Using Homotopy Perturbation and Analysis Methods for Solving Different-dimensions Fractional Analytical Equations

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Abstract: The aim of the research, we extended the one-dimensional to multi-dimensional, we applied the homotopy perturbation and analysis methods to solve Volterra integral equations and to obtain approximate analytical solutions of systems of the second kind multi-dimensional Volterra integral equations. We proved the convergence of the homotopy analysis method (HAM). The HAM solutions contained an auxiliary parameter that provides a convenient way of controlling the convergence region of series solutions. It is shown that the solutions obtained by the homotopy-perturbation method (HPM) are only special cases of the HAM solutions. Several examples are given to illustrate the efficiency and implementation of the method. The results indicate that this method is efficient for the linear and non - linear models with the dissipative terms.

Key-Words: perturbation method, analysis method and systems of multi-dimensional Volterra integral equations.

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1 Introduction

Differential equations, integral-equations or combinations of them, integro-differential equations, are obtained in the modeling of real-life engineering phenomena that are inherently non-linear with variable coefficients. Most of these types of equations do not have an analytical solution.

Therefore, these problems should be solved by using numerical or semi-analytical techniques. Computers and more powerful treatments are required on digital roads to achieve accurate results. Acceptable results are obtained through semianalytical way that are computer and more powerful treatments are required on digital roads more suitable than numerical methods.

The main feature of semi-analytical roads, compared to other methods depend on the fact that they can be applied comfortably to solve various complex problems. Integral equations of two kinds have obtained serious content in mathematics, physics, biology, and another problem in the theory of elasticity.

The author, [1], introduced integral equations with some applications. Also, their results can be found analytically in the last realizations. At the same time, the meaning of numerical methods is taken a significant place in solve of these equations.
The two kinds of integral equations:

$$f(x) = g(x) + \lambda \int_a^x \int_a^b K(r,t). f(t)dtdr.$$

We will expand it to a multi-dimensional fractional integral equation.

In our paper, we presented different methods, homotopy perturbation and homotopy analysis way for solving the two kinds of multi-dimensional integral equations. The author, [2], studied a few problems with or without small parameters with the homotopy perturbation technique.

Also, in 2003 the author, [3], studied a new non-linear analytical technique by the homotopy perturbation method.

The authors, [4], extended the concept of homotopy extension property in homotopy theory for topological to its analogical structure in homotopy theory for topological semigroups. In this extension, the authors also gave some results concerning on absolutely retract and its properties.

The author, [5], used the method of homotopy perturbation by providing the numerical solutions going out discretization to deal with the fifth-order boundary value problems and the computation of the Adomian polynomials with the coefficients of sixth-degree B-spline functions.

The author, [6], presented the homotopy perturbation method of the partial differential equations in many dimensions with variable coefficients to find the exact solutions. The results prove that this method is an effective tool for solving partial differential equations with variable coefficients.

Many analytical methods including linear crossing technology: the authors, [7], variational iteration methods. The authors, [8], a reliable approach for higher-order integro-differential.

The authors, [9], used the homotopy perturbation method for solving higher dimensional with initial boundary value problems of variable coefficients.

The authors in [10] used the homotopy analysis method for multiple solutions of nonlinear boundary value problems.

The authors, [11], and the way of decomposing the authors, [12], developed solving partial or non-linear partial differential equations.

One of these semi-analytical solutions is the way to analyze homotopy, the authors, [13], the laplace decomposition way, the authors, [14], homotopy perturbation method, the authors, [15], the matrix exponential way, the authors, [16], the exp-Function.

The authors, [17], used the variational iteration way and homotopy perturbation way to solve the fractional Fredholm integral differential equations with constant coefficients.

The authors, [18], proved the convergence of the homotopy analysis and applied this method to obtain approximative analytical solutions of systems of the second kind integral equations.

The authors, [19], applied coupling of the two methods (variational iteration and homotopy perturbation) to solve non-linear mixed integro - differential equations.

The authors, [20], used the homotopy analysis method to solve two-dimensional non-linear fuzzy integral equations of two kinds. The authors, [21], used a hybrid method to solve the analysis of the fractional-order Navier Stokes equation. It is proven that the hybrid method is reliable, efficient and easy to apply for varied contact problems of engineering and science. In recent years, the homotopy analysis method has been used to get approximate solutions for a wide category of differential, integrated, and integrated equations.

The method provides the solution in a chain quickly with the components that are elegantly calculated. The main feature of the method is it can be used directly without using assumptions or transformations. In this work, we aim to implement this reliable technique to solve multi-dimensional integral equation systems. The authors, [22], presented work, the homotopy perturbation way to solve the non-linear differential fractional equation with the help of He's Polynomials provided as the transformation plays an essential role in solving differential linear and non-linear equations.
The authors, [23], solved the Black-Scholes (B-S) model for the European options pricing problem using a hybrid way called fractional generalized homotopy analysis way (FGHAM).

2 Basic Idea of HAM

We consider the following differential equation $N[u(\tau)] = 0$. Where N is a nonlinear operator, denotes an independent variable, and $u(\tau)$ is an unknown function, respectively.

For simplicity, we ignore all boundaries or initial conditions, which can be treated similarly.
Using generalizing the traditional homotopy method, Liao [2003] construct the so-called zero-order deformation equation:

$$(1-p)L(\phi(\tau,p)-u_0(\tau)) = phH(\tau)N(\phi(\tau,p)).$$

Where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, $u_0(\tau)$ is an initial guess of $u(\tau)$ and $\phi(\tau;p)$ is an unknown function and L an auxiliary linear operator with the property $L[f(\tau)] = 0$ when $f(\tau) = 0$.
It is important, that one has great freedom to choose auxiliary things in HAM. obviously, when p=0 and p=1, it holds:

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau)$$

respectively.

Thus, as p increases from 0 to 1, the solution $\phi(\tau,p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have:

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m,$$

where,

$$u_m(\tau) = \left[\frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\right]_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the above series converges at p=1, then we have:

$$u(t) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau).$$

Which must be one of solutions of original nonlinear equation, as proved by, [24]. As h=-1 and $H(\tau) = 1$, becomes:

$$(1-p)L(\phi(\tau,p) - u_0(\tau)) + pN(\phi(\tau,p))$$

Which is used mostly in the homotopy perturbation method, [9], where as the solution obtained directly, without using Taylor series, [7]. The governing equation can be deduced from the zero-order deformation. Define the vector:

$$u_n = u_0(\tau), u_1(\tau), ..., u_n(\tau).$$

Differentiating above equation m-time with respect to the embedding parameter p and then setting p=0 and finally dividing them by m!, we have the so-called mth-order deformation equation:

$$L(u_m(\tau) - x_m u_{m-1}(\tau)) = \hbar H(\tau) R_m(u_{m-1}^{\rightarrow}) = 0.$$

Where

$$R_m(u_{m-1}^{\rightarrow}) = \left|\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(t,p)}{\partial p^{m-1}}\right|$$

Where q=0, and

$$x_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1 \end{cases}.$$

It should be emphasized that $um(\tau)$ for $m \geq 1$ is governed by the linear equation:

$$(1-p)L(\phi(\tau,p) - u_0(\tau)) + pN(\phi(\tau,p))$$

under the linear boundary conditions that come from the original problem which can be solved by symbolic computation software such as Matlab.

For the convergence of the above method we refer the reader to Liao's work, [24].

If $N[u(\tau)] = 0$, admits a unique solution, then this method will produce the unique solution.

If the equation does not possess a unique solution, the HAM will give a solution among many other (possible) solutions.

3 The Solution Series Convergent:

In this section, we will prove that, as long as the solution series

$$u(t) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau).$$

given by the homotopy analysis method is convergent, it must be the solution of the considered nonlinear problem.

Theorem [3.1]:-
As long as the series:

$$u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau),$$

is convergent, where $u_m(\tau)$ is governed by the high-order deformation equation

$$L[u_m(\tau) - x_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(u_{m-1}^{\rightarrow}) = 0,$$

under the definitions:

$$R_m(u_{m-1}^{\rightarrow}) = \left| \frac{1}{(m-1)!} \frac{\partial^{m-1}\phi(t,p)}{\partial p^{m-1}} \right|$$

where q=0 and

$$x_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1, \end{cases}$$

it must be a solution of equation
$N[u(\tau)] = 0$.

Proof:-
Let

$$u(t) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau).$$

Denote the convergent series by using the above definitions, we have:

$$\begin{aligned} &\hbar H(t) \sum_{m=1}^{+\infty} \Re_m u_{m-1} = \sum_{m=1}^{+\infty} \\ &L(u_m(\tau) - x_m u_{m-1}(\tau)) = \\ &L \sum_{m=1}^{+\infty} (u_m(\tau) - x_m u_{m-1}(\tau)) \\ &= L((1-x_2) \sum_{m=1}^{+\infty} u_m(\tau)) \\ &= L((1-x_2)(s(t) - u_0(t))), \end{aligned}$$

which gives, since $\hbar = 0, H(t) = 0$,

$$\sum_{m=1}^{+\infty} \Re_m u_{m-1} = 0.$$

On the other side, we have according to the above definition, that:

$$\sum_{m=1}^{+\infty} \Re_m u_{m-1} = R_m(u_{m-1}^{\rightarrow}) = \left| \frac{1}{(m-1)!} \frac{\partial^{m-1}\phi(t,q)}{\partial q^{m-1}} \right|_{q=0},$$

In general, $\Phi(t,q)$ does not satisfy the original non-linear equation $N[u(\tau)] = 0$.

Let $E(t;q) = N[\Phi(t,q)] = 0$, denote the residual error of equation
$N[u(\tau)] = 0$. Clearly $E(t;q) = 0$.

Corresponds to the exact solution of the original equation $N[u(\tau)] = 0$.

According to above definition, the Maclaurin series of the residual error $E(t;q)$ about the embedding parameter q is:

$$\sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m E(t,q)}{\partial q^m} q^m = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m N(t,q)}{\partial q^m} q^m,$$

When q=1, the above expression gives

$$E(t;q) = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m E(t,q)}{\partial q^m} q^m = 0.$$

This means, according to the definition of $E(t;q)$ that we gain the exact solution of the original equation $N[u(\tau)] = 0$, when q.

Thus, as long as the series:

$$u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau),$$

is convergent, it must be the solution of the original equation $N[u(\tau)] = 0$.

4 Applications:

In order to assess the advantages and the accuracy of homotopy analysis way for solving system of multi-dimensional integral equations of the second kind, we will consider the following two examples.

Example[1]:-
Consider the following linear system of two-dimensional Volterra integral equations:

$$\begin{aligned} &f_1(x,y) = ysiny - coshx \\ &+ \int_0^a \int_0^b [e^{-(s-y)} f_1(s_1,s_2) \\ &+cos(s-x) f_2(s_1,s_2)] ds_1 ds_2 \end{aligned}$$

$$\begin{aligned} &f_2(x,y) = 2siny + x(sin^2 x + e^x) \\ &- \int_0^a \int_0^b [e^{(s+y)} f_1(s_1,s_2) \\ &+xcos(s) f_2(s_1,s_2)] ds_1 ds_2. \end{aligned}$$

Where a and b are any constant, the exact solutions to above equations are given below:

$$f_1(x,y) = ysiny - coshx,$$
$$f_2(x,y) = 2siny + x(sin^2x + e^x).$$

The homotopy analysis way, the linear operators:

$$L_i\left[\phi_i(x,p)\right] = \left[\phi_i(x,p)\right] L_j\left[\phi_j(y,q)\right] = \left[\phi_j(y,q)\right],$$

for i=1,2 and j=1,2 We now define a non-linear operators as

$$\aleph_1\left[\Phi_1,\phi_2\right] = \left[\phi_1(x,y;p,q)\right] - (ysiny - coshx) + \int_0^a\int_0^b \left[e^{-(s-y)}\phi_1(s_1,s_2;p,q) + cos(s-x)\Phi_2(s_1,s_2;p,q)ds_1ds_2\right.$$

$$\aleph_2\left[\Phi_1,\phi_2\right] = \left[\phi_2(x,y;p,q)\right] - \left(2siny + x(sin^2x + e^x)\right) + \int_0^a\int_0^b e^{(s+y)}\Phi_1(s_1,s_2;p,q) + xcos(s)\Phi_2(s_1,s_2;p,q)ds_1ds_2.$$

by using the above definition, we construct the zeroth-order deformation equations:

$$(1-p)L_{11}[\Phi_{11}(x;p) - f_{11,0}(x)](1-q) L_{21}[\Phi_{21}(y;q) - f_{21,0}(y)] = p\hbar_{11}H_{11}(x)N_{11}[\Phi_1,\Phi_2] q\hbar_{21}H_{21}(y)N_{21}[\Phi_1,\Phi_2],$$

$$(1-p)L_{21}[\Phi_{21}(x;p) - f_{21,0}(x)](1-q) L_{22}[\Phi_{22}(y;q)f_{22,0}(y)] = p\hbar_{21}H_{21}(x)N_{21}[\Phi_1,\Phi_2] q\hbar_{22}H_{22}(y)N_{22}[\Phi_1,\Phi_2].$$

Thus, we obtain the mth-order ($m \geq 1$) deformation equations

$$L_1\left[f_{11,m_{11}}(x) - x_{11}f_{11,(m_{11}-1)}\right] [f_{21,m_{21}}(y) - f_{21,m_{21}-1}(y)] = \hbar_1H_1(x)\Re_{1,m} [f_{11,m_{11}-1}, f_{12,m_{12}-1}],$$

$$L_2\left[f_{21,m_{21}}(y) - y_{21}f_{21,m_{21}-1}(y)\right] \left[f_{22,m_{22}}(y) - y_{22}f_{22,(m_{22}-1)}(y)\right] = \hbar_{22}H_{22}(y)\Re_{22,m}[f_{21,m-1}, f_{22,m-1}].$$

Where

$$\Re_{1,m}[f_{11,m-1}, f_{12,m-1}] = f_{1,m-1}(x,y) - \int_0^a\int_0^b [e^{-(s-y)}f_1(s_1,s_2) + cos(s-x)f_2(s_1,s_2)]ds_1ds_2$$

$$\Re_{2,m}[f_{21,m-1}, f_{22,m-1}] = f_{2,m-1}(x,y) + \int_0^a\int_0^b [e^{(s+y)}f_1(s_1,s_2) + xcos(s)f_2(s_1,s_2)]ds_1ds_2.$$

Now, the solution of the mth-order $m \geq 1$ zeroth-order deformation equations becomes:

$$f_{11,m_{11}}(x) = x_{11}f_{11,(m_{11}-1)} [f_{21,m_{21}}(y) - f_{21,m_{21}-1}(y)] + \hbar_{11}H_{11}(x)\Re_{11,m} [f_{21,m_{21}-1}, f_{22,m_{22}-1}],$$

$$f_{21,m_{21}}(y) = y_{21}f_{21,m_{21}}(y)$$
$$f_{22,m_{22}}(y) - y_{22}f_{22,(m_{22}-1)}(y)$$
$$+\hbar_{22}H_{22}(y)\Re_{22,m}[f_{21,m-1}, f_{22,m-2}].$$

By start with an initial approximations:

$$f_{1,0}(x,y) = ysiny - coshx,$$

$$f_{2,0}(x,y) = 2siny + x(sin^2x + e^x).$$

and by choose $H_i = 1, i = 1,2$, we suppose:

$$f_1(x,y) \approx \sum_{5}^{m=0} f_{1,m},$$

$$f_2(x,y) \approx \sum_{5}^{m=0} f_{2,m}.$$

The comparison of the results of the HAM and the HPM, [8], are presented in Table 1.

Table 1, shows the absolute error between the HAM and HPM when (h =-1)by the comparison and the exact solution.

If	$e_{1,2}(f_1(HAM = HPM))$
x,y=0.0	0
x,y=0.1	$1.4E-07$
x,y=0.2	$3.5E-06$
x,y=0.3	$5.5E-05$
x,y=0.4	$3.8E-04$
x,y=0.5	$1.6E-03$

$e_{1,2}(f_2(HAM))$	$e_{1,2}(f_2(HPM))$
5.0e-8	$5.0e-8$
3.2E-7	$3.2E-7$
1.1E-5	$1.1E-5$
1.2E-4	$1.2E-4$
6.3E-4	$6.3E-4$
2.2E-3	$2.2E-3$

Example[2]:-

Let us solve the following non-linear system of two-dimensional Volterra integral equations:

$$\begin{aligned} f_1(x,y) &= sinx - y \\ &+ \int_0^x \int_0^y \left[f_1^2(s_1,s_2)\right. \\ &+ f_2^2(s_1,s_2)ds_1ds_2 \end{aligned}$$

$$\begin{aligned} f_2(x,y) &= cosx - \frac{1}{2}sin^2y \\ &+ \int_0^x \int_0^y f_1(s_1,s_2) \\ & f_2(s_1,s_2)ds_1ds_2 \end{aligned}$$

With the exact solutions:

$$\begin{aligned} f_1(x,y) &= sinx - y, \\ f_2(x,y) &= cosx - \frac{1}{2}sin^2y. \end{aligned}$$

To solve the system (deformation equation) by means of homotopy analysis method, we choose the linear operators:

$$L_i\left[\phi_i(x,p)\right] = \left[\phi_i(x,p)\right] L_j\left[\phi_j(y,q)\right] = \left[\phi_j(y,q)\right],$$

for i=1,2 and j=1,2.

We now define a nonlinear operators as:

$$\begin{aligned} & \aleph_1\left[\Phi_1,\phi_2\right] = \left[\phi_1(x,y;p,q)\right] \\ - & (sinx-y) \\ + & \int_0^x \int_0^y \left[\phi_1^2(s_1,s_2;p,q)\right. \\ + & \Phi_2^2(s_1,s_2;p,q)ds_1ds_2 \end{aligned}$$

$$\begin{aligned} & \aleph_2\left[\Phi_1,\phi_2\right] = \left[\phi_2(x,y;p,q)\right] \\ - & \left(cosx - \frac{1}{2}sin^2y\right) \\ + & \int_0^x \int_0^y \Phi_1(s_1,s_2;p,q) \\ + & \Phi_2(s_1,s_2;p,q)ds_1ds_2 \end{aligned}$$

Using the above definition, we construct the zeroth-order deformation equations:

$$\begin{aligned} (1-p)L_{11}[\Phi_{11}(x;p) - f_{11,0}(x)](1-q) \\ L_{21}[\Phi_{21}(y;q) - f_{21,0}(y)] = \\ p\hbar_{11}H_{11}(x)N_{11}[\Phi_1,\Phi_2] \\ q\hbar_{21}H_{21}(y)N_{21}[\Phi_1,\Phi_2], \end{aligned}$$

$$\begin{aligned} (1-p)L_{21}[\Phi_{21}(x;p) - f_{21,0}(x)](1-q) \\ L_{22}[\Phi_{22}(y;q)f_{22,0}(y)] = \\ p\hbar_{21}H_{21}(x)N_{21}[\Phi_1,\Phi_2] \\ q\hbar_{22}H_{22}(y)N_{22}[\Phi_1,\Phi_2]. \end{aligned}$$

Thus, we obtain the mth-order $(m \geq 1)$ deformation equations:

$$\begin{aligned} L_1\left[f_{11,m_{11}}(x) - x_{11}f_{11,(m_{11}-1)}\right] \\ \left[f_{21,m_{21}}(y) - f_{21,m_{21}-1}(y)\right] = \\ \hbar_1 H_1(x)\Re_{1,m}[f_{11,m_{11}-1}, f_{12,m_{12}-1}], \end{aligned}$$

$$\begin{aligned} L_2\left[f_{21,m_{21}}(y) - y_{21}f_{21,m_{21}-1}(y)\right] \\ \left[f_{22,m_{22}}(y) - y_{22}f_{22,(m_{22}-1)}(y)\right] = \\ \hbar_{22}H_{22}(y)\Re_{22,m}[f_{21,m-1}, f_{22,m-1}]. \end{aligned}$$

Where

$$\begin{aligned} \Re_{1,m}[f_{11,m-1}, f_{12,m-1}] = f_{1,m-1}(x,y) \\ -\int_0^x \int_0^y \left[f_1^2(s_1,s_2)\right. \\ +f_2^2(s_1,s_2)ds_1ds_2 \end{aligned}$$

$$\begin{aligned} \Re_{2,m}[f_{21,m-1}, f_{22,m-1}] = f_{2,m-1}(x,y) \\ \int_0^x \int_0^y f_1(s_1,s_2)f_2(s_1,s_2)ds_1ds_2 \end{aligned}$$

Now, the solution of the mth - order $m \geq 1$ zeroth-order deformation equations becomes:

$$\begin{aligned} & f_{11,m_{11}}(x) = x_{11}f_{11,(m_{11}-1)} \\ & [f_{21,m_{21}}(y) - f_{21,m_{21}-1}(y)] + \hbar_{11} \\ & H_{11}(x)\Re_{11,m}[f_{21,m_{21}-1}, f_{22,m_{22}-1}], \end{aligned}$$

$$\begin{aligned} & f_{21,m_{21}}(y) = y_{21}f_{21,m_{21}}(y) \\ & [f_{22,m_{22}}(y) - f_{22,(m_{22}-1)}(y)] + \hbar_{22} \\ & H_{22}(y)\Re_{22,m}[f_{21,m-1}, f_{22,m-2}]. \end{aligned}$$

By start with an initial approximations:

$$f_{1,0}(x,y) = sinx - y$$

$$f_{2,0}(x,y) = cosx - \frac{1}{2}sin^2y,$$

and by choose $H_i = 1, i = 1, 2,$
we suppose:

$$f_1(x,y) \approx \sum_{5}^{m=0} f_{1,m},$$

$$f_2(x,y) \approx \sum_{5}^{m=0} f_{2,m}.$$

The comparison of the results of the HAM and the HPM, [8], are presented in Table 2.

Table 2, shows the absolute error between the HAM (h=-0.98), the HPM (h=-1) and the exact solution:

$E_{1,2}(HAM)$	$E_{1,2}(HPM)$
$ifx, y = (0,0), 0$	$8.8818e-16$
(0,1), 2.0163E-09	$2.3210E-08$
$(0,2), 1.1357E-07$	$1.9188E-06$
$(0,3), 6.2312E-06$	$2.8336E-05$
$(0,4), 7.3007E-05$	$2.0695E-04$
(0,5),4.6483E-04	$1.0278E-03$

$E_{1,2}(HAM)$	$E_{1,2}(HPM)$
$1.3878e-17$	$4.7878e-16$
$3.1167E-09$	$1.7048E-08$
$4.2050E-08$	$1.6652E-06$
$6.0365E-06$	$2.8771E-05$
$8.5015E-05$	$2.4354E-04$
$6.2664E-04$	$1.3895E-03$

5 Conclusion:

In this paper, the HAM was used to obtain analytical solutions for systems of linear and non-linear Volterra integral equations of the second kind.

We studied multi-dimensional integral equations of the second kind by using the homotopy analysis method, and the homotopy perturbation way, which can degenerate to an approximate solution in the limit case. A comparison was made between HAM and HPM and found that HAM is more effective than HPM. In addition, the advantage of this method is the rapid convergence of solutions by the additional parameter H and the freedom to choose $\hbar$ for HAM a smoothness gives us more accuracy from HPM. Hence, it may be concluded that this method is a powerful and efficient technique for finding the analytic solutions for wide classes of problems.

Our results show that homotopy perturbation and analysis methods are applicable to solve the Volterra integral equations, how to apply this method to different-dimensions fractional analytical equations remains to be further studied.

References

[1] Jerri, A.J., Introduction to integral equation with application. Marcel Dekker, New York and Basel, 1985.

[2] J.-H. He, Homotopy perturbation technique,omp uter Methods in Applied Mechanics and Engineering, vol. 178, no. 3-4, pp. 25762, 1999.View at: Publisher Site Google Scholar-MathSciNet.

[3] J. H. He, homotopy perturbation method: a new nonlinear analytical technique, Applied Mathematics and Computation, vol. 135, no. 1, pp. 739, 2003.View at: Publisher Site-Google Scholar-Zentralblatt MATH.

[4] Adem, K. and Amin, S., Homotopy Extension Property in Homotopy Theory for Topological Semigroups, Hindawi Publishing Corporation, International Scholarly Research Network, Volume 2012, 2012. Article, ID 359403.https:

[5] SH. SHEN, pplication of homotopy perturbation method to the fifth-order boundary value problems. Int. J. Contemp. Math. Sci. Vol, 2, pp. 227-236, 2007.

[6] Jin L., Homotopy Perturbation Method for Solving Partial Differential Equations with Variable Coefficients. International Journal of Contemporary Mathematical Sciences, 2008.

[7] H. JAFARI, M. ZABIHI AND M. SAIDY, Application of homotopy perturbation method for solving gas dynamic equation, Appl. Math. Sci. Vol. 2, pp. 2393–2396, 2008.

[8] M. A. NOOR AND S. T. MOHYUD-DIN, A reliable approach for higher-order integro-differential

equations, Appl. Appl. Math., Vol. 3, No. (2), pp. 188–199, 2008.

[9] M. MATINFAR AND M. SAEIDY, The homotopy perturbation method for solving higher dimensional initial boundary value problems of variable coefficients, World. J. Model. Sim., pp. 72–80, 2009.

[10] W. X. MA, T. W. HUANG AND Y. ZHANG, A multiple exp-function method for nonlinear differential equations and its application, Phys. Scr., Vol. 82, 2010.

[11] W. X. MA, X. GU AND L. GAO, A note on exact solutions to linear differential equations by the matrix exponential, Adv. Appl. Math. Mech., Vol. 1, pp. 573–580, 2009.

[12] J. L. LI, Adomian's decomposition method and homotopy perturbation method in solving nonlinear equations, J. Comput. Appl. Math., 228 (2009), pp. 168–173.

[13] Abbasbandy, S.; Magyari, E.; Shivanian, E. he homotopy analysis method for multiple solutions of nonlinear boundary value problems Commun. Nonlinear Sci. Numer. Simul. Vol. 14, pp.353-536, 2009.

[14] X. M. CHENA AND L. J. WANG, The variational iteration method for solving a neutral functional-differential equation with proportional delays, Comput. Math. Appl., 59 (2010), pp. 2696–2702.

[15] Y. KHAN AND N. FARAZ, Application of modified Laplace decomposition method for solving boundary layer equation, Journal of King Saud University (Science), 23 (2011), pp. 115–119.

[16] W. X. MA AND E. G. FAN, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl., 61 (2011), pp. 950–959.

[17] Abdelouahab Kadem and Adem Kilicman, he Approximate Solution of Fractional Fredholm Integrodifferential Equations by Variational Iteration and Homotopy Perturbation Methods Abstract and Applied Analysis, Volume 2012, Article ID 486193, 10 pages, 2012

[18] M. Matinfar, M. Saeidy and J. Vahidi, pplication of Homotopy Analysis Method for Solving Systems of Volterra Integral Equations Advances in Applied Mathematics and Mechanics, Vol. 4, No. 1, pp. 36-45, 2012.

[19] F. A. Hendi and M. M. Al-Qarni, n Accelerated Homotopy Perturbation Method for Solving Nonlinear Two-Dimensional Volterra-Fredholm Integrodifferential Equations Advances in Mathematical Physics, Volume 2017, Article ID 9385040, 8 pages, 2017.

[20] Atanaska Georgieva and Snezhana Hristova, homotopy Analysis Method to Solve Two-Dimensional Nonlinear Volterra-Fredholm Fuzzy Integral Equations Fractal and Fractional, 2020.

[21] Chu et al., sed Analysis of fractional multidimensional Naviertokes equation Chu et al. Advances in Difference Equations 2021.

[22] Ali M. et al., -Sumudu Transformation Homotopy Perturbation Technique on Fractional Gas Dynamical Equation, volume 36, No. 1, 2023.

[23] Saratha S.R. et al., Solving Black–Scholes equations using fractional generalized homotopy analysis method, Computational and Applied Mathematics, 39:262, 2020. https://doi.org/10.1007/s40314-020-01306-4.

[24] S. J. LIAO, Beyond Perturbation: Introduction to the Homotopy Analysis Method, CRC Press, Boca Raton: Chapman Hall, 2003.

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we applied the homotopy perturbation and analysis methods to solve Volterra integral equations and to obtain approximate analytical solutions of systems of the second kind multi-dimensional Volterra integral equations

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