Optimal Constant-Stress Accelerated Life Tests for the Marshall-Olkin Extended Lindley Distribution under Complete Sampling

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Abstract: - This study introduces an extended Lindley distribution utilizing a new family of Marshall-Olkin distributions, providing a more flexible framework for modeling lifetime data. The distribution is examined within the context of complete sampling for k-level constant stress accelerated life testing. A reliability analysis of the proposed model and a discussion on parameter estimation using maximum likelihood estimation are presented. Asymptotic confidence intervals for the parameters are derived through the Fisher information matrix. Bayesian estimation procedures and the Markov Chain Monte Carlo (MCMC) approach are also explored. Real data is analyzed to assess the model's fit, followed by simulation studies to illustrate its performance. The paper concludes with a summary of findings and implications for future research.

Key-Words: - Reliability, Accelerated Life Testing, Constant-Stress, Maximum Likelihood Estimation, Bayesian Estimation, MCMC approach.

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1 Introduction

In the field of reliability engineering, traditional life testing experiments are conducted to analyze failure time data obtained under normal operating conditions. However, the high cost of these experiments can be difficult to get failure information under certain conditions. This problem makes it difficult to get the life data for devices like insulating materials, lasers, engine electronics, and power cables. To solve this problem, accelerated life testing (ALT) has become a commonly used strategy to obtain failure data quickly. ALT is subjecting products to higher-than-normal stress levels (e.g., voltage, humidity, vibration, temperature, or pressure) in order to get early failures. By examining the life data from accelerated life tests, researchers can estimate the life characteristics of the product under normal operating conditions. The most regularly used ALT techniques include constant stress testing, step stress testing, and combinations of both techniques. These methods allow researchers to get more valuable, reliable information within less time compared to traditional life testing methods.

[1], stated a discussion of the advantages and disadvantages of constant-stress and step-stress accelerated life testing (ALT) methods. In constant-stress ALT, units are preserved at a consistently high-stress level until all units fail or the test is concluded. On the other hand, However, step-stress ALT slowly increases the stress level on each unit at established times or after a specific number of failures. many researchers have investigated constant-stress and step-stress models, as well as their combinations, in addition to handling problems related to optimal ALT planning and result interpretation. For further information on this topic, refer to works by [2], [3], [4], [5], [6] and [7].

The exponential distribution is commonly used in survival analysis and reliability theory for analyzing lifetime data due to its simplicity and extensive usage throughout multiple fields.

However, The exponential distribution has restricted when it comes to effectively modeling variable failure rates. As a result, Alternative distributions have been introduced, including the Lindley distribution.

[8] in 1958 introduced the Lindley distribution offered advantages over the exponential distribution, including an increasing risk rate and decreasing mean residual life function. The Lindley distribution, which was first introduced, has been

examined further by researchers like as [9], who have proved a benefit over the exponential distribution for certain industries. [10] have been developed, Methods for estimating reliability under progressive type-II censoring for the Lindley distribution, while [11] have explored classical estimation techniques for constant stress-accelerated life tests under the exponential Lindley distribution. Researchers have been offering developments to the lifetime model's application by various generalizations and extensions of the Lindley distribution, such as weighted Lindley [12], extended Lindley [13], generalized Lindley [14], and power Lindley distributions [15].

[16] have been the first introduction of a transformation of the cumulative distribution function (cdf) by the development of a family of distributions by adding a new parameter. This method has been beneficial for various researchers to derive new distributions and investigate their properties. For example, [17] explored Marshall-Olkin logistic processes, [18] introduced the Marshall-Olkin power lognormal distribution along with its statistical properties, while [19] examined the mathematical properties of the Marshall-Olkin extended Weibull distribution. Additionally, [20] applied the Marshall-Olkin extended Uniform distribution, [21] studied the Marshall-Olkin Extended Lomax distribution, and [22] incorporated the Marshall-Olkin extended generalized linear exponential distribution. [23] investigated the Marshall-Olkin extended Pareto distribution, while [24] conducted a comprehensive analysis of the Marshall-Olkin extended Weibull distribution as a compound distribution mixed with exponential distribution for modeling censored data. Furthermore, [25] proposed an extension of the inverse Weibull distribution using the Marshall-Olkin method to provide a more flexible option for modeling lifetime data.

The paper's structure is as follows: The model definition and test assumptions are discussed in the subsequent section. The maximum likelihood estimates (MLEs) of the model parameters and the corresponding Fisher information matrix for k-level constant-stress accelerated life tests (ALTs) are derived in the subsequent section. Additionally, Bayesian estimation procedures for the unknown parameters and the Markov Chain Monte Carlo (MCMC) approach are detailed. Subsequently, two real data sets are analyzed to validate the theoretical findings presented earlier. The results obtained are then illustrated and compared using simulated data generated from the proposed model. Finally, conclusions drawn from the study are presented.

2 Describe the Model and Assumptions for Testing

2.1 Lindley Distribution

The probability density function (pdf) of the Lindley (λ) distribution is given by:

$$f(t)=\frac{\lambda^2}{(1+\lambda)}(1+t)e^{-\lambda t} \quad , \quad t>0,\lambda>0\,. \qquad (1)$$

The first moment $E(t)=\frac{\lambda+2}{\lambda(1+\lambda)}$, the second moment $E(t^2)=\frac{2(\lambda+3)}{\lambda^2(1+\lambda)}$.

The corresponding cumulative distribution function (cdf) is given by:

$$F(t)=1-\frac{(\,1+\lambda+\lambda t)}{(1+\lambda)}e^{-\lambda t} \quad , \quad t>0,\lambda>0. \qquad (2)$$

The corresponding hazard rate function is given by:

$$h(t)=\frac{\lambda^2}{(\,1+\lambda+\lambda t)}(1+t) \quad , \quad t>0,\lambda>0.$$

2.2 Marshall-Olkin Method

[16], proposed a transformation of the baseline (cdf) by adding a new parameter to obtain a family of distributions.

$$G(x,\theta)=\frac{F(x)}{1-\bar{\theta}\big(1-F(x)\big)} \quad , \quad -\infty<x<\infty,$$
$$\theta>0,\bar{\theta}=1-\theta. \qquad (3)$$

2.3 New Model

In this section, we will give the Marshall-Olkin Extended Lindley distribution (MOEL)

$$G(t,\theta)=\frac{F(t)}{1-\bar{\theta}\bar{F}(t)} \quad where \ \ t>0\,,\theta>0,$$
$$\bar{\theta}=1-\theta,\bar{F}(t)=1-F(t)\ .$$

Then

$$G(t,\theta)=\ \frac{(1+\lambda)-(\,1+\lambda+\lambda t)e^{-\lambda t}}{(1+\lambda)-\bar{\theta}(\,1+\lambda+\lambda t)e^{-\lambda t}}\ . \qquad (4)$$

and,

$$g(t,\theta)=\frac{\lambda^2(1+\lambda)(1+t)(1-\bar{\theta})e^{-\lambda t}}{[(1+\lambda)-\bar{\theta}(\,1+\lambda+\lambda t)e^{-\lambda t}]^2}. \qquad (5)$$

2.3.1 Reliability Analysis

The Reliability function (survival function) of (MOEL) distribution is given by

$$\bar{G}(t,\theta)=\frac{\theta\bar{F}(t)}{1-\bar{\theta}\bar{F}(t)}\ ,\ where\ \bar{F}(t)=\frac{(\,1+\lambda+\lambda t)}{(1+\lambda)}e^{-\lambda t}.$$

Then

$$\bar{G}(t,\theta)=\frac{\theta(\,1+\lambda+\lambda t)e^{-\lambda t}}{(1+\lambda)-\bar{\theta}(\,1+\lambda+\lambda t)e^{-\lambda t}}\,.$$

The Hazard rate function of MOEL distribution is given by:

$$h(t,\theta)=\frac{g(t,\theta)}{\bar{G}(t,\theta)}=\frac{\lambda^2(1+\lambda)(1+t)}{[(1+\lambda)-\bar{\theta}(1+\lambda+\lambda t)e^{-\lambda t}](1+\lambda+\lambda t)}.$$

Note that:

- $g(0,\theta)=\frac{\lambda^2}{\theta(1+\lambda)}$
- $g(\infty,\theta)=0$
- The pdf $g(t,\theta)$ is decreasing (unimodal) if $\theta\le\frac{2\lambda^2}{\lambda^2+1}$ or (if $\theta>\frac{2\lambda^2}{\lambda^2+1}$) see [15].
- $h(0,\theta)=g(0,\theta)=\frac{\lambda^2}{\theta(1+\lambda)}$
- $h(\infty,\theta)=\lambda$

2.3.2 Test Assumptions

Let $s_0<s_1<\cdots<s_k$ be the ordered stress levels in the tests and s_0 be the use stress. Consider the following k levels of constant stress accelerated life tests (CSALTs) scheme with the complete sample.

Under each stress level $s_i, i=1,\dots,k, n_i$ identical units are tested until all the n_i units fail such that $n=\sum_{i=1}^{k} n_i$, $let\ t_{i1},\ t_{i2},\dots,\ t_{in_i}$ be the observed failure times at stress level s_i such that $0<t_{i1}<\ t_{i2}<\cdots<t_{in_i}, i=1,2,\dots,k$.

Note that the failure time t_{in_i} at stress s_i is random the objective here is to specify $n_1,\ n_2,\dots,\ n_k$ according to some optimality criteria, the following assumptions are used through the our analysis proceeds:

A1: Under each stress level $s_i, i=1,\dots,k, n_i$ identical units are allocated under constant stress loading such that:
$n_i=\pi_i n, \sum_{i=1}^{k} n_i=n, \sum_{i=1}^{k}\pi_i=1, 0\le\pi_i\le 1, where\ \pi_i$ is the proportion of test units allocated to the stress level s_i , [26].

A2: Under each constant stress level $s_i, i=1,\dots,k$, the failure time t_{ij} of jth unit $(j=1,\dots,n_i)$ follows the MOEL distribution.

A3: [27], gives some life stress relationships between the life characteristic λ and the stress loading s as the following:

- Arrhenius model: $\ln\lambda=a+\frac{b}{-s}, b>0$, where s is the absolute temperature.
- Inverse power model: $\ln\lambda=a+b\ln s, b>0$, where s is the voltage.
- Exponential model: $\ln\lambda=a+bs, b>0$, where s is a weathering variable.

Thus: $\ln(\lambda)$ is a linear function of the transformed stress $\varphi(s)=\frac{1}{-s}$, $\ln s, s$ for the above three models. We assume that the relationship between the parameter λ_i and the stress s_i is given by:

$$\ln(\lambda_i)=a+b\varphi_i, , i=0,1,\dots,k,$$

where a and b (> 0) are unknown parameters, and $\varphi_i=\varphi(s_i)$ is increased function of s , λ_i satisfies $\lambda_0<\ \lambda_1<\cdots<\lambda_k$.

3 Maximum Likelihood Estimation (MLE)

The MLE of the model parameters and the associated fisher information matrix for k-level constant stress ALT are derived in the following section.

3.1 Point Estimation

In this subsection, the likelihood function of our model parameters is obtained. Let t_{ij} be the failure time of j[th] unit under stress level (i), based on assumptions 1, 2, and 3, the likelihood function can be written as:

$$L(a,b,\theta)=\prod_{i=1}^{k} g(t,\lambda,\theta)$$

$$L(a,b,\theta)=\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{\begin{pmatrix}(\exp(a+b\varphi_i))^2(1+\exp(a+b\varphi_i))\\ .(1+t_{ij})(1-\bar{\theta})e^{-t_{ij}\exp(a+b\varphi_i)}\end{pmatrix}}{\begin{pmatrix}(1+\exp(a+b\varphi_i))-\bar{\theta}(1+\exp(a+b\varphi_i)\\ +t_{ij}\exp(a+b\varphi_i))e^{-t_{ij}\exp(a+b\varphi_i)}\end{pmatrix}^2}. \quad (6)$$

$\lambda_0=\exp(a+b\varphi_0), \lambda_i=\exp(a+b\varphi_i)$,
$i=\ 1,2,\dots,k.\ \frac{\lambda_i}{\lambda_0}=\exp(\mathrm{a}+\mathrm{b}\varphi_\mathrm{i}-\mathrm{a}-\mathrm{b}\varphi_0)$,
$\lambda_i=\lambda_0\exp(b\{\varphi_i-\varphi_0\})=\lambda_0\alpha^{h_i}$.

Where λ_0 is the parameter of Lindley distribution under use stress s_0, $\alpha^{h_i}=\exp(b\{\varphi_i-\varphi_0\})=\frac{\lambda_i}{\lambda_0}$, is the acceleration factor from s_i to s_0.

Then $h_i(\ln\alpha)=b\{\varphi_i-\varphi_0\}$, $h_i=\frac{\varphi_i-\varphi_0}{\varphi_1-\varphi_0}$,
$h_k>h_{k-1}>\cdots>h_1=1$,

Therefore

$$L(\lambda_0,\alpha,\theta)=\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{(\lambda_0\alpha^{h_i})^2(1+\lambda_0\alpha^{h_i})(1+t_{ij})\theta e^{-t_{ij}\lambda_0\alpha^{h_i}}}{[(1+\lambda_0\alpha^{h_i})-\bar{\theta}(1+\lambda_0\alpha^{h_i}+t_{ij}\lambda_0\alpha^{h_i})e^{-t_{ij}\lambda_0\alpha^{h_i}}]^2}$$
$$=\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{(\lambda_0\alpha^{h_i})^2(1+\lambda_0\alpha^{h_i})(1+t_{ij})\theta e^{t_{ij}\lambda_0\alpha^{h_i}}}{[(1+\lambda_0\alpha^{h_i})\left(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\right)+(\theta-1)t_{ij}\lambda_0\alpha^{h_i}]^2}. \quad (7)$$
$$=\theta^{\sum_{i=1}^{k} n_i}\lambda_0{}^{2\sum_{i=1}^{k} n_i}\alpha^{2\sum_{i=1}^{k} n_i h_i}(1+\lambda_0\alpha^{h_i})^{\sum_{i=1}^{k} n_i}e^{\lambda_0\sum_{i=1}^{k}\sum_{j=1}^{n_i}t_{ij}\alpha^{h_i}}\prod_{i=1}^{k}\prod_{j=1}^{n_i}(1+t_{ij})\prod_{i=1}^{k}\prod_{j=1}^{n_i}[(1+\lambda_0\alpha^{h_i})\left(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\right)+(\theta-1)t_{ij}\lambda_0\alpha^{h_i}]^{-2}. \quad (8)$$

The logarithm of the likelihood function is given by:

$$l(\lambda_0,\alpha,\theta)=\sum_{i=1}^{k} n_i \log\,\theta+2\sum_{i=1}^{k} n_i \log\lambda_0+2\sum_{i=1}^{k} n_i\,h_i \log\,\alpha+\sum_{i=1}^{k} n_i \log\,(1+\lambda_0\alpha^{h_i})+\lambda_0\sum_{i=1}^{k}\sum_{j=1}^{n_i} t_{ij}\,\alpha^{h_i}+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\log\,(1+t_{ij})-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\log\,\Big((1+\lambda_0\alpha^{h_i})\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\lambda_0\alpha^{h_i}\Big). \tag{9}$$

On differentiating the previous equation (9) concerning $\lambda_0,\alpha, and\,\theta$ to get the MLEs of $\lambda_0,\alpha, and\,\theta$ as following:

$$\frac{\partial l(\lambda_0,\alpha,\theta)}{\partial\lambda_0}=\frac{2\sum_{i=1}^{k}n_i}{\lambda_0}+\sum_{i=1}^{k}\frac{\alpha^{h_i}n_i}{1+\alpha^{h_i}\lambda_0}+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha^{h_i}t_{ij}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\alpha^{h_i}\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+\alpha^{h_i}(\theta-1)t_{ij}+\alpha^{h_i}(1+\alpha^{h_i}\lambda_0)t_{ij}\,e^{\alpha^{h_i}t_{ij}\lambda_0}}{(1+\lambda_0\alpha^{h_i})\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\lambda_0\alpha^{h_i}} \tag{10}$$

$$\frac{\partial l(\lambda_0,\alpha,\theta)}{\partial\,\alpha}=\frac{2\sum_{i=1}^{k}h_in_i}{\alpha}+\sum_{i=1}^{k}\frac{\alpha^{-1+h_i}\lambda_0h_in_i}{1+\alpha^{h_i}\lambda_0}+\lambda_0\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha^{-1+h_i}t_{ij}\,h_i-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\alpha^{-1+h_i}\,\lambda_0h_i\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+\alpha^{-1+h_i}\,h_i(\theta-1)t_{ij}\,\lambda_0+h_i\alpha^{-1+h_i}e^{t_{ij}\lambda_0\alpha^{h_i}}\lambda_0t_{ij}\,(1+\lambda_0\alpha^{h_i})}{(1+\lambda_0\alpha^{h_i})\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\lambda_0\alpha^{h_i}} \tag{11}$$

$$\frac{\partial l(\lambda_0,\alpha,\theta)}{\partial\,\theta}=\frac{\sum_{i=1}^{k}n_i}{\theta}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{1+\lambda_0\alpha^{h_i}+\alpha^{h_i}\lambda_0t_{ij}}{(1+\lambda_0\alpha^{h_i})\big(\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\lambda_0\alpha^{h_i}}. \tag{12}$$

$$\frac{2\sum_{i=1}^{k}n_i}{\widehat{\lambda_0}}+\sum_{i=1}^{k}\frac{\alpha^{h_i}n_i}{1+\alpha^{h_i}\widehat{\lambda_0}}+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha^{h_i}t_{ij}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\alpha^{h_i}\big(\theta-1+e^{t_{ij}\widehat{\lambda_0}\alpha^{h_i}}\big)+\alpha^{h_i}(\theta-1)t_{ij}+\alpha^{h_i}(1+\alpha^{h_i}\widehat{\lambda_0})t_{ij}\,e^{\alpha^{h_i}t_{ij}\widehat{\lambda_0}}}{(1+\widehat{\lambda_0}\alpha^{h_i})\big(\theta-1+e^{t_{ij}\widehat{\lambda_0}\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\widehat{\lambda_0}\alpha^{h_i})}=0\,. \tag{13}$$

$$\frac{2\sum_{i=1}^{k}h_in_i}{\hat\alpha}+\sum_{i=1}^{k}\frac{\hat\alpha^{-1+h_i}\lambda_0h_in_i}{1+\hat\alpha^{h_i}\lambda_0}+\lambda_0\sum_{i=1}^{k}\sum_{j=1}^{n_i}\hat\alpha^{-1+h_i}t_{ij}\,h_i-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\hat\alpha^{-1+h_i}\,\lambda h_i\big(\theta-1+e^{t_{ij}\lambda_0\hat\alpha^{h_i}}\big)+\hat\alpha^{-1+h_i}\,h_i(\theta-1)t_{ij}\,\lambda_0+h_i\hat\alpha^{-1+h_i}e^{t_{ij}\lambda_0\hat\alpha^{h_i}}\lambda_0t_{ij}\,(1+\lambda_0\hat\alpha^{h_i})}{(1+\lambda_0\hat\alpha^{h_i})\big(\theta-1+e^{t_{ij}\lambda_0\hat\alpha^{h_i}}\big)+(\theta-1)t_{ij}\,\lambda_0\hat\alpha^{h_i})}=0\,. \tag{14}$$

$$\frac{\sum_{i=1}^{k}n_i}{\hat\theta}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{1+\lambda_0\alpha^{h_i}+\alpha^{h_i}\lambda_0t_{ij}}{(1+\lambda_0\alpha^{h_i})\big(\hat\theta-1+e^{t_{ij}\lambda_0\alpha^{h_i}}\big)+(\hat\theta-1)t_{ij}\,\lambda_0\alpha^{h_i})}=0\,. \tag{15}$$

As shown in the 3 previous equations are nonlinear functions of random quantities $\widehat{\Theta}=(\widehat{\lambda_0},\hat\alpha,\hat\theta)$ and thus, statistical inference with these MLEs can be used on the asymptotic distribution theorem, that is the vector $\widehat{\Theta}=(\widehat{\lambda_0},\ \hat\alpha,\hat\theta)$ is approximately distributed as multivariate normal with mean vector $\Theta=(\lambda_0,\alpha,\theta)$ and variance covariance matrix $I^{-1}(\lambda_0,\alpha,\theta)$, we have $\sqrt{n}(\widehat{\Theta}-\Theta)\backsim N_3\big(0,K_\Theta^{-1}\big)$ where $\backsim$ means approximately distributed, K_Θ^{-1} is unit expected information matrix, the asymptotic behavior remains valid if $K_\Theta^{-1}=\lim_{n\to\infty}n^{-1}I(\Theta)$, where $I(\lambda_0,\alpha,\theta)$ is the fisher information matrix .

The elements of the fisher information matrix for the MLE can be obtained as the negative of the second partial derivatives, [22]:

$$I=\begin{pmatrix} I_{11} & I_{12} & I_{13}\\ & I_{22} & I_{23}\\ & & I_{33}\end{pmatrix}=-\begin{pmatrix}\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial{\lambda_0}^2} & \frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\lambda_0\,\partial\,\alpha} & \frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\lambda_0\,\partial\theta}\\ & \frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\alpha^2} & \frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\alpha\,\partial\,\theta}\\ & & \frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\theta^2}\end{pmatrix}$$

$$I_{11}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial{\lambda_0}^2}=\frac{2\sum_{i=1}^{k}n_i}{{\lambda_0}^2}+\sum_{i=1}^{k}\frac{\alpha^{2h_i}n_i}{(1+\alpha^{h_i}\lambda_0)^2}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{\Big(\alpha^{h_i}\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)+\alpha^{h_i}(-1+\theta)t_{ij}+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{h_i}(1+\alpha^{h_i}\lambda_0)t_{ij}\Big)^2}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}-\frac{2e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{2h_i}t_{ij}+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{2h_i}(1+\alpha^{h_i}\lambda_0){t_{ij}}^2}{\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}}\right). \tag{16}$$

$$I_{22}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\alpha^2}=\frac{2\sum_{i=1}^{k}h_in_i}{\alpha^2}-\sum_{i=1}^{k}\left(\frac{\alpha^{-2+h_i}\lambda_0(-1+h_i)h_in_i}{1+\alpha^{h_i}\lambda_0}-\frac{\alpha^{-2+2h_i}{\lambda_0}^2h_i^2n_i}{(1+\alpha^{h_i}\lambda_0)^2}\right)-\lambda_0\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha^{-2+h_i}t_{ij}\,(-1+h_i)h_i+2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{-\Big(\alpha^{-1+h_i}\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)\lambda_0h_i+\alpha^{-1+h_i}(-1+\theta)\lambda_0t_{ij}\,h_i+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+h_i}\lambda_0(1+\alpha^{h_i}\lambda_0)t_{ij}\,h_i\Big)^2}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}+\frac{\begin{pmatrix}\alpha^{-2+h_i}\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)\lambda_0(-1+h_i)h_i+\alpha^{-2+h_i}(-1+\theta)\lambda_0t_{ij}\,(-1+h_i)h_i\\+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-2+h_i}\lambda_0(1+\alpha^{h_i}\lambda_0)t_{ij}\,(-1+h_i)h_i+2e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-2+2h_i}{\lambda_0}^2t_{ij}\,h_i^2+\\e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-2+2h_i}{\lambda_0}^2(1+\alpha^{h_i}\lambda_0){t_{ij}}^2h_i^2\end{pmatrix}}{(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij})}. \tag{17}$$

$$I_{33}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\theta^2}=\frac{\sum_{i=1}^{k}n_i}{\theta^2}-2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\big(1+\alpha^{h_i}\lambda_0+\alpha^{h_i}\lambda_0t_{ij}\big)^2}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}. \tag{18}$$

$$I_{12}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\lambda_0\,\partial\,\alpha}=-\sum_{i=1}^{k}\left(-\frac{\alpha^{-1+2h_i}\lambda_0h_in_i}{(1+\alpha^{h_i}\lambda_0)^2}+\frac{\alpha^{-1+h_i}h_in_i}{1+\alpha^{h_i}\lambda_0}\right)-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha^{-1+h_i}t_{ij}\,h_i+2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\begin{matrix}-(((\alpha^{h_i}(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)+\alpha^{h_i}(-1+\theta)t_{ij}+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{h_i}(1\\+\alpha^{h_i}\lambda_0)t_{ij})(\alpha^{-1+h_i}(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)\lambda_0h_i+\alpha^{-1+h_i}(-1+\theta)\lambda_0t_{ij}\,h_i\\+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+h_i}\lambda_0(1+\alpha^{h_i}\lambda_0)t_{ij}\,h_i))\end{matrix}}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}+\frac{\begin{matrix}\alpha^{-1+h_i}\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)h_i+\alpha^{-1+h_i}(-1+\theta)t_{ij}\,h_i+2e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+2h_i}\lambda_0t_{ij}\,h_i\\+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+h_i}(1+\alpha^{h_i}\lambda_0)t_{ij}\,h_i+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+2h_i}\lambda_0(1+\alpha^{h_i}\lambda_0){t_{ij}}^2h_i)\end{matrix}}{(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij})}. \tag{19}$$

$$I_{13}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\lambda_0\,\partial\theta}=2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\Bigg(\frac{\alpha^{h_i}+\alpha^{h_i}t_{ij}}{\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}}-\frac{\begin{matrix}((1+\alpha^{h_i}\lambda_0+\alpha^{h_i}\lambda_0t_{ij})(\alpha^{h_i}(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)+\alpha^{h_i}(-1+\theta)t_{ij}\\+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{h_i}(1+\alpha^{h_i}\lambda_0)t_{ij}))\end{matrix}}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}. \tag{20}$$

$$I_{23}=-\frac{\partial^2 l(\lambda_0,\alpha,\theta)}{\partial\,\alpha\,\partial\,\theta}=2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\Bigg(\frac{\alpha^{-1+h_i}\lambda_0h_i+\alpha^{-1+h_i}\lambda_0t_{ij}\,h_i}{\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}}-\frac{\begin{matrix}(1+\alpha^{h_i}\lambda_0+\alpha^{h_i}\lambda_0t_{ij})(\alpha^{-1+h_i}(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta)\lambda_0h_i+\alpha^{-1+h_i}(-1+\theta)\lambda_0t_{ij}\,h_i\\+e^{\alpha^{h_i}\lambda_0t_{ij}}\alpha^{-1+h_i}\lambda_0(1+\alpha^{h_i}\lambda_0)t_{ij}\,h_i))\end{matrix}}{\Big(\big(-1+e^{\alpha^{h_i}\lambda_0t_{ij}}+\theta\big)(1+\alpha^{h_i}\lambda_0)+\alpha^{h_i}(-1+\theta)\lambda_0t_{ij}\Big)^2}. \tag{21}$$

3.2 Interval Estimation

In this subsection, the approximate confidence intervals (CIs) of the parameter are derived based on the asymptotic distributions of MLEs of the elements of the vector of unknown parameters $(\lambda_0, \alpha, \theta)$.

The asymptotic distribution of the MLEs of $(\lambda_0, \alpha, \theta)$ is given by:

$$\left((\widehat{\lambda_0} - \lambda_0), (\hat{\alpha} - \alpha), (\hat{\theta} - \theta)\right) \rightarrow N(0, I^{-1}(\lambda_0, \alpha, \theta)),$$

where $I^{-1}(\lambda_0, \alpha, \theta) =$ the variance – covariance matrix of unknown parameters $(\lambda_0, \alpha, and\ \theta)$.

The approximate $100(1-\delta)\%$ confidence interval (CIs) for the parameters $\lambda_0, \alpha, and\ \theta$ are:

$$\left(\widehat{\lambda_{0l}}, \widehat{\lambda_{0u}}\right) = \widehat{\lambda_0} \pm Z_{1-\frac{\delta}{2}}\sqrt{V(\widehat{\lambda_0})}$$

$$\left(\widehat{\alpha_l}, \widehat{\alpha_u}\right) = \hat{\alpha} \pm Z_{1-\frac{\delta}{2}}\sqrt{V(\hat{\alpha})}$$

$$\left(\widehat{\theta_l}, \widehat{\theta_u}\right) = \hat{\theta} \pm Z_{1-\frac{\delta}{2}}\sqrt{V(\hat{\theta})}$$

Respectively, where $V(\widehat{\lambda_0})$, $V(\hat{\alpha})$, and $V(\hat{\theta})$ are the variance of $\widehat{\lambda_0}$, $\hat{\alpha}$, and $\hat{\theta}$, which are given by the diagonal elements of $I^{-1}(\Theta)$ and $Z_{1-\frac{\delta}{2}}$ is the upper $(1-\frac{\delta}{2})$ percentile of standard normal distribution.

4 Bayesian Approach for Estimation and Prediction

In this section, we will introduce Bayesian estimation of the unknown parameters $\lambda_0, \alpha, and\ \theta$. The prior knowledge about the parameters are represented by independent informative prior distributions. The parameters $\lambda_0, \alpha, and\ \theta$ are assumed to be independent and follow the gamma prior distributions as follows:

$$\pi_1(\lambda_0) \propto {\lambda_0}^{a_1-1} e^{-b_1\lambda_0} \qquad \lambda_0 > 0, a_1 > 0, b_1 > 0,$$
$$\pi_2(\alpha) \propto \alpha^{a_2-1} e^{-b_2\,\alpha} \qquad \alpha > 0, a_2 > 0, b_2 > 0,$$
$$\pi_3(\theta) \propto \theta^{a_3-1} e^{-b_3\theta} \qquad \theta > 0, a_3 > 0, b_3 > 0. \quad (22)$$

Where the hyperparameters a_i and b_i i=1,2,3 are assumed to be known, and chosen to reflect the prior knowledge about the unknown parameters. [28], established the Bayesian estimation for their parameters models based on informative gamma priors.

Using Baye's theorem, we can obtain the posterior distribution of the parameters $\lambda_0, \alpha, and\ \theta$ by using the likelihood function (7) with the priors (22) and denoted it by $\pi^*(\lambda_0, \alpha, \theta)$ as follows:

$$\pi^*(\lambda_0, \alpha, \theta) = \frac{\pi_1(\lambda_0)\pi_2(\alpha)\pi_3(\theta)L(\lambda_0, \alpha, \theta|data)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\lambda_0)\pi_2(\alpha)\pi_3(\theta)L(\lambda_0, \alpha, \theta|data)d\lambda_0 d\alpha d\theta}. \quad (23)$$

The squared error loss (SEL) function, this type of loss function ensures that the model is penalized equally for both types of errors. The SEL function will be given as:

$$L(\Phi, \widehat{\Phi}) = \left(\widehat{\Phi} - \Phi\right)^2$$

Therefore, $g(\lambda_0, \alpha, \theta)$ which is Bayes estimate of any function of $\lambda_0,\ \alpha, and\ \theta$under SEL function is given by:

$$\hat{g}_{BS}(\lambda_0, \alpha, \theta) = E_{(\lambda_0, \alpha, \theta|data)}\left(g(\lambda_0, \alpha, \theta)\right)$$
$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\lambda_0, \alpha, \theta)\pi_1(\lambda_0)\pi_2(\alpha)\pi_3(\theta)L(\lambda_0, \alpha, \theta|data)d\lambda_0 d\alpha d\theta}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\lambda_0)\pi_2(\alpha)\pi_3(\theta)L(\lambda_0, \alpha, \theta|data)d\lambda_0 d\alpha d\theta}. \quad (24)$$

It is difficult to solve the multiple integrals in (24) analytically due to the complexity of the likelihood function (7), the Bayes estimate of $\lambda_0, \alpha, and\ \theta$ can be computed using the MCMC approximation method which is used to generate samples from joint posterior density function (23) and also to obtain the associated credible intervals. The joint posterior distribution can be written as:

$$\pi^*(\lambda_0, \alpha, \theta) \propto {\lambda_0}^{2\sum_{i=1}^k n_i + a_1 - 1} \alpha^{2\sum_{i=1}^k n_i h_i + a_2 - 1} \theta^{\sum_{i=1}^k n_i + a_3 - 1}$$
$$.\exp[-(b_1\lambda_0 + b_2\,\alpha + b_3\theta + \lambda_0 \sum_{i=1}^k \sum_{j=1}^{n_i} t_{ij}\,\alpha^{h_i})]$$
$$.\prod_{i=1}^k \prod_{j=1}^{n_i} \frac{(1+\lambda_0\alpha^{h_i})(1+t_{ij})}{\left[(1+\lambda_0\alpha^{h_i}) - \bar{\theta}(1+\lambda_0\alpha^{h_i} + t_{ij}\,\lambda_0\alpha^{h_i})e^{-t_{ij}\lambda_0\alpha^{h_i}}\right]^2}. \quad (25)$$

The conditional posterior distribution for $\lambda_0, \alpha, and\ \theta$

$$\pi_1^*(\lambda_0|\alpha, \theta, data) \propto {\lambda_0}^{2\sum_{i=1}^k n_i + a_1 - 1} \exp[-\lambda_0(b_1 + \sum_{i=1}^k \sum_{j=1}^{n_i} t_{ij}\,\alpha^{h_i})]$$
$$.\prod_{i=1}^k \prod_{j=1}^{n_i} \frac{(1+\lambda_0\alpha^{h_i})}{\left[(1+\lambda_0\alpha^{h_i}) - \bar{\theta}(1+\lambda_0\alpha^{h_i} + t_{ij}\,\lambda_0\alpha^{h_i})e^{-t_{ij}\lambda_0\alpha^{h_i}}\right]^2}. \quad (26)$$

$$\pi_2^*(\alpha|\lambda_0, \theta, data) \propto \alpha^{2\sum_{i=1}^k n_i h_i + a_2 - 1} \exp[-(b_2\alpha + \lambda_0 \sum_{i=1}^k \sum_{j=1}^{n_i} t_{ij}\,\alpha^{h_i})]$$
$$.\prod_{i=1}^k \prod_{j=1}^{n_i} \frac{(1+\lambda_0\alpha^{h_i})}{[(1+\lambda_0\alpha^{h_i}) - \bar{\theta}(1+\lambda_0\alpha^{h_i} + t_{ij}\,\lambda_0\alpha^{h_i})e^{-t_{ij}\lambda_0\alpha^{h_i}}]^2}. \quad (27)$$

$$\pi_3^*(\theta|\lambda_0, \alpha, data) \propto \theta^{\sum_{i=1}^k n_i + a_3 - 1} \exp[-b_3\theta]$$
$$.\prod_{i=1}^k \prod_{j=1}^{n_i} [(1+\lambda_0\alpha^{h_i}) - \bar{\theta}(1+\lambda_0\alpha^{h_i} + t_{ij}\,\lambda_0\alpha^{h_i})e^{-t_{ij}\lambda_0\alpha^{h_i}}]^{-2}. \quad (28)$$

We observed that the conditional posteriors of $\lambda_0, \alpha, and\ \theta$ in equations (26), (27), and (28) are not known distributions, so a better choice to make the MCMC approach is to use Metropolis- Hasting (H-M) sampler.

The algorithm that explains the process of the Metropolis-Hasting within Gibbs sampling:

(1) Start with initial guess $({\lambda_0}^{(0)}, \alpha^{(0)}, \theta^{(0)})$
(2) Set u=1
(3) Using the following M-H algorithm, generate ${\lambda_0}^{(u)}, \alpha^{(u)}, and\ \theta^{(u)}$ from $\pi_1^*({\lambda_0}^{(u-1)}|\alpha^{(u-1)}, \theta^{(u-1)}, data)$, $\pi_2^*(\alpha^{(u-1)}|{\lambda_0}^{(u-1)}, \theta^{(u-1)}, data)$, and $\pi_3^*(\theta^{(u-1)}|{\lambda_0}^{(u-1)}, \alpha^{(u-1)}, data)$ with the normal proposal distributions $N({\lambda_0}^{(u-1)}, var(\lambda)), N(\alpha^{(u-1)}, var(\alpha)), and\ N(\theta^{(u-1)}, var(\theta))$
(i) Generate proposal ${\lambda_0}^*$from $N({\lambda_0}^{(u-1)}, var(\lambda)), \alpha^*\ from\ N(\alpha^{(u-1)}, var(\alpha))$, $and\ \theta^* from\ N(\theta^{(u-1)}, var(\theta))$.
(ii) Evaluate the acceptance probabilities

$$\zeta_{\lambda_0} = \min[1, \frac{\pi_1^*({\lambda_0}^*|\alpha^{(u-1)}, \theta^{(u-1)}, data)}{\pi_1^*({\lambda_0}^{(u-1)}|\alpha^{(u-1)}, \theta^{(u-1)}, data)}],$$

$$\zeta_\alpha = \min[1, \frac{\pi_2^*(\alpha^*|{\lambda_0}^{(u-1)}, \theta^{(u-1)}, data)}{\pi_2^*(\alpha^{(u-1)}|{\lambda_0}^{(u-1)}, \theta^{(u-1)}, data)}],$$

$$\zeta_\theta = \min[1, \frac{\pi_3^*(\theta^*|{\lambda_0}^{(u-1)}, \alpha^{(u-1)}, data)}{\pi_3^*(\theta^{(u-1)}|{\lambda_0}^{(u-1)}, \alpha^{(u-1)}, data)}].$$

(iii) Generate a $\nu_1, \nu_2, and\ \nu_3$ from a uniform (0,1) distribution.
(iv) If $\nu_1 < \zeta_{\lambda_0}$, accept the proposal and set ${\lambda_0}^* = {\lambda_0}^{(u)}$, else set ${\lambda_0}^{(u)} = {\lambda_0}^{(u-1)}$.
(v) If $\nu_2 < \zeta_\alpha$, accept the proposal and set $\alpha^* = \alpha^{(u)}$, else set $\alpha^{(u)} = \alpha^{(u-1)}$.
(vi) If $\nu_3 < \zeta_\theta$, accept the proposal and set $\theta^* = \theta^{(u)}$, else set $\theta^{(u)} = \theta^{(u-1)}$.
(4) Set u = u+1.
(5) Repeat steps (3) – (4) N- times and obtain ${\lambda_0}^{(i)}, \alpha^{(i)}, and\ \theta^{(i)} where\ i = 1,2, \dots, N$.

The first M-simulated varieties will be discarded to ensure the convergence and the removal of the effect of the selection of initial values. Then the selected samples are ${\lambda_0}^{(u)}, \alpha^{(u)}, and\ \theta^{(u)}$ for u = M+1, …, N, for sufficiently large N, form an approximate posterior samples which can be used to get the Bayesian inferences.

The proposed distributions are chosen to be normal distributions as proposals for generating samples in the Metropolis-Hasting (M-H) algorithm, as one of the assumptions to apply MCMC is that the proposed distribution should be symmetric, [29]. The accepted function involved in the (M-H) algorithm ensures that the proposed distribution is the target posterior that we are interested in [30].

Based on SEL, the approximate Bayes estimates of $\Phi = \lambda_0, \alpha, and\ \theta$ is given by:

$$\widehat{\Phi}_{BS} = \frac{1}{N-M} \sum_{u=M+1}^{N} \Phi^i$$

The credible intervals (CRIs) of $\lambda_0, \alpha, and\ \theta$ can be computed by sorting ${\lambda_0}^{(u)}, \alpha^{(u)}, and\ \theta^{(u)} where\ u = M+1, \dots, N$ Then the 100(1 $-\vartheta$)% CRIs of $\Phi = \lambda_0, \alpha, and\ \theta$ will be $\left(\Phi_{\left(N_{\vartheta/2}\right)}, \Phi_{\left(N_{1-\vartheta/2}\right)}\right)$.

5 Application

Here, we explain the proposed procedure previously with two real data sets.

Example 1

The failure time in hours of 40 motors with new Class-H insulation run at 190° C, 220° C, 240° C, and 260° C is represented in Table 1 (Appendix) from [27]. For each temperature stage, ten motors were inspected for failure over some time, with the assigned failure time being the halfway point between the inspection time when the failure was discovered and the previous inspection time. Our test aimed to determine how long such insulation would last at its design temperature of 180º C.

The Arrhenius relationship is expected to characterize temperature acceleration dependent on engineering practice. Thus, the acceleration model can be as:
$\ln(\lambda_i) = a + \frac{b}{S_i},\ b > 0, i = 0, 1, 2, 3, 4.$ In this example, $S_0 = 180^\circ C,\ S_1 = 190^\circ C,\ S_2 = 220^\circ C,\ S_3 = 240^\circ,$ $S_4 = 260^\circ C\ and\ \varphi_i = \frac{1}{S_i},\ i = 0, 1, 2, 3, 4.$

We compute the Kolmogorov–Smirnov (K-S) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by maximum likelihood estimation.

Example 2

The data in Table 4 (Appendix) is the times in hours of transformer life testing at high voltage, see page 161, of [27]. In this test, the accelerated stress is the voltage from 35.4 to 46.7 KV and the design voltage is 14.4 KV.

Based on engineering experience, the inverse power model is expected to be adequate to describe the acceleration voltage relationship. Thus, the acceleration model can be expressed as $\mathrm{Ln}(\lambda_i) = a + b \ln S_i,\ b > 0, i = 0, 1, 2, 3.$ In this example, S_0 = 14.4 KV, S_1 = 35.4 KV, S_2 = 42.4 KV, S_3 = 46.7 KV, and $\varphi_i = \ln S_i, i = 0, 1, 2, 3.$

From the results, the following notices can be observed from Table 2, Table 3, Table 5 and Table 6 in Appendix:
(1) The values of K-S distances and the corresponding P-values of each stress level are presented in Table 2 and Table 5 in Appendix. It is clear that the estimated MOEL distribution gives a good fit to the given data because all P-values are greater than 0.05.
(2) The MLEs of parameters based on complete data for the estimated MOEL distribution and the Bayes estimates relative to the SEL function for the parameters λ_0, $\alpha\ and\ \theta$ with 95% ACIs and CRIs are displayed in Table 3 and Table 6 in Appendix. We note that:
(i) The values of estimates are close together which indicates the good performance of the estimators for different values of stress levels.
(ii) the Bayes estimates have the smallest values than the MLEs.
(iii) the CRIs give more accurate results than the ACIs for different values of stress levels.

6 Simulation Studies

In this section simulation studies are conducted to compare the performance of the MLEs in terms of their biases and mean squared errors (MSEs) for different choices of n values and different parameter values.

So, the biases and MSEs based on 10000 simulations are estimated and reported in Table 7 and Table 8 in Appendix, we carried out this simulation study according to the following algorithm:

1. Specify the values of k, $n_i, h_i, s_i, i = 0, 1, \dots, k$.
2. Take prior parameters $\lambda_0, \alpha_0, and\ \theta_0$ to generate $\lambda_k = \lambda_0 {\alpha_0}^{h_k}$.
3. Determine the MOEL $[\lambda_k, \theta]$ probability distribution as follows:
   MOEL $[\lambda_k, \theta] = \frac{\lambda^2(1+\lambda)(1+t)\theta e^{\lambda t}}{[(1+\lambda)(\theta - 1 + e^{\lambda t}) + (\theta-1)\lambda t]^2}$.
4. Generate a random sample of size $n_i, i = 0, 1, \dots, k$, from the variable that we determined in MOEL $[\lambda_k, \theta]$ distribution and sort it.
5. Solve the nonlinear systems to obtain the MLEs of the parameters.
6. Replicate the steps 3-5 10000 times.
7. Compute the average values of $\widehat{\lambda_0}$, $\hat{\alpha}$, and $\hat{\theta}$.
8. Compute the biases and MSEs associated with MLEs of the parameters $\widehat{\lambda_0}$, $\hat{\alpha}$, and $\hat{\theta}$.
9. Obtain the fisher information matrix by using MLEs of the parameters $\widehat{\lambda_0}$, $\hat{\alpha}$, and $\hat{\theta}$ and compute the asymptotic variance and length of 95% Cis of MLEs.
10. Steps 1-9 are done with different values of k, $n_i, h_i, s_i, \lambda_0, \alpha_0, and\ \theta_0, i = 0, 1, \dots, k$.

The results show the following observations:
(1) in Table 7 and Table 9 in Appendix, it is observed that as the values of n increase, the MSEs decrease and Bayes estimates have the smallest MSEs for λ_0, $\alpha\ and\ \theta$, also if we increase the stress levels, the average of MLEs and the Bayesian estimates decrease.
(2) From Table 8 and Table 10 in Appendix, it can be noticed that the average lengths of the approximate confidence intervals for the parameters are very high compared to those in the Bayesian case, as the values of n increase, the average lengths of both decrease, also if we increase the stress levels, Average lengths of the ACIs and the CRIs for the estimates decrease.

7 Conclusion

In this study, it is considered that the lifetimes of the units follow our new model MOEL distribution. To estimate the acceleration factor and parameters of the distribution classical and Bayesian inference procedures are discussed based on a complete sample under constant stress ALT method. Their performance is numerically compared and appropriate comments are finally provided. The computational results proved that increasing the sample size and level of stress improves the performances of all estimators. As a future work, classical and Bayesian inference procedures under the same distribution assuming other progressively hybrid censoring schemes will be considered.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used quillbot.com in order to check grammar, and spelling, and improve the readability and language of this manuscript. After using this tool, the authors reviewed and edited the content as needed and took full responsibility for the content of the publication.

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Conflict of Interest

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APPENDIX

Table 1. Time (H) to failure of 40 motors

$S_1 = 190^{\circ}C$	$S_2 = 220^{\circ}C$	$S_3 = 240^{\circ}C$	$S_4 = 260^{\circ}C$
7228	1764	1175	600
7228	2436	1175	744
7228	2436	1521	744
8448	2436	1569	744
9167	2436	1617	912
9167	2436	1665	1128
9167	3108	1665	1320
9167	3108	1713	1464
10511	3108	1761	1608
10511	3108	1953	1896

Table 2. K-S distances and the corresponding P-values of each stress level for MOEL distribution

Stress (Temperature)	**190º C**	**220º C**	**240º C**	**260º C**
K-S distances	0.3132	0.2437	0.1762	0.2785
P-values	0.2804	0.59277	0.9155	0.4201

Table 3. Point estimate and 95% CIs for the parameters $\lambda_0, \alpha\ and\ \theta$

	k	MLE	SEL
λ_0	2	0.0001 $[-1.9\times10^{-4}, 3.9\times10^{-4}]$	0.0001 $[9.779\times10^{-5}, 9.785\times10^{-5}]$
	3	0.0010 $[-0.0404, 0.0424]$	0.0010 $[1.012\times10^{-3}, 1.016\times10^{-3}]$
	4	0.0003 $[-3.6\times10^{-5}, 5.3\times10^{-4}]$	0.0003 $[2.5052\times10^{-4}, 2.5058\times10^{-4}]$
α	2	0.5720 $[0.4076, 0.7364]$	0.5720 $[0.57197, 0.57199]$
	3	0.4510 $[-56.5050, 57.4069]$	0.4525 $[0.4496, 0.4576]$
	4	0.7370 $[0.6164, 0.8576]$	0.7370 $[0.73697, 0.73699]$
θ	2	0.0069 $[-0.0252, 0.0391]$	0.0069 $[6.945\times10^{-3}, 6.947\times10^{-3}]$
	3	0.0205 $[-28.1637, 28.2047]$	0.0226 $[0.0206, 0.0240]$
	4	0.0108 $[-0.0094, 0.0310]$	0.0108 $[1.0758\times10^{-2}, 1.0760\times10^{-2}]$

Table 4. The number of failure times in hours of 30 transformer life testing at high voltage

S_1 = 35.4 KV	S_2 = 42.4 KV	S_3 = 46.7 KV
40.1	0.6	3.1
59.4	13.4	8.3
71.2	15.2	8.9
166.5	19.9	9.0
204.7	25.0	13.6
229.7	30.2	14.9
308.3	32.8	16.1
537.9	44.4	16.9
1002.3	50.2	21.3

1002.3	56.2	48.1

Table 5. K-S distances and the corresponding P-values of each stress level for MOEL distribution

Stress (Voltage)	**35.4 KV**	**42.4 KV**	**46.7 KV**
K-S distances	0.1743	0.2503	0.2655
P-values	0.9215	0.5582	0.4815

Table 6. Point estimate and 95% CIs for the parameters $\lambda_0, \alpha\ and\ \theta$

	k	MLE	SEL
λ_0	2	0.0028 $[-0.0022, 0.0078]$	0.0028 $[2.8071 \times 10^{-3}, 2.8077 \times 10^{-3}]$
	3	0.0064 $[-0.0025, 0.0154]$	0.0064 $[6.4196 \times 10^{-3}, 6.4206 \times 10^{-3}]$
α	2	0.9745 $[-0.1917, 2.1408]$	0.9745 $[0.9745, 0.9746]$
	3	0.5054 $[0.0726, 0.9382]$	0.5054 $[0.50537, 0.50541]$
θ	2	0.0130 $[-0.0215, 0.0476]$	0.0130 $[1.3030 \times 10^{-2}, 1.3033 \times 10^{-2}]$
	3	0.0044 $[-0.0058, 0.0146]$	0.0044 $[4.4055 \times 10^{-3}, 4.4066 \times 10^{-3}]$

Table 7. The comparison between the average of MLEs and the Bayesian estimates according to the MSE with true values $\lambda_0 = 0.01, \alpha_0 = 0.9, and\ \theta_0 = 0.0001$ in Arrhenius model

		$\widehat{\lambda_0}$		$\hat{\alpha}$		$\hat{\theta}$	
K	n	MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
4	36	0.0585 (0.0034)	0.0585 (0.0034)	0.8891 (0.0057)	0.8891 (0.0057)	0.0031 (0.00001)	0.0031 (0.00001)
	60	0.0542 (0.0028)	0.0542 (0.0028)	0.8980 (0.0030)	0.8980 (0.0030)	0.0030 (0.00001)	0.0030 (0.00001)
	96	0.0509 (0.0024)	0.0509 (0.0024)	0.9000 (0.0022)	0.9000 (0.0022)	0.0027 (0.00001)	0.0027 (0.00001)
	192	0.0489 (0.0021)	0.0489 (0.0021)	0.8989 (0.0011)	0.8989 (0.0011)	0.0026 (0.00001)	0.0026 (0.00001)
3	36	0.0571 (0.0032)	0.0571 (0.0032)	0.8929 (0.0078)	0.8929 (0.0078)	0.0031 (0.00002)	0.0031 (0.00002)
	60	0.0546 (0.0028)	0.0546 (0.0028)	0.8984 (0.0046)	0.8984 (0.0046)	0.0030 (0.00001)	0.0030 (0.00001)
	96	0.0505 (0.0023)	0.0505 (0.0023)	0.9032 (0.0031)	0.9032 (0.0031)	0.0028 (0.00001)	0.0028 (0.00001)
	192	0.0477 (0.0020)	0.0477 (0.0020)	0.9002 (0.0015)	0.9002 (0.0015)	0.0025 (0.00001)	0.0025 (0.00001)
2	36	0.0549 (0.0029)	0.0549 (0.0029)	0.8964 (0.0120)	0.8964 (0.0120)	0.0030 (0.00001)	0.0030 (0.00001)
	60	0.0546 (0.0028)	0.0546 (0.0028)	0.8983 (0.0067)	0.8983 (0.0067)	0.0030 (0.00001)	0.0030 (0.00001)
	96	0.0501 (0.0022)	0.0501 (0.0022)	0.9010 (0.0045)	0.9010 (0.0045)	0.0027 (0.00001)	0.0027 (0.00001)
	192	0.0471 (0.0019)	0.0471 (0.0019)	0.9006 (0.0024)	0.9006 (0.0024)	0.0024 (0.00001)	0.0024 (0.00001)

Table 8. Average lengths of the approximate confidence intervals (ACIs) and the credible intervals (CRIs) for the estimates

		$\widehat{\lambda_0}$		$\hat{\alpha}$		$\hat{\theta}$	
K	n	MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
4	36	0.5317	0.00004	0.2986	0.00002	0.0376	0.000002
	60	0.3033	0.00002	0.2292	0.00002	0.0281	0.000001
	96	0.2505	0.00001	0.1808	0.00001	0.0230	0.000001
	192	0.1894	0.00001	0.1271	0.00001	0.0172	0.000001
3	36	0.4688	0.00003	0.3448	0.00002	0.0360	0.000002
	60	0.3193	0.00002	0.2661	0.00002	0.0288	0.000002
	96	0.2431	0.00002	0.2105	0.00001	0.0228	0.000002
	192	0.1838	0.00001	0.1478	0.00001	0.0165	0.000001
2	36	0.4187	0.00003	0.4370	0.00003	0.0336	0.000002
	60	0.3089	0.00002	0.3365	0.00002	0.0284	0.000002
	96	0.2411	0.00002	0.2661	0.00002	0.0224	0.000002
	192	0.1815	0.00001	0.1873	0.00001	0.0161	0.000001

Table 9. The comparison between the average of MLEs and the Bayesian estimates according to the MSE with true values $\lambda_0 = 0.007; \alpha_0 = 0.3; \theta_0 = 3$ in the Inverse power model

		$\widehat{\lambda_0}$		$\hat{\alpha}$		$\hat{\theta}$	
K	n	MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
6	60	0.0060 (0.000003)	0.0060 (0.000003)	0.3355 (0.0039)	0.3355 (0.0039)	2.7565 (0.5377)	2.7565 (0.5377)
	120	0.0062 (0.000001)	0.0062 (0.000001)	0.3242 (0.0021)	0.3242 (0.0021)	2.8420 (0.4476)	2.8420 (0.4476)
	180	0.0063 (0.000001)	0.0063 (0.000001)	0.3192 (0.0015)	0.3192 (0.0015)	2.8705 (0.3868)	2.8704 (0.3868)
	240	0.0065 (0.000001)	0.0065 (0.000001)	0.3126 (0.0010)	0.3126 (0.0010)	2.8943 (0.3289)	2.8943 (0.3288)
5	60	0.0059 (0.000003)	0.0059 (0.000003)	0.3402 (0.0050)	0.3402 (0.0050)	2.7556 (0.5267)	2.7556 (0.5267)
	120	0.0061 (0.000001)	0.0061 (0.000001)	0.3277 (0.0026)	0.3277 (0.0026)	2.8084 (0.4420)	2.8083 (0.4420)
	180	0.0064 (0.000001)	0.0064 (0.000001)	0.3180 (0.0015)	0.3180 (0.0015)	2.9083 (0.3774)	2.9083 (0.3774)
	240	0.0064 (0.000001)	0.0064 (0.000001)	0.3159 (0.0013)	0.3159 (0.0013)	2.9126 (0.3266)	2.9126 (0.3266)
4	60	0.0058 (0.000003)	0.0058 (0.000003)	0.3467 (0.0065)	0.3467 (0.0065)	2.7740 (0.4932)	2.7740 (0.4932)
	120	0.0061 (0.000001)	0.0061 (0.000001)	0.3332 (0.0040)	0.3332 (0.0040)	2.8445 (0.4456)	2.8445 (0.4456)
	180	0.0063 (0.000001)	0.0063 (0.000001)	0.3253 (0.0026)	0.3253 (0.0026)	2.8438 (0.3666)	2.8438 (0.3666)
	240	0.0064 (0.000001)	0.0064 (0.000001)	0.3192 (0.0017)	0.3192 (0.0017)	2.9101 (0.3360)	2.9101 (0.3360)
3	60	0.0059 (0.000003)	0.0059 (0.000003)	0.3529 (0.0095)	0.3529 (0.0095)	2.7985 (0.4988)	2.7985 (0.4988)
	120	0.0061 (0.000001)	0.0061 (0.000001)	0.3370 (0.0056)	0.3370 (0.0056)	2.8520 (0.3934)	2.8520 (0.3934)
	180	0.0062 (0.000001)	0.0062 (0.000001)	0.3352 (0.0046)	0.3352 (0.0046)	2.8981 (0.3647)	2.8981 (0.3647)
	240	0.0064 (0.000001)	0.0064 (0.000001)	0.3226 (0.0021)	0.3226 (0.0021)	2.9283 (0.2627)	2.9283 (0.2627)
2	60	0.0060 (0.000003)	0.0060 (0.000003)	0.3528 (0.0108)	0.3528 (0.0108)	2.8229 (0.5169)	2.8229 (0.5169)
	120	0.0061 (0.000002)	0.0061 (0.000002)	0.3505 (0.0108)	0.3505 (0.0108)	2.9020 (0.4182)	2.9020 (0.4182)
	180	0.0062 (0.000002)	0.0062 (0.000002)	0.3433 (0.0077)	0.3433 (0.0077)	2.9069 (0.3630)	2.9069 (0.3630)

	240	0.0062 (0.000002)	0.0062 (0.000002)	0.3382 (0.0065)	0.3382 (0.0065)	2.9173 (0.3386)	2.9173 (0.3385)

Table 10. Average lengths of the approximate confidence intervals (ACIs) and the credible intervals (CRIs) for the estimates

		$\widehat{\lambda_0}$		$\hat{\alpha}$		$\hat{\theta}$	
K	n	MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
6	60	0.0095	0.000001	0.3287	0.00002	5.6233	0.0004
	120	0.0068	0.0000004	0.2157	0.000014	4.0335	0.0003
	180	0.0073	0.0000005	0.1831	0.00001	4.9022	0.0003
	240	0.0076	0.0000004	0.1494	0.00001	5.1331	0.0004
5	60	0.0103	0.000001	0.3928	0.00003	5.6250	0.0004
	120	0.0117	0.000001	0.2629	0.00002	7.4750	0.0005
	180	0.0062	0.0000004	0.2014	0.00001	3.3708	0.0002
	240	0.0053	0.0000003	0.1721	0.00001	2.9140	0.0002
4	60	0.0118	0.000001	0.5001	0.00003	5.7634	0.0004
	120	0.0087	0.000001	0.3363	0.00002	4.1255	0.0003
	180	0.0068	0.0000004	0.2534	0.00002	3.2951	0.0002
	240	0.0060	0.0000004	0.2139	0.00001	2.9126	0.0002
3	60	0.0137	0.000001	0.6464	0.00004	5.7088	0.0004
	120	0.0099	0.000001	0.4332	0.00003	4.0746	0.0003
	180	0.0081	0.000001	0.3500	0.00002	3.3663	0.0002
	240	0.0062	0.0000004	0.2345	0.00002	2.3909	0.0002
2	60	0.0279	0.000002	1.4292	0.0001	8.2452	0.0005
	120	0.0141	0.000001	0.7157	0.00005	4.1652	0.0003
	180	0.0116	0.000001	0.5724	0.00004	3.3782	0.0002
	240	0.0102	0.000001	0.4835	0.00003	2.9283	0.0002