and we could not give any information about the persistent and stability property of a system (58).



Figure 1: Global asymptotical stability of E_2 , the initial conditions (H(0), P(0)) = (0.1, 0.8), (1, 0.1), (1.2, 0.5), (1.2, 0.8), (1.2, 0.1), (0.4, 0.8) and (0.2, 0.8), respectively.

Example 7.2 Let us take the following parameter set

$$r_{11} = 3,$$

 $r_{12} = b_1 = a_2 = r_1 = a_1 = r_2 = 1.$
(66)

By computation, $E_2(H^*, P^*)$ satisfies the equation

$$3m + \frac{3 - 3m}{-k(-1+m)H^* + 1} - 1$$

-H* + (1 - m) (-1 + m) H* = 0, (67)
P* = (1 - m)H*.

Numeric simulations (Fig. 2 and 3) show that in this case, H^* is the increasing function of m and decreasing function of k.

Also, in this case, by computation, P^* satisfies the

following equation

$$3m + \frac{3 - 3m}{kP^* + 1} - 1 + \frac{P^*}{-1 + m} - (1 - m) P^* = 0.$$
(68)

Numeric simulation (Fig. 4) shows that P^* is the decreasing function of k. Fixed k = 0.5, Figure 5 shows that there exists a m^* , such that P^* is increasing in $(0, m^*)$ and decreasing in $(m^*, 1)$.



Figure 2: Relationship of H^* , m and k.

Example 7.3 In system (9), let us take the following parameter set

$$r_{11} = 3, r_{12} = 1, k = 10, m = 0.1,$$

$$b_1 = a_2 = r_1 = a_1 = r_2 = 1.$$
(69)

By computation, $P_1 = 2(1 - m) = 1.8$, hence

$$r_{11}m + \frac{r_{11}(1-m)}{1+kP_1} = 0.3 + \frac{2.7}{1+18}$$

$$< 0.3 + 0.15 = 0.45$$

$$< 1 + 1.62 = 1 + 2(1-m)^2$$

$$= r_{12} + a_1(1-m)P_1.$$
(70)

According to Theorem 2.1 and 2.2, the system admits a locally asymptotically stable positive equilibrium. Additionally, since inequalities (24) and (41) are not met, we have unsure about the positive equilibrium's



Figure 3: Relationship of H^* and k, here we choose $m = \frac{1}{2}$.

global stability feature. Numeric simulation (Fig. 6), however, reveals that the positive equilibrium in this situation is globally asymptotically stable.

8 Discussion

The fear effect of the predator on prey species is widespread, and recently, many scholars have been working in this direction([5]-[26]). On the other hand, prey species could live in the shelter to reduce direct killing of predator species. The influence of refuge remains a long and important research subject of investigation in the predator-prey system. Recently, several scholars ([14]-[16], [18], [20]-[22]) tried to combine these two aspects and to propose some new modeling of predator-prey system. They had made some critical progress in this direction, but all of these studies involved that the prey species inside or outside of the sanctuary all suffer from the effect of fear. Such an assumption seems unreasonable since there are no predator species within the sanctuary.

Stimulated by this fact, beginning with our earlier research ([5]), we suggest the system (9). Under the permits that $r_{11} > r_{12}$, i.e., the birth rate of prey species is greater than its mortality rate, we discover the following: The system is capable of supporting both a boundary equilibrium E_1 and a positive equilibrium E_2 . One could easily see that E_1 is unstable while E_2 is locally asymptotically stable (see Theorem 2.2 for more detailed discussion).

It is natural to explore the property of global stability of the equilibrium as this property reflects the



Figure 4: Relationship of P^* , m and k.

long-run coexistence of the two species. In system (1) and (2), by developing some proper Lyapunov functions, the authors demonstrated the stability property of the positive equilibrium. However, with the introduction of the fear effect, the Lyapunov function used in [3] and [5] could not be applied directly to the system (9). In this research, we began by investigating the system's persistence. Then, by applying the Dulac criterion, we derived sufficient conditions to ensure the globally asymptotical stability of positive equilibrium. Intriguingly, the condition that ensures the permanence of the system is sufficient to assure the globally asymptotically stable of positive equilibrium, which indicates that if (24) holds, the system could not exhibit bifurcation behaviors and could not have a periodic solution.

The main innovation of this article is the assumption that the prey population in the refuge is unaffected by the fear effect. As can be seen from Section 6, such an assumption, compared to the classical hypothesis, where scholars assumed that the prey population is affected by the fear effect no matter where it is located, and the dynamic behavior is vastly different. Under our assumption, predator and prey populations can always coexist as long as the refuge is large enough. In contrast, under the classical hypothesis, prey populations tend to go extinct if the fear effect is too large.

Example 7.3 demonstrates that Theorems 3.1 and 4.1 have space for improvement; we want to note this at the conclusion of the study. With the methodology used in this research, this seems implausible. This is



Figure 5: Relationship of P^* and m, here we choose $k = \frac{1}{2}$.

left for further research.

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Figure 6: Global asymptotical stability of E_2 , the initial condition (H(0), P(0)) = (0.1, 0.4), (0.4, 0.1), (0.4, 0.4) and (0.2, 0.4), respectively.

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Sijia Lin, Yanbo Chong wrote the draft. Shangming Chen carried out the simulation. Fengde Chen proposed the problem.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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