Global Existence and Finite Time Blow-Up for the Laplacian Equation with Variable Exponent Sources

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Abstract: - In this paper, we study a class of semilinear m(.)-Laplacian equations with variable exponent sources. By using the potential well method, we discuss this problem at three different initial energy levels. When the initial energy is sub-critical, we obtain the blow-up result and estimate the lower and upper bounds of the blow-up time. In the case of critical initial energy, we prove global existence, asymptotic behavior, and finite-time blow-up and determine the lower bound of the blow-up time. For super-critical initial energy, we establish the finite-time blow-up and estimate the lower and upper bounds of the blow-up time.

Key-Words: - m(.)-Laplacian parabolic equation, blow-up time, bounds of blow-up time, global existence, critical exponents, variable nonlinearity.

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1 Introduction

The diffusion problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems corresponding to several physical phenomena. The diffusion and reaction process is of great importance because it explains the behavior of a range of chemical systems where the diffusion of material competes with the production of that material by some form of chemical reaction. The most common is the change in space and time of the concentration of one or more chemical substances: local chemical reactions in which the substances transform into each other, and diffusion causes the substances to spread out over a surface in space. However, the system can also describe dynamical processes of non-chemical nature. Examples are biology, geology, physics (neutron diffusion theory), and ecology. Mathematically, reaction-diffusion systems take the form of semi-linear parabolic partial differential equations. The known standard model can be represented by an equation that includes a divergence operator, [1].

$$\frac{\partial u}{\partial t} = div(D(u, \nabla u)\nabla u) + f(x, t, u, \nabla u), u(x, 0) = u_0(x). \quad (1)$$

The function $u(x, t)$ represents the mass concentration in chemical reaction processes or temperature in heat conduction at a specific position x in the diffusion medium and time t. The function D is known as the diffusion coefficient or the thermal diffusivity. The term $div(D(u, \nabla u)\nabla u)$ represents the rate of change due to diffusion while $f(x, t, u, \nabla u)$ represents the rate of change due to reaction. The diffusion problem is the challenge of spreading and adopting new ideas, innovations, or research findings into practical applications in various fields, including applied science. In applied science, the diffusion problem is crucial because it determines how effectively research findings can be translated into real-world solutions, products, or services that benefit society.

For example, in materials science, researchers may develop new materials with unique properties. However, the diffusion problem arises when trying to scale up the production process, ensure consistency, and integrate these materials into existing manufacturing systems. Similarly, in environmental science, the diffusion problem occurs when trying to implement sustainable practices or technologies in different regions or communities, where local factors such as infrastructure, policy, or cultural norms can hinder adoption. The reaction-diffusion equation with variable exponent is a mathematical model with special diffusion process $D(u,\nabla u)=|\nabla u|^{m(.)-2}$. The variable exponent in the equation introduces nonlinearity, which can lead to interesting pattern formation and behavior in the system. This type of equation has been studied in various fields and used to model a variety of physical phenomena, including chemical reactions, heat transfer, population dynamics, biological sciences etc., [2]. A large part of this research centered around electrorheological fluids, [3], [4], [5], porous media [6], and image processing [7], [8] and typical defects considered are involve $p(x)$-Laplacian operators in which the exponent p depends on the spatial variable, It has been shown that spatial defects can hinder traveling waves in nonlinear wave equations [9], [10], [11] in which, the function spaces with spatially dependent exponents are required along with new mathematical techniques. In this paper, we establish the global existence and finite-time blow-up, lower and upper bounds of blow-up time, at three different initial energy levels for a non-autonomous $m(.)$-Laplacian parabolic equation.

$$\begin{aligned} &u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u, (x,t)\in\Omega\times(0,T),\\ &u(x,0)=u_0(x), \qquad x\in\Omega,\\ &u(x,t)=0, \qquad (x,t)\in\partial\Omega\times(0,T). \end{aligned} \tag{2}$$

The nonlinear operator $\Delta_{m(.)}u$ is called $m(.)$-Laplace operator, where $m(x)$ is a measurable function, and Ω is a bounded Lipschitz domain of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. The operator is defined as:

$$\Delta_{m(.)}u = -div\left(|\nabla u|^{m(x)-2}\nabla u\right) = -\nabla.\left(|\nabla u|^{m(x)-2}\nabla u\right),$$

where

$$2\le m(x) < p(x) < \infty, \tag{3}$$

Furthermore, it should be noted that $p(.)$ also fulfills the condition (H) in the following manner: The given exponent measurable function $p(.)$ satisfying

$$\begin{cases} 2 < p_{1,2} < \infty, n \le m_2,\\ m_2 < p_1 \le p(x)\\ \le p_2 < \dfrac{nm(x)}{\underset{x\in\Omega}{esssup}\left(n-m(.)\right)}, \qquad n > m_2, \end{cases}$$

where

$$p_2 = \underset{x\in\Omega}{esssup}\,p(x), p_1 = \underset{x\in\Omega}{essinf}\,p(x). \tag{4}$$

We also assume that p, and m satisfies log-Hölder continuity condition:

$$|p(x)-p(y)| + |m(x)-m(y)| \le M(|x-y|),\ x,y\in\Omega, \tag{5}$$

where $M(r)$ satisfies:

$$\underset{r\to 0^+}{\limsup} M(r)\ln\left(\frac{1}{r}\right) = c < \infty.$$

A significant amount of effort has been dedicated to studying problem (1) in the case of constant and variable exponent nonlinearities. Let's begin with some classical results. The model of non-Newtonian elastic filtration type with the equation $u_t = div(|\nabla u|^{m-2}\nabla u)$, where m is a constant >2, has been discussed in various studies, including [12] and [13]. This model is used to describe non-stationary flow in a porous medium [14] of fluids that exhibit a power dependence of the tangential stress on the velocity of the displacement under elastic conditions. If there are heat sources or sinks in the medium whose power relies on temperature, then we need to consider the special case, of equation (1), written as $u_t = div(|\nabla u|^{m-2}\nabla u) + f(x,t,u,\nabla u)$, where m is constant >2.

It appears that this parabolic equation was first introduced in reference, [15]. It was called the n-diffusion equation and is a generalized form of diffusion. This equation is related to the unsteady vertical heat transfer from horizontal surfaces by turbulent free convection and the unsteady turbulent flow of a liquid with a free surface over a plane. If, in addition, $f_u'(x,t,u,\nabla u) < 0$ for $u>0$, then we refer to equation (1) as the nonlinear heat equation with absorption. If $f_u'(x,t,u,\nabla u) > 0$ for at least some interval $(0,u)_{t=0}$, we refer to equation (1) as the nonlinear heat equation with sources, [16]. For the Cauchy problem with nonlinear inhomogeneous source term:

$$\begin{aligned} u_t = div\left(|\nabla u|^{m(x)-2}\nabla u\right) + |u|^{p(x)-2}u + f(t),\\ m(x)\ge m_1 > 2, \end{aligned} \tag{6}$$

In their discussion, the authors in [17] considered the nonlinear heat equation (6). They showed that for certain conditions on m and p, any solution with nontrivial initial datum blows up in finite time when $f=0$. They also provided numerical examples to illustrate their result in two dimensions. For the following pseudo-parabolic equation with $p(x)$-Laplacian and viscoelastic terms:

$$u_t - \Delta \mathrm{u} - \int_0^t g(t-s)\Delta_{p(x)}u(\mathrm{x},\mathrm{s})ds$$

$= |u|^{q(x)-2}u, \mathrm{x} \in \Omega, \mathrm{t} \geq 0,$
$u(x, t) = 0, x \in \partial\Omega, t \geq 0,$
$u(x, 0) = u_0(x), x \in \Omega.$

In a bounded domain $\Omega \subset \mathbb{R}^n$ ($\mathrm{n} \geq 1$) with a smooth boundary, where $u_0 \geq 0$, with $u_0 \in W_0^{1,p(.)}(\Omega)$, and the parameters $p(.)$, and $q(.)$ satisfy some conditions, the authors in [18] proved that this equation blows up in finite time in two cases. Subject to certain conditions on p(.), q(.), g, and the initial given data, they have established a new criterion for blow-up and provided lower and upper bounds on the solutions if blow-up occurs.

A pseudo-parabolic equation with nonlinearities of variable exponent type as in (2) was considered in [19]. By using a technique called differential inequality, the researchers were able to find an upper bound for the blow-up time (i.e. when the solution becomes unbounded or undefined), under certain conditions involving variable exponents $p(.)$, $m(.)$, and the initial data. They also determined a lower bound for blow-up time based on some other conditions. The author in [20] considered the same problem as in (2). They established an upper bound for blow-up time for certain solutions with positive initial energy, in case the solutions blow-up. Noticing that, the case where the initial energy is positive, namely $\mathrm{J}(u_0) > 0$, has not been discussed yet and remains an intriguing unsolved matter. Therefore, in this paper, our objective is to address a particular problem by utilizing the variational method, known as the potential well method. As a result, we will divide the scenario where $\mathrm{J}(u_0) > 0$ into two cases based on the value of $\mathrm{J}(u_0)$: $0 < J(u_0) < d$ and $\mathrm{J}(u_0) > 0$. Here, d refers to the depth of the potential well, which is also known as the mountain pass level and will be explained later in the paper.

2 Preliminaries and Main Result

To begin, we will review some basic results that will be required in the following sections. The results are often reported without supporting evidence, but we have included references to the relevant publications. In addition, some of our notation conventions are introduced. First, we denote $\|.\|_q$ to the usual $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and $\|\nabla.\|_k$ the Dirichlet norm in $W_0^{1,k}(\Omega)$. Furthermore, from now on, C represents different positive constants varying on the known numbers and may be different at each advent.

The results listed below can be found in [21].

Let $p: \Omega \to 1, \infty]$ be a measurable function. $L^{p(.)}(\Omega)$ denotes the set of the real measurable functions u on Ω such that:

$$\int_\Omega |\lambda u(x)|^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0.$$

The variable-exponent space $L^{p(.)}(\Omega)$ equipped with the Luxemburg-type norm:

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\},$$

is a Banach space. Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \leq q \leq +\infty$.

Next, we will define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ in the following manner:

$$W^{1,p(.)}(\Omega) = \left\{\begin{array}{l} u \in L^{p(.)}(\Omega): \\ \nabla u \text{ exists and } |\nabla u| \in L^{p(.)}(\Omega) \end{array}\right\}.$$

This space is a Banach space, which is defined by its norm:

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|u\|_{p(.),\Omega} + \|\nabla u\|_{p(.),\Omega}.$$

In addition, we have established that $W_0^{1,p(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. It is known that for the elements of $W_0^{1,p(.)}(\Omega)$ the Poincaré inequality holds:

$$\|u\|_{p(.),\Omega} \leq C(n, \Omega)\|\nabla u\|_{p(.),\Omega}, \tag{7}$$

and an equivalent norm of $W_0^{1,p(.)}(\Omega)$ can be defined by:

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.),\Omega}.$$

Next, we introduce some functionals and sets as follows:

$$\mathrm{J}(u) = \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} \mathrm{d}x - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \mathrm{d}x.$$

$$\mathrm{I}(u) = \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x - \int_\Omega |u|^{p(x)} \mathrm{d}x,$$

$$W = \left\{u \in W_0^{1,m(.)}(\Omega) | \mathrm{I}(u) > 0, J(u) < d\right\} \cup \{0\},$$

$$V = \left\{u \in W_0^{1,m(.)}(\Omega) | \mathrm{I}(u) < 0, J(u) < d\right\},$$

and the depth of potential well d is defined by:

$$d = \inf_{u \in \mathcal{N}} \mathrm{J}(u),$$

where the Nehari manifold is defined by:

$$\mathcal{N} = \left\{\begin{array}{l} u \in W_0^{1,m(.)}(\Omega) | \\ \mathrm{I}(u) = 0, \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x \neq 0 \end{array}\right\},$$

and $\mathcal{N}$ separates two unbounded sets:

$$\mathcal{N}_+ = \{u \in W_0^{1,m}(\Omega) | \mathrm{I}(u) > 0\} \cup \{0\},$$

and

$$\mathcal{N}_- = \left\{u \in W_0^{1,m(.)}(\Omega) | \mathrm{I}(u) < 0\right\}.$$

A weak solution of problem (2) can be defined as follows. We denote:

$$\widetilde{W} := \begin{cases} u: u \in L^{m_1}\left(0, T; W_0^{1,m(x)}(\Omega)\right) \\ \cap L^\infty(0, T; L^2(\Omega)) \cap L^{p(x)}(\Omega \times (0, T)) \\ \text{with } \nabla u \in L^{m(x)}(\Omega \times (0, T))\}. \end{cases}$$

Definition 1 (Weak solution) *A solution to the problem (2) is a function $u = u(x,t) \in \widetilde{W}$ such that*

$$\int_0^t \int_\Omega \left(-u \frac{\mathrm{d}\varphi}{\mathrm{d}t} + |\nabla u|^{m(x)-2} \nabla u \nabla \varphi - |u|^{p(x)-2} u \varphi\right) \mathrm{d}x \mathrm{d}\tau = -\int_\Omega u \varphi \mathrm{d}x \Big|_0^t, t \in (0, T), \tag{8}$$

holds for any $t \leq T$ and all $\varphi \in \widetilde{W}$ with $\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in \widetilde{W}^*$, where $\widetilde{W}^*$ is the dual space of $\widetilde{W}$.

$$u(x, 0) = u_0(x) \text{ in } W_0^{1,m(.)}(\Omega).$$

for $0 \leq t < T$

$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \mathrm{J}(u) \leq \mathrm{J}(u_0). \tag{9}$$

Before stating our main result, we first give the following theorem of existence and uniqueness, as well as the regularity, It should be noted by using the Faedo-Galerkin arguments combined with the fixed point Theorem, we can easily establish the well-posedness of the solution to the problem (2).

Theorem 2 (Local solution) *Let $u_0 \in W_0^{1,m(.)}(\Omega) \setminus \{0\}$ and $p(.)$ satisfy (H). Then there exist a $T_{max} > 0$ and a unique weak solution u of (2) satisfying $u \in C\left(0, T; W_0^{1,m(.)}(\Omega)\right)$, and the energy inequality*

$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \mathrm{J}(u(t)) \leq \mathrm{J}(u_0), 0 \leq t \leq T_{\max},$$

where $T_{\max}$ is the maximum existence time of solution $u(t)$.

Moreover,

If $T_{\max} < \infty$, then

$$\lim_{t \to T} \| u \|_q = \infty \text{ for all } q > 1 \text{ such that } q > \frac{n(p_1 - m_2)}{m_2}.$$

If $T_{\max} = \infty$, then $u(t)$ is a global solution of problem (2).

Here, we have the following qualitative analysis about $\mathrm{J}(u)$ and $\mathrm{I}(u)$. We then have the following lemma.

Lemma 3 *For $p(.)$ satisfy (3), (H) and $u \in H_0^1(\Omega) \setminus \{0\}$. Let $F: [0, +\infty) \to \mathbb{R}$ the Euler functional defined by*

$$F(\lambda) = \int_\Omega \frac{\lambda^{m(x)}}{m(x)} |\nabla u|^{m(x)} \mathrm{d}x - \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} \mathrm{d}x,$$

then, F keeps the following properties:

$\lim_{\lambda \to 0^+} F(\lambda) = 0$ and $\lim_{\lambda \to +\infty} F(\lambda) = -\infty$.

There is at least one solution to the equation $F'(\lambda) = 0$ on the interval $[\lambda_1, \lambda_2]$, where

$$\lambda_1 = \min\left\{\rho(u)^{\frac{-1}{m_1 - p_1}}, \rho(u)^{\frac{-1}{m_2 - p_2}}\right\},$$

$$\lambda_2 = \max\left\{\rho(u)^{\frac{-1}{m_1 - p_1}}, \rho(u)^{\frac{-1}{m_2 - p_2}}\right\}, \tag{10}$$

and

$$\rho(u) := \frac{\int_\Omega |\nabla u(x)|^{m(x)} \mathrm{d}x}{\int_\Omega |u(x)|^{p(x)} \mathrm{d}x}.$$

There exists a $\lambda^* = \lambda^*(u) > 0$ such that $F(\lambda)$ gets its maximum at $\lambda = \lambda^*$. Furthermore, we have that $0 < \lambda^* < 1$, $\lambda^* = 1$ and $\lambda^* > 1$ provided $\mathrm{I}(u) < 0$, $\mathrm{I}(u) = 0$ and $\mathrm{I}(u) > 0$, respectively.

Proof. Since $p(x) \in C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}): \inf_{x \in \overline{\Omega}} p(x) > 2\right\}$, the assertion (i) is shown by the following:

$$F(\lambda) \leq \max\{\lambda^{m_1}, \lambda^{m_2}\} \int_\Omega \frac{1}{m(x)} |\nabla u(x)|^{m(x)} \mathrm{d}x$$

$$-\min\{\lambda^{p_1}, \lambda^{p_2}\} \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x,$$

and

$$F(\lambda) \geq \min\{\lambda^{m_1}, \lambda^{m_2}\} \int_\Omega \frac{1}{m(x)} |\nabla u(x)|^2 \mathrm{d}x$$

$$-\max\{\lambda^{p_1}, \lambda^{p_2}\} \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x.$$

For (ii).

We have

$$F'(\lambda) = \int_\Omega \lambda^{m(x)-1} |\nabla u(x)|^{m(x)} \mathrm{d}x - \int_\Omega \lambda^{p(x)-1} |u(x)|^{p(x)} \mathrm{d}x$$

$= \int_\Omega \lambda^{m(x)-1}\left[|\nabla u(x)|^{m(x)} - \lambda^{p(x)-m(x)}|u(x)|^{p(x)}\right]\mathrm{d}x,$

which implies that $F'(\lambda)$ lies in the following two inequalities

$$F'(\lambda) \geq \min\{\lambda^{m_1-1}, \lambda^{m_2-1}\}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x - \max\{\lambda^{p_1-1}, \lambda^{p_2-1}\}\int_\Omega |u(x)|^{p(x)}\mathrm{d}x,$$

and

$$F'(\lambda) \leq \max\{\lambda^{m_1-1}, \lambda^{m_2-1}\}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x - \min\{\lambda^{p_1-1}, \lambda^{p_2-1}\}\int_\Omega |u(x)|^{p(x)}\mathrm{d}x.$$

Since $p_2 \geq p_1 > m_2 \geq m_1 \geq 2$, we signify that $F'(\lambda)$ has at least one zero point λ satisfying (10). So we get (ii). The definition of λ^* and the relation $\mathrm{I}(\lambda u) = \lambda F'(\lambda)$ and

$$F'(\lambda) \leq (\lambda^{m_1-1} - \lambda^{p_2-1})\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x + \lambda^{p_2-1}\mathrm{I}(u), \qquad \text{for } \lambda \in (0,1),$$

and

$$F'(\lambda) \geq (\lambda^{m_1-1} - \lambda^{p_2-1})\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x + \lambda^{p_2-1}\mathrm{I}(u), \qquad \text{for} \lambda \in (1,\infty),$$

lead to the last claim (iii). Completeness of the proof.

In the following Lemma, we find a ball in the $W_0^{1,m(.)}(\Omega)$ space with a radius of $\int_\Omega |\nabla u(x)|^{m(x)}dx$. This helps us understand the relationship between $\mathrm{I}(u)$, $\int_\Omega |\nabla u(x)|^{m(x)}dx$, and the depth of the potential well d.

Lemma 4 *Let $u \in W_0^{1,m(.)}(\Omega)$ and assume that (3), (H) and $J(u) \leq d$ hold.*
If $0 < \int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x < r$, then $\mathrm{I}(u) > 0$ and

$$\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x < \frac{m_2 p_1}{p_1 - m_2} d.$$

If

$$\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x > \frac{m_2 p_1}{p_1 - m_2} d,$$

then $\mathrm{I}(u) < 0$ and $\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x > r$.

If $\mathrm{I}(u) = 0$, then $\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x = 0$ or

$$r \leq \int_\Omega |\nabla u|^{m(x)}\mathrm{d}x \leq \frac{p_1 m_2}{p_1 - m_2} d,$$

where $r = \min\left(\left(\frac{1}{C_*^{p_2}}\right)^{\frac{m_1}{p_2-m_1}}, \left(\frac{1}{C_*^{p_2}}\right)^{\frac{m_2}{p_1-m_2}}\right)$ and $C_* = \max(B,1)$, where B is the best embedding constant from $W_0^{1,m(x)}(\Omega)$ to $L^{p(x)}(\Omega)$, i.e.,$\frac{1}{B} = \inf_{u\in W_0^{1,m(x)}(\Omega), u\neq 0}\left(\frac{\|\nabla u\|_{m(x)}}{\|u\|_{p(x)}}\right)$.

Proof. (i) From (H), (3) and

$$0 < \int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x < r,$$

we have

$$\begin{aligned}
\int_\Omega |u|^{p(x)}\mathrm{d}x &\leq \max\left(\|u\|_{p(x)}^{p_1}, \|u\|_{p(x)}^{p_2}\right) \\
&\leq \max\left\{\begin{matrix} C_*^{p_2}\max\left\{\begin{matrix}\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_2}{m_2}}, \\ \left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_2}{m_1}}\end{matrix}\right\}, \\ C_*^{p_1}\max\left\{\begin{matrix}\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_1}{m_2}}, \\ \left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_1}{m_1}}\end{matrix}\right\}\end{matrix}\right\} \\
&\leq C_*^{p_2}\max\left\{\begin{matrix}\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_2}{m_1}}, \\ \left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_1}{m_2}}\end{matrix}\right\} \\
&= C_*^{p_2}\max\left\{\begin{matrix}\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_2-m_1}{m_1}}, \\ \left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x\right)^{\frac{p_1-m_2}{m_2}}\end{matrix}\right\} \times \int_\Omega |\nabla u|^{m(x)}\mathrm{d}x \\
&< C_*^{p_2}\max\left\{r^{\frac{p_2-m_1}{m_1}}, r^{\frac{p_1-m_2}{m_2}}\right\}\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x \\
&\leq \int_\Omega |\nabla u|^{m(x)}\mathrm{d}x, \qquad (11)
\end{aligned}$$

which gives $\mathrm{I}(u) > 0$. According to (3), $\mathrm{I}(u) > 0$, and the definition of $\mathrm{J}(u)$, we check that

$$\begin{aligned}
\mathrm{J}(u) &= \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x \\
&\geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x + \frac{1}{p_1}\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x - \int_\Omega |u|^{p(x)}\mathrm{d}x\right) \\
&= \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x + \frac{1}{p_1}\mathrm{I}(u) \\
&> \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x, \qquad (12)
\end{aligned}$$

since $\mathrm{J}(u) \leq d$ gives that

$$\left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x < d,$$

i.e.

$$\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x < \frac{m_2 p_1}{p_1 - m_2} d.$$

(ii) By (12) and $\int_\Omega |\nabla u|^{m(x)}dx > \frac{m_2 p_1}{p_1-m_2} d$, we have

$$J(u) \geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}dx + \frac{1}{p_1}I(u) > d + \frac{1}{p_1}I(u),$$

then $J(u) \leq d$ gives

$$I(u) < 0,$$

due to the Sobolev inequality this means $\int_\Omega |\nabla u|^{m(x)}dx \neq 0$. Then $I(u) < 0$ gives

$$C_*^{p_2}\max\left\{r^{\frac{p_2-m_1}{m_1}}, r^{\frac{p_1-m_2}{m_2}}\right\}\int_\Omega |\nabla u|^{m(x)}dx < \int_\Omega |u|^{p(x)}dx \leq C_*^{p_2}\max\left\{\begin{array}{l}\left(\int_\Omega |\nabla u|^{m(x)}dx\right)^{\frac{p_2-m_1}{m_1}},\\ \left(\int_\Omega |\nabla u|^{m(x)}dx\right)^{\frac{p_1-m_2}{m_2}}\end{array}\right\} \times \int_\Omega |\nabla u|^{m(x)}dx,$$

which gives $\int_\Omega |\nabla u|^{m(x)}dx > r$.

(iii) As $I(u) = \int_\Omega |\nabla u|^{m(x)}dx - \int_\Omega |u|^{p(x)}dx = 0$. If $\int_\Omega |\nabla u|^{m(x)}dx \neq 0$, then by

$$\int_\Omega |\nabla u|^{m(x)}dx = \int_\Omega |u|^{p(x)}dx \leq C_*^{p_2}\max\left\{\begin{array}{l}\left(\int_\Omega |\nabla u|^{m(x)}dx\right)^{\frac{p_2-m_1}{m_1}},\\ \left(\int_\Omega |\nabla u|^{m(x)}dx\right)^{\frac{p_1-m_2}{m_2}}\end{array}\right\}\int_\Omega |\nabla u|^{m(x)}dx,$$

we get $\int_\Omega |\nabla u|^{m(x)}dx \geq r$. By (12) and $I(u) = 0$, we infer

$$J(u) \geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)}dx,$$

combining $J(u) \leq d$ yields

$$r \leq \int_\Omega |\nabla u|^{m(x)}dx \leq \frac{p_1 m_2}{p_1 - m_2}d.$$

In this lemma, we provide the expression of d in terms of r, prove the non-increasing nature of $J(u)$, and establish a relationship between $J(u)$, $I(u)$, and d.

Lemma 5 If r is defined as in Lemma 4, then we can conclude that

$$d \geq \frac{p_1-m_2}{p_1 m_2}r. \tag{13}$$

The functional energy J(u) is nonincreasing.

Let $u \in W_0^{1,m(.)}(\Omega)$ with $I(u) < 0$. We have

$$I(u) < p_1(J(u) - d). \tag{14}$$

Proof. (i) Let $u \in \mathcal{N}$, according to Lemma 4, (iii), we know that $\int_\Omega |\nabla u|^{m(x)}dx \geq r$. From the definition of d it follows that for each $k = 1,2,\ldots$ there exists a $u_k \in \mathcal{N}$ such that

$$d \leq J(u_k) < d + \frac{1}{k}, k = 1,2,\ldots \tag{15}$$

From Lemma 4 and

$$J(u_k) \geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u_k|^{m(x)}dx + \frac{1}{p_1}I(u_k) \geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u_k|^{m(x)}dx,$$

we get

$$r \leq \int_\Omega |\nabla u_k|^{m(x)}dx \leq \frac{p_1 m_2}{p_1 - m_2}\left(d + \frac{1}{k}\right), \quad k = 1,2,\ldots. \tag{16}$$

From (16) and Sobolev embeding Lemma it follows that there exists a $u \in \mathcal{N}$ and a subsequence $\{u_\nu\}$ of $\{u_k\}$ such that as $\nu \to \infty$, $u_\nu \to u$ in $H_0^1(\Omega)$ weakly, $u_\nu \to u$ in $L^{\frac{2n}{n-2}}(\Omega)$ strongly. Hence we have:

$$\left|\int_\Omega \frac{1}{p(x)}|u_\nu|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx\right| \leq \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\left|\int_\Omega |u_\nu|^{p(x)}dx - \int_\Omega |u|^{p(x)}dx\right|.$$

Using the fact that, for any $x \in \Omega$ fixed, we have:

$$|u_\nu|^{p(.)} - |u|^{p(.)} = p(x)\zeta^{p(x)-1}v,$$

with $v = u_\nu - u$, and $\zeta = su_\nu + (1-s)u$, $s \in (0,1)$. Young's inequality implies

$$\begin{aligned} I &= \left|\int_\Omega \left(|u_\nu|^{p(.)} - |u|^{p(.)}\right)dx\right| \\ &\leq c\int_\Omega \left|p(x)\zeta^{p(x)-1}\right|^2 |v|^2 dx \\ &\leq c\int_\Omega |su_\nu + (1-s)u|^{2(p(x)-1)}|v|^2 dx \\ &\leq c\left(\int_\Omega |v|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}\left[\left(\int_\Omega \left|su_\nu + (1-s)u\right|^{n(p_2-1)}\right)^{\frac{2}{n}}dx + \left(\int_\Omega |su_\nu + (1-s)u|^{n(p_1-1)}\right)^{\frac{2}{n}}dx\right] \\ &\leq c\|\nabla(u_\nu - u)\|_2^2\left(\|\nabla u_\nu\|_2^{2(p_2-1)} + \|\nabla u\|_2^{2(p_1-1)}\right). \end{aligned}$$

so we get

$$\lim_{\nu\to\infty}\int_\Omega |u_\nu|^{p(x)}dx = \int_\Omega |u|^{p(x)}dx.$$

Thus from (15) we obtain:

$$\int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}\mathrm{d}x \le \liminf_{\nu\to\infty}\int_\Omega \frac{1}{m(x)}|\nabla u_\nu|^{m(x)}\mathrm{d}x \le \liminf_{\nu\to\infty}\left(d+\frac{1}{\nu}+\int_\Omega\frac{1}{p(x)}|u_\nu|^{p(x)}\mathrm{d}x\right) = \lim_{\nu\to\infty}\left(d+\frac{1}{\nu}+\int_\Omega\frac{1}{p(x)}|u_\nu|^{p(x)}\mathrm{d}x\right) = d+\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x,$$

which gives $J(u)\le d$. On the other hand, we have $J(u)\ge d$. Hence we get $J(u)=d$, which in turn give $d\ge\frac{p_1-m_2}{m_2p_1}r$.

(ii) Let $v=u_t$ in (8), we obtain:

$$\int_\Omega|u_t|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{m(x)}|\nabla u|^{m(x)}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x,$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{1}{m(x)}|\nabla u|^{m(x)}\mathrm{d}x-\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x\right) = -\int_\Omega|u_t|^2\mathrm{d}r,$$

or

$$\mathrm{J}'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{J}(u)=-\int_\Omega|u_t|^2\mathrm{d}x\le 0.$$

(iii) Lemma 3, (iii) and $\mathrm{I}(u)<0$ implies existence of $\lambda^*\in(0,1)$ such that $\mathrm{I}(\lambda^*u)=0$. Let

$$h(\lambda):=p_1\mathrm{J}(\lambda u)-\mathrm{I}(\lambda u),\quad \lambda>0.$$

From Lemma 4, by the definition $\mathrm{J}(u)$, $\mathrm{I}(u)$, (3) and (ii), we obtain:

$$h'(\lambda)=p_1\frac{\mathrm{dJ}(\lambda u)}{\mathrm{d}\lambda}-\frac{\mathrm{dI}(\lambda u)}{\mathrm{d}\lambda} \ge p_1\left(\int_\Omega\lambda^{m(x)-1}|\nabla u(x)|^{m(x)}\mathrm{d}x-\int_\Omega\lambda^{p(x)-1}|u(x)|^{p(x)}\mathrm{d}x\right) -m_2\int_\Omega\lambda^{m(x)-1}|\nabla u(x)|^{m(x)}\mathrm{d}x +p_1\int_\Omega\lambda^{p(x)-1}|u(x)|^{p(x)}\mathrm{d}x = (p_1-m_2)\int_\Omega\lambda^{m(x)-1}|\nabla u(x)|^{m(x)}\mathrm{d}x > (p_1-m_2)\min(\lambda^{m_1-1},\lambda^{m_2-1})r>0.$$

Thus, $h(\lambda)$ is strictly increasing for $\lambda>0$. Therefore, $h(1)>h(\lambda^*)$ for $\lambda^*\in(0,1)$. Using the definition of d and the fact that $\mathrm{I}(\lambda^*u)=0$, we obtain:

$$p_1\mathrm{J}(u)-\mathrm{I}(u)>p_1\mathrm{J}(\lambda^*u)-\mathrm{I}(\lambda^*u) = p_1\mathrm{J}(\lambda^*u)\ge p_1d,$$

so (14).

3 Global Existence, Asymptotic Behavior, and Blow-up in Finite Time with $\mathbf{J}(u_0)<d$

In this section, we find that the solution to Problem (2) yields an explosion in finite time and evaluate the upper and lower bounds of the explosion time when $\mathrm{J}(u_0)<d$. The global existence and asymptotic behavior of the solution to problem (8) with $\mathrm{J}(u_0)<d$ and $\mathrm{I}(u_0)>0$ can similarly obtaining it using reference [20]. The proof has been ignored, and we only mention it to illustrate the routine results.

Theorem 6 *[22], Theorem 4. Eq.14]Assuming that $J(u_0)<d$ and $I(u_0)>0$, let $p(.)$ satisfy (H) and $u_0\in W_0^{1,m(.)}(\Omega)$. Then problem (2) admits a global weak solution $u(t)\in L^\infty\left(0,\infty;W_0^{1,m(.)}(\Omega)\right)$. Furthermore, $u_t\in L^2(0,\infty;L^2(\Omega))$ and $u(t)\in W$ for $0\le t<\infty$. Additionally, there exists a constant $\kappa>0$ such that*

$$\| u\|_2^2\le\left(\|u_0\|_2^{2-m_1}+(m_1-2)\kappa t\right)^{\frac{-2}{m_1-2}}.$$

In the following Lemma, we needed to give the invariant set V as follows:

Lemma 7 (Invariant set for $\mathbf{J}(u_0)<d$.) *Assuming (H) holds for $p(.)$, and $u_0\in W_0^{1,m(.)}(\Omega)$, let T_{max} be the maximal existence time. Then, for $J(u_0)<d$, the weak solution u of problem (2) is belongs to V for $0\le t<T_{max}$, as long as $I(u_0)<0$.*

Proof. We know that $u_0\in V$ because $\mathrm{J}(u_0)<d$ and $\mathrm{I}(u_0)<0$. Our goal is to prove that $u(t)\in V$ for $0<t<T_{\max}$. Let's assume the contrary and suppose that there exists $\mathrm{t}_0\in(0,T_{\max})$ such that $\mathrm{J}(u(\mathrm{t}_0))=d$ or $\mathrm{I}(u(\mathrm{t}_0))=0$ and $\int_\Omega|\nabla u(x,\mathrm{t}_0)|^{m(x)}\mathrm{d}x\ne0$. Since $\mathrm{J}(u)$ and $\mathrm{I}(u)$ are continuous in t, we can assume that t_0 is the first time such that $\mathrm{J}(u(\mathrm{t}_0))=d$ or $\mathrm{I}(u(\mathrm{t}_0))=0$ and $\int_\Omega|\nabla u(x,\mathrm{t}_0)|^{m(x)}\mathrm{d}x\ne0$. By Definition 1 (iii) and the fact that $\mathrm{J}(u_0)<d$, we have

$$\int_0^t\|u_r\|_2^2\mathrm{d}\tau+\mathrm{J}(u)\le\mathrm{J}(u_0)<d,\quad 0\le t<T_{\max}. \tag{17}$$

Therefore, $\mathrm{J}(u(\mathrm{t}_0))\ne d$. If $\mathrm{I}(u(\mathrm{t}_0))=0$ and $\int_\Omega|\nabla u(x,\mathrm{t}_0)|^{m(x)}\mathrm{d}x\ne0$, then by the definition of d, we have $\mathrm{J}(u(\mathrm{t}_0))\ge d$, which contradicts (17). Thus, we have proven that $u(t)\in V$ for $0<t<T_{\max}$. The proof has been achieved.

In the following Theorem 8, we establish the blow-up in finite time of solution and provide a enough condition by introducing a elementary auxiliary function.

Theorem 8 (Blow-up for $\mathrm{J}(\boldsymbol{u_0}) < \boldsymbol{d}$) *Assuming that $p(.)$ satisfies (H) and $u_0 \in W_0^{1,m(.)}(\Omega)$, if $J(u_0) < d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem (2) blows up in a finite time.*

Proof. As per Theorem 2, problem (2) has a unique local weak solution $u \in C\left(0, T; W_0^{1,m(.)}(\Omega)\right)$, where $T_{\max}$ denotes the maximum existence time of $u(t)$. We can prove that the existence of time is finite. To do this, we assume the time of existence $T_{\max} = +\infty$ and proceed by contradiction. We then define

$$\mathrm{M}(t) := \int_0^t \| u \|_2^2 \, \mathrm{d}\tau, \quad t \in 0, +\infty), \tag{18}$$

then $\mathrm{M}'(t) = \| u \|_2^2$.

If we substitute $\nu = u_s$ in equation (8), we get:

$$\begin{aligned} \mathrm{M}''(\mathrm{t}) &= 2(u, u_s) \\ &= 2\left(|u|^{p(x)-2}u, u\right) - 2\left(|\nabla u|^{m(x)-2}\nabla u, \nabla u\right) \\ &= -2\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x - \int_\Omega |u|^{p(x)}\mathrm{d}x\right) = -2I(u). \end{aligned} \tag{19}$$

By combining (9) and (12), we get:

$$\begin{aligned} \mathrm{J}(u_0) &\geq \mathrm{J}(u) + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau \\ &\geq \frac{p_1 - m_2}{p_1 m_2}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x \\ &\quad + \frac{1}{p_1}\mathrm{I}(u) + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau, \end{aligned}$$

which implies

$$\frac{1}{p_1}\mathrm{I}(u) \leq \mathrm{J}(u_0) - \frac{p_1 - m_2}{p_1 m_2}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x - \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau,$$

i.e.,

$$\mathrm{I}(u) \leq p_1\mathrm{J}(u_0) - \frac{p_1 - m_2}{m_2}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x - p_1\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau. \tag{20}$$

By substituting (20) into (19), we can derive:

$$\mathrm{M}''(t) \geq -2p_1\mathrm{J}(u_0) + 2\frac{p_1 - m_2}{m_2}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x + 2p_1\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau. \tag{21}$$

From

$$\int_0^t (u_\tau, u)\mathrm{d}\tau = \frac{1}{2}\| u \|_2^2 - \frac{1}{2}\|u_0\|_2^2,$$

we derive

$$\begin{aligned} \left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 &= \left(\frac{1}{2}\| u \|_2^2 - \frac{1}{2}\|u_0\|_2^2\right)^2 \\ &= \frac{1}{4}\left(\| u \|_2^4 - 2\| u \|_2^2 \|u_0\|_2^2 + \|u_0\|_2^4\right) \\ &= \frac{1}{4}\left((\mathrm{M}'(t))^2 - 2\|u_0\|_2^2\mathrm{M}'(t) + \|u_0\|_2^4\right), \end{aligned}$$

then

$$(\mathrm{M}'(t))^2 = 4\left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 + 2\|u_0\|_2^2\mathrm{M}'(t) - \|u_0\|_2^4. \tag{22}$$

Therefore, by combining (21) and (22) we deduce:

$$\begin{aligned} &\mathrm{M}(t)\mathrm{M}''(t) - \tfrac{p_1}{2}(\mathrm{M}'(t))^2 \\ &\geq \mathrm{M}(t)\begin{pmatrix} 2\frac{p_1 - m_2}{m_2}\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x \\ -2p_1\mathrm{J}(u_0) + 2p_1\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \end{pmatrix} \\ &\quad - \tfrac{p_1}{2}\begin{pmatrix} 4\left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 \\ +2\|u_0\|_2^2\mathrm{M}'(t) - \|u_0\|_2^4 \end{pmatrix} \\ &= -2p_1\mathrm{J}(u_0)\mathrm{M}(t) \\ &\quad + 2\tfrac{p_1 - m_2}{m_2}\mathrm{M}(t)\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x \\ &\quad + 2p_1\begin{pmatrix} \int_0^t \| u \|_2^2 \,\mathrm{d}\tau \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \\ -\left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 \end{pmatrix} \\ &\quad - p_1\|u_0\|_2^2\mathrm{M}'(t) + \tfrac{p_1}{2}\|u_0\|_2^4 \\ &> -2p_1\mathrm{J}(u_0)\mathrm{M}(t) \\ &\quad + 2\tfrac{p_1 - m_2}{m_2}\mathrm{M}(t)\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x \\ &\quad + 2p_1\begin{pmatrix} \int_0^t \| u \|_2^2 \,\mathrm{d}\tau \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \\ -\left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 \end{pmatrix} \\ &\quad - p_1\|u_0\|_2^2\mathrm{M}'(t). \end{aligned} \tag{23}$$

Using the Cauchy-Schwarz inequality, we have:

$$\left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 \leq \int_0^t \| u \|_2^2 \,\mathrm{d}\tau \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau, \tag{24}$$

which drives together (23)

$$\begin{aligned} &\mathrm{M}(t)\mathrm{M}''(t) - \frac{p_1}{2}(\mathrm{M}'(t))^2 \\ &> 2\frac{p_1 - m_2}{m_2}\mathrm{M}(t)\int_\Omega |\nabla u(x)|^{m(x)}\mathrm{d}x \\ &\quad - 2p_1\mathrm{J}(u_0)\mathrm{M}(t) - p_1\|u_0\|_2^2\mathrm{M}'(t). \quad (3.5) \end{aligned} \tag{25}$$

Recalling the embedding inequality from $W_0^{1,m(.)}(\Omega)$ to $W_0^{1,2}(\Omega)$ for $2 \leq m(x)$, as follows;

$$\int_\Omega |\nabla u(x)|^{m(x)} dx \geq \min\left(C_1^{m_1}\|\nabla u\|_2^{m_1}, C_1^{m_2}\|\nabla u\|_2^{m_2}\right), \tag{26}$$

and the Poinearé inequality

$$C_2\|u\|_2 \leq \|\nabla u\|_2. \tag{27}$$

Let

$$C_3 = \min(C_1C_2, 1), \tag{28}$$

based on the (3), (26) and (27), it can be deduced that:

$$\begin{aligned}&\frac{p_1 - m_2}{m_2}\mathrm{M}(t)\int_\Omega |\nabla u(x)|^{m(x)} dx \\ &\geq \frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\left(\|u\|_2^{m_1}, \|u\|_2^{m_2}\right)\mathrm{M}(t) \\ &= \frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}\|u\|_2^2\mathrm{M}(t),\end{aligned}$$

hence (3.9) becomes:

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t) - \frac{p_1}{2}(\mathrm{M}'(t))^2 \\ &> \frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t)\mathrm{M}'(t) \\ &-2p_1\mathrm{J}(u_0)\mathrm{M}(t) - p_1\|u_0\|_2^2\mathrm{M}'(t).\end{aligned} \tag{29}$$

Now, we distinguish the following two issues for the level, i.e. $\mathrm{J}(u_0) \leq 0$ and $0 < J(u_0) < d$.

(i) If $\mathrm{J}(u_0) \leq 0$, from (29) we derive

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t) - \frac{p_1}{2}(\mathrm{M}'(t))^2 \\ &> \frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t)\mathrm{M}'(t)(\\ &-p_1\|u_0\|_2^2\mathrm{M}'(t).\end{aligned} \tag{30}$$

By combining (8), (12) and $\mathrm{J}(u_0) \leq 0$, we arrive at:

$$\begin{aligned}0 &\geq \mathrm{J}(u_0) > J(u) \\ &\geq \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega |\nabla u|^{m(x)} dx + \frac{1}{p_1}\mathrm{I}(u),\end{aligned}$$

which means $\mathrm{I(u)} < 0$. Using this and (19), we deduce that $\mathrm{M}''(t) > 0$ for $\mathrm{t} \geq 0$, this implies that $\mathrm{M}'(t) = \| \mathrm{u} \|_2^2$ is increasing with $t \in (0, \infty)$. Since $\mathrm{M}'(0) = \| u_0 \|_2^2 > 0$ and $\mathrm{M}''(t) > 0$, we conclude that $\mathrm{M}'(t) > \mathrm{M}'(0) > 0$ for $t > 0$. This means $\mathrm{M}(t)$ is increasing over $[0, \infty)$, which leads to $\mathrm{M}(t) > M(0) = 0$. Therefore, we can conclude that

$$\mathrm{M}(t) - \mathrm{M}(0) = \int_0^t \mathrm{M}'(\tau)d\tau > \int_0^t \mathrm{M}'(0)d\tau = \mathrm{M}'(0)t,$$

that is

$$\mathrm{M}(t) > \mathrm{M}'(0)t, \quad t > 0.$$

Hence, choosing sufficiently large t, for $\mathrm{M}'(t) > \mathrm{M}'(0) > 0$, we find:

$$\begin{aligned}&\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\left(\|u\|_2^{m_1-2}, \|u\|_2^{m_2-2}\right)\mathrm{M}(t) \\ &> \frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\left(\|u_0\|_2^{m_1-2}, \|u_0\|_2^{m_2-2}\right)\mathrm{M}(t) \\ &> p_1\|u_0\|_2^2,\end{aligned}$$

which (30) take the form:

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t) - \frac{p_1}{2}(\mathrm{M}'(t))^2 \\ &> \mathrm{M}'(t)\begin{pmatrix}\frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t) \\ -p_1\|u_0\|_2^2\end{pmatrix} \\ &> 0.\end{aligned}$$

(ii) If $0 < J(u_0) < d$, then Lemma 7 implies $u(t) \in V$ for $t \geq 0$. By (14), (9) and $0 < J(u_0) < d$, (19) becomes

$$\begin{aligned}\mathrm{M}''(t) &= -2\mathrm{I(u)} \\ &> 2p_1(d - \mathrm{J(u)}) \\ &\geq 2p_1\left(d - \mathrm{J}(u_0) + \int_0^t \|u_\tau\|_2^2 d\tau\right) \\ &> 2p_1\left(d - \mathrm{J}(u_0)\right) \\ &=: \mathrm{C} > 0.\end{aligned} \tag{31}$$

Therefore, (31) and $\mathrm{M}'(0) = \|u_0\|_2^2 > 0$ gives:

$$\mathrm{M}'(t) - \mathrm{M}'(0) = \int_0^t \mathrm{M}''(\tau)d\tau > Ct, \quad 0 < t < \infty,$$

that is

$$\mathrm{M}'(\mathrm{t}) > Ct + \mathrm{M}'(0) > Ct. \tag{32}$$

By the same manner, since $\mathrm{M}''(t) > 0$, $\mathrm{M}(0) = 0$ and (32), for $t \in (0, \infty)$ we conclude:

$$\begin{aligned}\mathrm{M}(t) - \mathrm{M}(0) &= \int_0^t \mathrm{M}'(\tau)d\tau \\ &> \int_0^t \mathrm{C}\tau d\tau = \frac{1}{2}\mathrm{C}t^2,\end{aligned}$$

i.e.,

$$\mathrm{M}(t) > \frac{1}{2}\mathrm{C}t^2 + \mathrm{M}(0) = \frac{1}{2}\mathrm{C}t^2. \tag{33}$$

Thus, the fact $\mathrm{M}'(t) > \mathrm{M}'(0) > 0$, (32) and (33), for sufficiently large t, gives:

$$\begin{aligned}&\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t) \\ &> \frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u_0\|_2^{m_1-2}, \\ \|u_0\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t) \\ &> p_1\|u_0\|_2^2,\end{aligned} \tag{34}$$

and

$$\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2},\\ \|u\|_2^{m_2-2}\end{pmatrix}\mathrm{M}'(t)$$
$$>\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u_0\|_2^{m_1-2},\\ \|u_0\|_2^{m_2-2}\end{pmatrix}\mathrm{M}'(t)$$
$$>2p_1\mathrm{J}(u_0). \tag{35}$$

Therefore, using (34) and (35), (29), for sufficiently large t, becomes

$$\mathrm{M}(t)\mathrm{M}''(t)-\frac{p_1}{2}(\mathrm{M}'(t))^2$$
$$\geq\begin{pmatrix}\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u_0\|_2^{m_1-2},\\ \|u_0\|_2^{m_2-2}\end{pmatrix}\mathrm{M}(t)\\ -p_1\|u_0\|_2^2\end{pmatrix}\mathrm{M}'(t)$$
$$+\begin{pmatrix}\frac{C_3^{m_2}(p_1-m_2)}{m_2}\min\begin{pmatrix}\|u_0\|_2^{m_1-2},\\ \|u_0\|_2^{m_2-2}\end{pmatrix}\mathrm{M}'(t)\\ -2p_1\mathrm{J}(u_0)\end{pmatrix}$$
$$\times\mathrm{M}(t)>0 \tag{36}$$

Because of $\mathrm{M}(t)$, $\mathrm{M}'(t)$ and $\mathrm{M}''(t)$ are all positive for sufficiently large t_*, then (36) leading to:

$$\frac{\mathrm{M}''(t)}{\mathrm{M}'(t)}>\frac{\mathrm{p}_1\mathrm{M}'(t)}{2\mathrm{M}(t)},\quad t\in t_*,\infty).$$

Integrating above inequality on (t_*,t), we get:

$$\int_{t_*}^{t}\frac{\mathrm{dM}'(\tau)}{\mathrm{M}'(\tau)}>\frac{p_1}{2}\int_{t_*}^{t}\frac{\mathrm{dM}(\tau)}{\mathrm{M}(\tau)},$$

which give:

$$\frac{\mathrm{M}'(t)}{(\mathrm{M}(t))^{\frac{p_1}{2}}}>\frac{\mathrm{M}'(t_*)}{(\mathrm{M}(t_*))^{\frac{p_1}{2}}}.$$

Integrating once more on (t_*,t) gives:

$$\mathrm{M}(t)^{-\frac{p_1-2}{2}}(\mathrm{t})$$
$$<M(t_*)^{-\frac{p_1-1}{2}}\left(1-\frac{(p_1-2)\mathrm{M}'(t_*)}{2\mathrm{M}(t_*)}(t-t_*)\right),$$

i.e.,

$$\mathrm{M}(t)>M(t_*)\left(1-\frac{(p_1-2)\mathrm{M}'(t_*)}{2\mathrm{M}(t_*)}(t-t_*)\right)^{-\frac{2}{p_1-2}}. \tag{37}$$

Observe that, for the time $\bar{t}$

$$0<\bar{t}\leq t_*+\frac{2\mathrm{M}(t_*)}{(p_1-2)\mathrm{M}'(t_*)},$$

we have

$$\lim_{t\to\bar{t}}\mathrm{M}(t)=+\infty,$$

this contradicts the assumption $T_{\max}=+\infty$.

To estimate the upper bound of the blow-up time, we need to consider the following lemmas

Lemma 9 *Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies the inequality*

$$\varphi''(t)\varphi(t)-(1+\theta)(\varphi'(t))^2\geq 0,\qquad t>0,$$

where $\theta>0$ is some constant. If $\varphi(0)>0$ and $\varphi'(0)>0$, then there exists $0<\mathrm{t}_1\leq\frac{\varphi(0)}{\theta\varphi'(0)}$ such that $\varphi(t)$ tends to infinity as $t\to t_1$.

A different auxiliary function is used to prove the blow- up in finite time given by Theorem 8. Moreover, we evaluate the upper bound of the blow-up time.

Noticing that Theorems 8 and 10 present two different proofs of the similar conclusion to the finite time blow-up results.

Theorem 10 *Assuming $p(.)$ satisfy (H) and $u_0\in W_0^{1,m}(\Omega)$ such that $J(u_0)<d$ and $I(u_0)<0$. Then the weak solution $u(t)$ of problem (2) blows up in finite time. The upper bound of the blow-up time estimated as follows.*

$$0<T\leq\frac{4\|u_0\|_2^2}{(p_1-2)^2\beta},$$

where $0<\beta\leq\frac{p_1(d-\mathrm{J}(\mathbf{u}_0))}{p_1-1}$ is a constant.

Proof. We will prove that the existence of time is finite by contradiction. Suppose $T_{\max}=+\infty$. For a suitable $T_0>0$, we define the positive function

$$\mathrm{F}(t):=\frac{1}{2}\int_0^t\|u\|_2^2\,\mathrm{d}\tau+\frac{1}{2}(T_0-t)\|u_0\|_2^2$$
$$+\frac{1}{2}\beta(t+t_0)^2\quad\text{for}\,t\in[0,T_0],t_0>0, \tag{38}$$

where t_0, and T_0 are positive constants to be determined later. Using the definition of $\mathrm{J}(u)$, $\mathrm{I}(u)$ and (12), we get:

$$\mathrm{J}(u)\geq\left(\frac{1}{m_2}-\frac{1}{p_1}\right)\int_\Omega|\nabla u|^{m(x)}\mathrm{d}x+\frac{1}{p_1}\mathrm{I}(u)$$
$$\geq\frac{1}{p_1}\mathrm{I}(u)+\frac{p_1-m_2}{p_1m_2}\int_\Omega|\nabla u|^{m(x)}\mathrm{d}x,$$

that is

$$\mathrm{I}(u)\leq p_1\mathrm{J}(u)-\frac{p_1-m_2}{m_2}\int_\Omega|\nabla u|^{m(x)}\mathrm{d}x. \tag{39}$$

For $v=u$ in (8), we obtain

$$(u,u_t)=-\mathrm{I}(u) \tag{40}$$

From (38)-(40) and (9), we have for any $t\in 0,T)$ that

$$\mathrm{F}'(t)=\frac{1}{2}\|u\|_2^2-\frac{1}{2}\|u_0\|_2^2+\beta(t+t_0)$$

$$= \int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t + t_0), \quad (41)$$

and

$$\begin{aligned} \mathrm{F}''(t) &= (u, u_t) + \beta \\ &\geq -p_1 \mathrm{J(u)} + \tfrac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x + \beta \\ &\geq \tfrac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x \\ &\quad - p_1 \left(\mathrm{J}(u_0) - \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau \right) + \beta \\ &= \tfrac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x - p_1 \mathrm{J}(u_0) \\ &\quad + p_1 \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta. \end{aligned} \quad (42)$$

Therefore, by (38) and (41), it comes:

$$\begin{aligned} \mathrm{FF}'' - \alpha(\mathrm{F}')^2 &= \mathrm{FF}'' - \alpha \begin{pmatrix} \int_0^t (u, u_r)\mathrm{d}\tau \\ +\beta(t + t_0) \end{pmatrix}^2 \\ &= \mathrm{FF}'' - \alpha \left(\int_0^t (u, u_r)\mathrm{d}\tau + \beta(t + t_0) \right)^2 \\ &\quad + \alpha \begin{pmatrix} \int_0^t \| u \|_2^2 \, \mathrm{d}\tau \\ +\beta(t + t_0)^2 \end{pmatrix} \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \right) \\ &\quad - \alpha(2\mathrm{F} - (T - t)\|u_0\|_2^2) \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \right). \end{aligned} \quad (43)$$

Using the Young's inequality from (24), we obtain for any $t \in 0, T)$ that:

$$\begin{aligned} &\left(\int_0^t \| u \|_2^2 \, \mathrm{d}\tau + \beta(t + t_0)^2 \right) \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \right) \\ &- \left(\int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t + t_0) \right)^2 \\ &= \begin{pmatrix} \int_0^t \|u\|_2^2 \mathrm{d}\tau \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau \\ -\left(\int_0^t (u_\tau, u)\mathrm{d}\tau \right)^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \beta \int_0^t \|u\|_2^2 \mathrm{d}\tau + \beta(t + t_0)^2 \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau \\ -2\beta(t + t_0) \int_0^t (u, u_\tau)\mathrm{d}\tau \end{pmatrix} \\ &\geq 2\beta(t + t_0) \left(\int_0^t \| u \|_2^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &-2\beta(t + t_0) \int_0^t (u, u_\tau)\mathrm{d}\tau = 0. \end{aligned} \quad (44)$$

Then by (44) and (42), (43) yields

$$\begin{aligned} &\mathrm{FF}'' - \alpha(\mathrm{F}')^2 \\ &\geq \mathrm{FF}'' - \alpha \begin{pmatrix} 2\mathrm{F} \\ -(T - t)\|u_0\|_2^2 \end{pmatrix} \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \right) \\ &\geq \mathrm{F} \left(\mathrm{F}'' - 2\alpha \left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \right) \right) \\ &\geq \mathrm{F} \begin{pmatrix} \frac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x - p_1 \mathrm{J}(u_0) \\ +p_1 \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta \\ -2\alpha \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau - 2\alpha\beta \end{pmatrix} \\ &\geq \mathrm{F} \begin{pmatrix} \frac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x - p_1 \mathrm{J}(u_0)\tau \\ +(p_1 - 2\alpha) \int_0^t \|u_\tau\|_2^2 \mathrm{d} - (2\alpha - 1)\beta \end{pmatrix}. \end{aligned} \quad (45)$$

Let $\alpha := \frac{p_1}{2}$. (42) give

$$\mathrm{FF}'' - \frac{p_1}{2}(\mathrm{F}')^2 \geq \mathrm{F} \begin{pmatrix} \frac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x \\ -p_1 \mathrm{J}(u_0) - (p_1 - 1)\beta \end{pmatrix}, \quad t \in [0, T].$$

According to Lemma 4 and (13), using property (ii), we can deduce that:

$$\frac{p_1 - m_2}{m_2} \int_\Omega |\nabla u|^{m(x)} \mathrm{d}x > \frac{p_1 - m_2}{m_2} r = p_1 d.$$

If we assume that $0 < \beta \leq \frac{p_1(d - \mathrm{J}(u_0))}{p_1 - 1}$, we can find that:

$$\mathrm{FF}'' - \frac{P_1}{2}(\mathrm{F}')^2 > F\big(p_1(d - \mathrm{J}(u_0)) - (p_1 - 1)\beta\big) \geq 0. \quad (46)$$

Now, using Lemma 9 to confirm that:

$$\lim_{t \to \bar{t}} \mathrm{F}(t) = +\infty,$$

then T cannot be infinite, meaning there is no weak solution at all times, and

$$0 < T \leq \frac{2\mathrm{F}(0)}{(p_1 - 2)\mathrm{F}'(0)} = \frac{T\|u_0\|_2^2 + \beta {t_0}^2}{(p_1 - 2)\beta t_0}. \quad (47)$$

Let's choose appropriate values for t_0 and T_0. We can set t_0 to any number that depends only on p_1, β and u_0

$$t_0 > \frac{\|u_0\|_2^2}{(p_1-2)\beta}.$$

If t_0 is fixed, then T_0 can be chosen as

$$T_0 = \frac{T\|u_0\|_2^2 + \beta t_0^2}{(p_1-2)\beta t_0},$$

so that

$$T_0 = \frac{\beta t_0^2}{(p_1-2)\beta t_0 - \|u_0\|_2^2}.$$

The lifespan of the solution $u(x,t)$ is bounded by a certain number as:

$$T_0 = \inf_{t\geq t_0} \frac{\beta t^2}{((p_1-2)\beta t - \|u_0\|_2^2)} = \frac{4\|u_0\|_2^2}{(p_1-2)^2\beta}. \tag{48}$$

due to the arbitrariness of $T_0 < T$ it follows that

$$T \leq \frac{4\|u_0\|_2^2}{(p_1-2)^2\beta}.$$

We shall estimate the lower bound of the blow-up time. To achieve this, we will use the necessary conditions mentioned in Theorem 8, Theorem 10 and Theorem 11.

Theorem 11 (Lower bound of blow-up time) *Let* $m_1 < p_2 < m_1 + \frac{2m_1}{n}$. *Assume that* $J(u_0) < d$ *and* $I(u_0) < 0$. *The lower bound of the blow-up time estimated as follows*

$$\bar{t} > max\left(\frac{\| u_0 \|_2^{2-\eta p_2}}{(\eta p_2 - 2)\delta}, \frac{\| u_0 \|_2^{2-\eta p_1}}{(\eta p_1 - 2)\delta}\right) > 0.$$

where $\delta = 2\max(\gamma^{p_2}, \gamma^{p_1})$,

$$\gamma = \min\left(\left(2c_g c_2^{\frac{\theta}{m_1}}\right)^{\frac{m_1}{(m_1-p_1\theta)}}, \left(2c_g c_2^{\frac{\theta}{m_1}}\right)^{\frac{m_1}{(m_1-p_2\theta)}}\right),$$

$c_2 = \left(1 + C_*^{\frac{p_2 m_1}{m_1-p_2}\frac{m_1-m_2}{m_2}}\right)$, and c_g is the constant of Gagiiardo-Nirenbery's inequality

$$\|u\|_{p_2} \leq c_g \|\nabla u\|_{m_1}^{\theta} \|u\|_2^{1-\theta},$$

$$\theta = \frac{(p_2-2)nm_1}{p_2(m_1 n + 2m_1 - 2n)} \in (0,1)$$

$$\text{and } 1 < \eta \in \left\{\frac{(1-\theta)m_1}{m_1 - p_2\theta}, \frac{(1-\theta)m_1}{m_1 - p_1\theta}\right\}.$$

Proof. Recall from Theorem 8 that u solution to problem (2) blows up in finite time in the sense $\lim_{t\to\bar{t}} \int_0^t \| u \|_2^2 \, d\tau = +\infty$, i.e.

$$\lim_{t\to\bar{t}} \| u \|_2^2 = +\infty. \tag{49}$$

By Lemma 7 we have $I(u) < 0$, then, by using (11) leads to.

$$\int_\Omega |\nabla u|^{m(x)} dx < \int_\Omega |u|^{p(x)} dx \leq \max\left(\|u\|_{p(x)}^{p_1}, \|u\|_{p(x)}^{p_2}\right) \leq C_*^{p_2} \max\left\{\begin{matrix}\left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{p_2}{m_1}}, \\ \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{p_1}{m_2}}\end{matrix}\right\},$$

which give that

$$\int_\Omega |\nabla u|^{m(x)} dx \geq \max\left\{\left(C_*^{p_2}\right)^{\frac{m_1}{m_1-p_2}}, \left(C_*^{p_2}\right)^{\frac{m_2}{m_2-p_1}}\right\} = C_*^{\frac{p_2 m_1}{m_1-p_2}}, \tag{50}$$

where C_* is the same defined in Lemma4.
In other hand we have

$$\int_\Omega |\nabla u|^{m(x)} dx \geq \int_{\Omega_-} |\nabla u|^{m_2} dx + \int_{\Omega_+} |\nabla u|^{m_1} dx \geq \left(\int_{\Omega_-} |\nabla u|^{m_1} dx\right)^{\frac{m_2}{m_1}} + \int_{\Omega_+} |\nabla u|^{m_1} dx,$$

where

$$\Omega_- = \{x \in \Omega: |\nabla u(x,t)| < 1\},$$
$$\Omega_+ = \{x \in \Omega: |\nabla u(x,t)| \geq 1\}.$$

This implies that:

$$\left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{m_1}{m_2}} \geq \int_{\Omega_-} |\nabla u|^{m_1} dx$$
$$\text{and } \int_\Omega |\nabla u|^{m(x)} dx \geq \int_{\Omega_+} |\nabla u|^{m_1} dx,$$

and, hence

$$\left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{m_1}{m_2}} + \int_\Omega |\nabla u|^{m(x)} dx \geq \int_{\Omega_-} |\nabla u|^{m_1} dx + \int_{\Omega_+} |\nabla u|^{m_1} dx = \| \nabla u \|_{m_1}^{m_1}. \tag{51}$$

Using (50), then (51) leads to:

$$\left(1 + C_*^{\frac{p_2 m_1}{m_1-p_2}\frac{m_1-m_2}{m_2}}\right) \int_\Omega |\nabla u|^{m(x)} dx \geq \| \nabla u \|_{m_1}^{m_1},$$

Thus

$$c_2 \int_\Omega |\nabla u|^{m(x)} dx \geq \| \nabla u \|_{m_1}^{m_1},$$

Then joining Gagliardo-Niremberg's inequality, we obtain:

$$\| u \|_{p_2} \leq c_g \|\nabla u\|_{m_1}^{\theta} \|u\|_2^{1-\theta}$$
$$\leq c_g c_2^{\frac{\theta}{m_1}} \left(\max\left(\| u \|_{p_1}^{p_1}, \| u \|_{p_2}^{p_2}\right)\right)^{\frac{\theta}{m_1}} \|u\|_2^{1-\theta},$$

which comes:

$$\| u \|_{p_2} < \gamma \| u \|_2^{\eta}, \qquad (52)$$

where

$$\gamma = \min\left(\left(2c_g c_2^{\frac{\theta}{m_1}}\right)^{\frac{m_1}{(m_1-p_1\theta)}}, \left(2c_g c_2^{\frac{\theta}{m_1}}\right)^{\frac{m_1}{(m_1-p_2\theta)}}\right),$$
$$1 < \eta \in \left\{\frac{(1-\theta)m_1}{m_1-p_2\theta}, \frac{(1-\theta)m_1}{m_1-p_1\theta}\right\},$$
$$\theta = \frac{(p_2-2)nm_1}{p_2(m_1n+2m_1-2n)} \in (0,1)$$

and

$$p_2\theta < m_1$$

due to $m_1 < p_2 < m_1 + \frac{2m_1}{n}$.

Substituting (52) into (19), yields:

$$\frac{d}{dt} \| u \|_2^2 \leq -2I(u)$$
$$= 2\int_\Omega |u|^{p(x)} dx - 2\int_\Omega |\nabla u|^{m(x)} dx$$
$$< 2\int_\Omega |u|^{p(x)} dx \leq 2\max\left(\|u\|_{p_2}^{p_1}, \|u\|_{p_2}^{p_2}\right)$$
$$< 2max(\gamma^{p_2}, \gamma^{p_1})\max\left(\| u \|_2^{\eta p_2}, \| u \|_2^{\eta p_1}\right)$$
$$= \delta\max\left((\| u \|_2^2)^{\frac{\eta p_2}{2}}, (\| u \|_2^2)^{\frac{\eta p_1}{2}}\right),$$

where $\delta = 2\max(\gamma^{p_2}, \gamma^{p_1})$.

After solving the differential inequality mentioned, we have obtained:

$$\begin{cases} \| u \|_2^{2-\eta p_2} - \| u_0 \|_2^{2-\eta p_2} > (2-\eta p_2)\delta t; \\ \| u \|_2^{2-\eta p_1} - \| u_0 \|_2^{2-\eta p_1} > (2-\eta p_1)\delta t \end{cases}$$

i.e.

$$\begin{cases} \| u \|_2^{2-\eta p_2} + (\eta p_2 - 2)\delta t > \| u_0 \|_2^{2-\eta p_2}; \\ \| u \|_2^{2-\eta p_1} + (\eta p_1 - 2)\delta t > \| u_0 \|_2^{2-\eta p_1}. \end{cases}$$

Since (49) and $p_2\eta \geq p_2\eta > 2$, letting $t \to \bar{t}$, we have

$$\bar{t} > max\left(\frac{\| u_0 \|_2^{2-\eta p_2}}{(\eta p_2 - 2)\delta}, \frac{\| u_0 \|_2^{2-\eta p_1}}{(\eta p_1 - 2)\delta}\right) > 0.$$

4 Global Existence, Asymptotic Behavior and Blow-up in Finite Time with $\mathbf{J}(\boldsymbol{u_0}) = \boldsymbol{d}$

Here we are only reviewing the theories of global existence, asymptotic behavior and blow-up in finite time to extend the results in the subcritical initial energy $J(u_0) < d$ to the critical initial energy $J(u_0) = d$. The proof is omitted because it can be obtained from previous papers referenced in this work with minor modifications.

Theorem 12 (Global existence for $\mathbf{J}(\boldsymbol{u_0}) < \boldsymbol{d}$) *Let $p(.)$ satisfy condition (H) and let $u_0 \in W_0^{1,m(.)}(\Omega)$. Suppose that $J(u_0) = d$ and $I(u_0) \geq 0$. Then the problem (2) admits a global weak solution $u(t) \in L^\infty\left(0, \infty; W_0^{1,m(.)}(\Omega)\right)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$.*

Theorem 13 (Asymptotic behavior of solution for $\mathbf{J}(\boldsymbol{u_0}) = \boldsymbol{d}$) *Let $p(.)$ satisfy (H), $u_0 \in W_0^{1,m(.)}(\Omega)$. Assume that $J(u_0) = d$ and $I(u_0) > 0$. Then for the global weak solution $u(x,t)$ of problem (2), there exists a constant $\kappa > 0$ such that*

$$\| u \|_2 \leq \left(\|u_0\|_2^{2-m_1} + (m_1 - 2)\kappa t\right)^{\frac{1}{2-m_1}}.$$

Theorem 14 (Blow-up for $\mathbf{J}(\boldsymbol{u_0}) = \boldsymbol{d}$) *Assuming that $p(.)$ satisfies (H) and $u_0 \in W_0^{1,m(.)}(\Omega)$, if $J(u_0) = d$ and $I(u_0) < 0$, then the weak solution for problem (2) blows up in a finite time.*

Proof. Firstly, using (18)-(29) and $J(u_0) = d$, we obtain:

$$M(t)M''(t) - \frac{p_1}{2}(M'(t))^2$$
$$\geq \left(\frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}M(t) - p_1\|u_0\|_2^2\right)M'(t)$$
$$+ \left(\frac{C_3^{m_2}(p_1 - m_2)}{m_2}\min\begin{pmatrix}\|u\|_2^{m_1-2}, \\ \|u\|_2^{m_2-2}\end{pmatrix}M'(t) - 2p_1 d\right)M(t).$$

From $J(u_0) = d > 0$, $I(u_0) < 0$, since both $J(u)$ and $I(u)$ are continuous in t, this implies that there exists a $t_1 > 0$ small enough such that $J(u(t_1)) > 0$ and $I(u) < 0$ for $t \in [0, t_1)$. By combining this with equation (40), we can obtain $(u, u_t) = -I(u) > 0$ for $t \in [0, t_1]$, which means that $u_t \neq 0$. Using equation (9), we can further conclude that:

$$0 < J(u(t_1)) \leq d - \int_0^{t_1} \|u_\tau\|_2^2 d\tau = d_1 < d.$$

By putting $t = t_1$ as a new initial time, we have $u \in V$ for $0 < t < \infty$. The rest proof is analogous to Theorem 8.

Theorem 15 *Assuming* $m_1 < p_2 < m_1 + \frac{2m_1}{n}$ *and* $J(u_0) = d$, *with* $I(u_0) < 0$, *we have a lower bound estimate for the blow-up time of the solution to problem (2)*

$$\bar{t} > max\left(\frac{\| u_0 \|_2^{2-\eta p_2}}{(\eta p_2 - 2)\delta}, \frac{\| u_0 \|_2^{2-\eta p_1}}{(\eta p_1 - 2)\delta}\right) > 0.$$

where c_g, η, δ and θ as in in Theorem 11.

Proof. According to Theorem 14, the solution of problem (2) blows up in finite time $T > 0$ and $I(u) < 0$ for $0 < t < T$. The continuing proof is similar to Theorem 11.

5 Blow-up and Blow-up Time with High (sup-critical) Initial Energy $J(u_0) > 0$

In this section, we will prove that the solution to problem (2) has a finite time blow-up. To estimate the upper bound of the blow-up time for the high initial energy, we will employ the concave function method. To prove the main results, we require the use of the following lemma.

Lemma 16 *Assuming that* u_0 *belongs to* $W_0^{1,m(.)}(\Omega)$ *and satisfies*

$$J(u_0) < Amin\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right), \quad (53)$$

where $A = \frac{C_3^{m_2}(p_1 - m_2)}{p_1 m_2}$, and C_3 are defined in (28).
Then

$$u \in \mathcal{N}_- = \left\{u \in W_0^{1,m(.)}(\Omega) | I(u) < 0\right\}$$

Proof. Let $u(t)$ be the weak solution of problem (2). Using the definition of $J(u)$, (12), (26) and (27), we deduce:

$$J(u_0) \geq \frac{1}{p_1} I(u_0) + \frac{p_1 - m_2}{p_1 m_2} \int_\Omega |\nabla u_0|^{m(x)} dx$$
$$\geq \frac{1}{p_1} I(u_0) + \frac{C_3^{m_2}(p_1 - m_2)}{p_1 m_2} \min\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)$$
$$= \frac{1}{p_1} I(u_0) + \text{Amin}\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right).$$

Due to (52), $I(u_0) < 0$. We then prove that $u(t) \in \mathcal{N}_-$ for all $t \in 0, T)$. By contradiction, and using the continuity of $I(u)$ in t, we assume that there exists an $s \in (0, T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, then (40) means:

$$\frac{d}{dt} \| u(t) \|_2^2 = -2I(u) > 0 \text{ for } t \in 0, s),$$

which give:

$$\|u_0\|_2^2 < \| u(s) \|_2^2. \quad (54)$$

By Lemma 5 $_{(ii)}$, we see that:

$$J(u(s)) < J(u_0). \quad (55)$$

From the definition of $J(u)$, $u(s) \in \mathcal{N}$, (26), (27) and (53), we stem:

$$J(u(s)) \geq \frac{1}{p_1} I(u(s)) + \frac{p_1 - m_2}{p_1 m_2} \int_\Omega |\nabla u(s)|^{m(x)} dx$$
$$= \frac{p_1 - m_2}{p_1 m_2} \int_\Omega |\nabla u(s)|^{m(x)} dx$$
$$\geq \frac{C_3^{m_2}(p_1 - m_2)}{p_1 m_2} \min\left(\|u(s)\|_2^{m_1}, \|u(s)\|_2^{m_2}\right)$$
$$= \text{Amin}\left(\|u(s)\|_2^{m_1}, \|u(s)\|_2^{m_2}\right),$$

then additional joining (52) and (53), we obtain

$$\text{Amin}\left(\|u(s)\|_2^{m_1}, \|u(s)\|_2^{m_2}\right) \leq J(u(s)) < J(u_0) < Amin\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right),$$

this contradicts (53).

Next, we prove the finite time blow-up of the solution under $J(u_0) > 0$. We estimate the upper and lower bounds of the blow-up time under the support of Lemma 9 and Theorem 11.

Theorem 17 *Assuming* $u_0 \in W_0^{1,m(.)}(\Omega)$, *and* $J(u_0) > 0$, *let* $p(.)$ *satisfy (H) and (52) hold. Then, the solution* $u(x,t)$ *of problem (2) blows up in finite time. An upper bound estimate of the blow-up time is provided.*

$$0 < t_* \leq \frac{c}{(\alpha - 1)\varepsilon^{-1}\|u_0\|_2^4},$$

where $1 < \alpha < \frac{\text{Amin}\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)}{J(u_0)}$,

$\varepsilon < \frac{2\left(\text{Amin}\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right) - \alpha J(u_0)\right)}{\alpha \|u_0\|_2^2}$, and $c > \frac{1}{4}\varepsilon^{-2}\|u_0\|_2^4$.

Proof. We first assume that u exists in the classical sense on $\Omega \times 0, \infty)$ i.e., $T_{\max} = +\infty$ (The interval of existence of u is unbounded, or u is defined in the whole interval $(0, +\infty)$), and then with the condition (52) show that this leads to a contradiction. We show an $\varphi(t)$ of the following form;

$$\varphi(t) := \int_0^t \| u \|_2^2 \, \mathrm{d}\tau, \qquad \text{for } 0 < t < \infty,$$

then we have

$$\varphi'(t) = \| u \|_2^2 \text{ for all } t \in 0, \infty).$$

From (40) and the definition of $\mathrm{J}(u)$, $\mathrm{I}(u)$, we have:

$$\begin{aligned}\varphi''(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \| u \|_2^2 = -2\mathrm{I}(u)\\ &= -2\left(\int_\Omega |\nabla u|^{m(x)}\mathrm{d}x - \int_\Omega |u|^{p(x)}\mathrm{d}x\right)\\ &\geq -2m_2\left(\int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x\right)\\ &\quad +2\frac{p_1-m_2}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x\\ &= -2m_2\mathrm{J}(u) + 2\frac{p_1-m_2}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x.\end{aligned} \tag{56}$$

we distinguish two cases:

Case 1 $\mathrm{J}(u) \geq 0$ for all $t > 0$. Through (52) we can choose α as such

$$1 < \alpha < \frac{\mathrm{A}\min\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)}{\mathrm{J}(u_0)}. \tag{57}$$

Injecting (9) into (54), we get:

$$\begin{aligned}\varphi''(t) &= 2m_2(\alpha-1)\mathrm{J}(u) - 2m_2\alpha\mathrm{J}(u)\\ &\quad +2\frac{p_1-m_2}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x\\ &> -2m_2\alpha\mathrm{J}(u) + 2\frac{p_1-m_2}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x\\ &\geq -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2\frac{p_1-m_2}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x.\end{aligned} \tag{58}$$

Using Lemma 16, (i.e., $\mathrm{I}(u) < 0$) and (40), we derive:

$$\varphi''(t) = \frac{\mathrm{d}}{\mathrm{d}t} \| u \|_2^2 > 0. \tag{59}$$

making us (26), (27) and (58), (57) becomes:

$$\begin{aligned}\varphi''(t) &> -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2\frac{p_1-m_2}{p_1}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x\\ &\geq -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2\frac{C_3^{m_2}(p_1-m_2)}{p_1}\min\begin{pmatrix}\|u\|_2^{m_1},\\ \|u\|_2^{m_2}\end{pmatrix}\\ &> -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2\frac{C_3^{m_2}(p_1-m_2)}{p_1}\min\begin{pmatrix}\|u_0\|_2^{m_1-2},\\ \|u_0\|_2^{m_2-2}\end{pmatrix}\|u\|_2^2\\ &= -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2Bm_2\|u\|_2^2,\end{aligned} \tag{60}$$

where

$$B = \mathrm{A}\min\left(\|u_0\|_2^{m_1-2}, \|u_0\|_2^{m_2-2}\right).$$

Which gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} \| u \|_2^2 - 2Bm_2\|u\|_2^2 > -2m_2\alpha\mathrm{J}(u_0).$$

Which by solving it, gives:

$$\| u \|_2^2 > \|u_0\|_2^2 e^{2Bm_2t} + \frac{\alpha}{B}\mathrm{J}(u_0)(1 - e^{2Bm_2t}). \tag{61}$$

Substituting (60) into (59) yields

$$\begin{aligned}\varphi''(t) &> -2m_2\alpha\mathrm{J}(u_0) + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\\ &\quad +2Bm_2\|u_0\|_2^2 e^{2Bm_2t}\\ &\quad +2m_2\alpha\mathrm{J}(u_0)(1 - e^{2Bm_2t})\\ &= 2m_2e^{2Bm_2t}\left(B\|u_0\|_2^2 - \alpha\mathrm{J}(u_0)\right)\\ &\quad +2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau.\end{aligned} \tag{62}$$

From (56) we can take $\varepsilon > 0$ such that:

$$0 < \varepsilon < 2\frac{B\|u_0\|_2^2 - \alpha\mathrm{J}(u_0)}{\alpha\|u_0\|_2^2},$$

which combining (61) leads to:

$$\varphi''(t) > \varepsilon\alpha m_2 e^{2Bm_2t}\|u_0\|_2^2 + 2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau. \tag{63}$$

Let ϕ be an auxiliary function defined as:

$$\phi(t) := \varphi^2(t) + \varepsilon^{-1}\|u_0\|_2^2\varphi(t) + \gamma,$$

and $\gamma > 0$ large enough (if needed), so that

$$4\varepsilon^2\gamma > (\varphi'(0))^2. \tag{64}$$

then

$$\phi'(t) = (2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi'(t), \tag{65}$$

and

$$\phi''(t) = (2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t) + 2\left(\varphi'(t)\right)^2. \tag{66}$$

From (64) we obtain:

$$(\phi'(t))^2 = (2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)^2\big(\varphi'(t)\big)^2 = \begin{pmatrix} 4y^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2\varphi(t) \\ +\varepsilon^{-2}\|u_0\|_2^4 \end{pmatrix}\big(\varphi'(t)\big)^2.$$

Let $\delta := 4\gamma - \varepsilon^{-2}(\varphi'(0))^2 > 0$, then

$$(\phi'(t))^2 = \begin{pmatrix} 4\varphi^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2\varphi(t) \\ +4\gamma - \delta \end{pmatrix}\big(\varphi'(t)\big)^2 = (4\phi(t) - \delta)\big(\varphi'(t)\big)^2, \tag{67}$$

i.e.,

$$4\phi(t)\big(\varphi'(t)\big)^2 = (\phi'(t))^2 + \delta\big(\varphi'(t)\big)^2. \tag{68}$$

Noting that

$$\int_0^t (u_t(.,s), u)\mathrm{d}s = \frac{1}{2}\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\|u\|_2^2\right)\mathrm{d}s = \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u_0\|_2^2.$$

Therefore,

$$\|u(t)\|_2^2 = \|u_0\|_2^2 + 2\int_0^t \int_\Omega u_t(.,s)u(s)\mathrm{d}x\mathrm{d}s.$$

Using Holder and Young's inequalities gives:

$$\begin{aligned}
\big(\varphi'(t)\big)^2 &= \| u \|_2^4 \\
&= \left(\|u_0\|_2^2 + 2\int_0^t (u, u_\tau)\mathrm{d}\tau\right)^2 \\
&\leq \left(\|u_0\|_2^2 + 2\left(\int_0^t \| u \|_2^2\, \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\right)^{\frac{1}{2}}\right)^2 \\
&= \|u_0\|_2^4 + 4\|u_0\|_2^2(\varphi(t))^{\frac{1}{2}}\left(\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad +4\varphi(t)\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \\
&\leq \|u_0\|_2^4 + 4\varphi(t)\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau + 2\varepsilon\|u_0\|_2^2\varphi(t) \\
&\quad +2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau.
\end{aligned} \tag{69}$$

From (65) and (67), we get:

$$\begin{aligned}
2\phi(t)\phi''(t) &= 2\begin{pmatrix} \begin{pmatrix} 2\varphi(t) \\ +\varepsilon^{-1}\|u_0\|_2^2 \end{pmatrix}\varphi''(t) \\ +2\big(\varphi'(t)\big)^2 \end{pmatrix}\phi(t) \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)\phi(t) \\
&\quad +4\big(\varphi'(t)\big)^2\phi(t) \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)\phi(t) \\
&\quad +(\phi'(t))^2 + \delta\big(\varphi'(t)\big)^2.
\end{aligned} \tag{70}$$

Now, from (69), 66) and the value of δ, we obtain:

$$\begin{aligned}
&2\phi(t)\phi''(t) - (1+\alpha)(\phi'(t))^2 \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)(t)\phi(t) \\
&+(\phi'(t))^2 + \delta\big(\varphi'(t)\big)^2 - (1+\alpha)(\phi'(t))^2 \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)\phi(t) \\
&-\alpha(\phi'(t))^2 + \delta\big(\varphi'(t)\big)^2 \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)\phi(t) \\
&-\alpha(4\phi(t) - \delta)\big(\varphi'(t)\big)^2 + \delta\big(\varphi'(t)\big)^2 \\
&= 2(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t)\phi(t) \\
&-4\alpha\phi(t)\big(\varphi'(t)\big)^2 + \delta(1+\alpha)\big(\varphi'(t)\big)^2 \\
&> 2\phi(t)\begin{pmatrix} 2\varphi(t) \\ +\varepsilon^{-1}\|u_0\|_2^2 \end{pmatrix}\varphi''(t) - 4\alpha\phi(t)\big(\varphi'(t)\big)^2 \\
&= 2\phi(t)\begin{pmatrix} (2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t) \\ -2\alpha\big(\varphi'(t)\big)^2 \end{pmatrix}.
\end{aligned}$$

Recalling 62), (68), (3), and the fact that $e^{2mBt} > 1$, it results:

$$\begin{aligned}
&(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\varphi''(t) - 2\alpha\big(\varphi'(t)\big)^2 \\
&> (2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2)\begin{pmatrix} m_2\varepsilon\alpha\|u_0\|_2^2\mathrm{e}^{2m_2Bt} \\ +2m_2\alpha\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \end{pmatrix} \\
&-2\alpha\big(\varphi'(t)\big)^2 \\
&> m_2\alpha(2\varphi(t) + \varepsilon^{-1}\|u_0\|_2^2) \\
&\times\left(\varepsilon\|u_0\|_2^2 + 2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\right) - 2\alpha\big(\varphi'(t)\big)^2 \\
&= m_2\alpha\begin{pmatrix} 2\varepsilon\|u_0\|_2^2\varphi(t) + \|u_0\|_2^4 \\ +4\varphi(t)\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \\ +2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau \end{pmatrix} \\
&-2\alpha\big(\varphi'(t)\big)^2 \\
&\geq (m_2\alpha - 2\alpha)\big(\varphi'(t)\big)^2 \geq 0,
\end{aligned}$$

that is

$$\phi(t)\phi''(t) - \frac{1+\alpha}{2}(\phi'(t))^2 > 0,$$

Now, in this case we show that T cannot be infinite, and therefore there is no weak solution all the time.
From Lemma 9, it follows that there exists a $0 < t_1 < +\infty$ such that $\phi(t) \to \infty$ as $t \to t_1$, where

$$0 < t_1 \le \frac{2\phi(0)}{(\alpha-1)\phi'(0)} = \frac{\gamma}{(\alpha-1)\varepsilon^{-1}\|u_0\|_2^4}.$$

Since $\varphi(t)$ is continuous with respect to $\phi(t)$, we conclude that there exists a $T_1 \le t_1$ such that $\lim_{t\to T_1}\|u(s)\|_2^2 ds = +\infty \Rightarrow \lim_{t\to T_1}\sup\|u(t)\|_2^2 = +\infty$.

Hence, $u(x,t)$ discontinuing at some finite time T_1. Now, by considering the continuity of ϕ with respect to y, we can conclude that $\varphi(t)$ tends to infinity at some finite time, that is to means, $u(x,t)$ not exist for all time, i.e. $u(x,t)$ blows up at a time T_1, which will lead to the nonexistence result stated in the theorem, then ϕ blows up at time T_1 in $L^2(\Omega)$-norm, which contradicts. Hence, for the data satisfy (52) any solution possesses finite explosion time.

Case 2 Assume that there exists $t_0 > 0$ such that $J(u(t_0)) < 0$, $(u(t_0) \neq 0)$. Noting that $J(0) > 0$ and considering the continuity of $J(t)$, we know that there exists $t_1 \in (0, t_0)$ such that $J(t_1) = 0$. In addition, we apply the monotonicity of $J(t)$ to obtain $J(t) \ge 0$, $0 < t \le t_1$. In a similar way as in **Case 1**, we can prove that the solution to problem (2) break down before the time t_0.

Combining **Case 1** and **Case 2**, we conclude the blow-up of solution in finite time. Since $\mathrm{J}(u_0) < Amin\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)$ indicates $\mathrm{I}(u) < 0$, we can get the same lower bound of blow-up time as $\mathrm{J}(u_0) \le d$. This ends the proof.

Theorem 18 *Under the assumptions*

$$m_1 < p_2 < m_1 + \frac{2m_1}{n},$$

$d < J(u_0) < Amin\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)$. *The lower bounded of blow-up time of solution for problem (2) estimate by*

$$\bar{t} > max\left(\frac{\|u_0\|_2^{2-\eta p_2}}{(\eta p_2 - 2)\delta}, \frac{\|u_0\|_2^{2-\eta p_1}}{(\eta p_1 - 2)\delta}\right) > 0.$$

where c_g, η, δ and θ are defined as in Theorem 11.
Proof. From Lemma 16, we see that $\mathrm{I}(u) < 0$. Then, the rest proof is analogous to Theorem 11.

Remark 19 *Conditions assumed in Theorem 17 are compatible. To show this, we fix* $u_0 \in W_0^{1,m(.)}(\Omega)$ *large enough so that* $J(u_0) > 0$ *and* $J(u_0) < Amin\left(\|u_0\|_2^{m_1}, \|u_0\|_2^{m_2}\right)$, *where* $A = \frac{C_3^{m_2}(p_1 - m_2)}{p_1 m_2}$, *and* C_3 *are defined by (28). Let* $u_0 = \lambda\phi$, *with* $\|u_0\|_2 \le 1$, *it is permissible because if it is not, we can take* $\left\|\frac{u_0}{|u_0|}\right\|_2$ *instead of* $\|u_0\|_2$, *and* $\lambda > 0$ *is some positive constant which will be defined later,* ϕ *is non-zero function in* $W_0^{1,p(.)}(\Omega)$ *that will be defined later. First, for* $\phi \in W_0^{1,p(.)}(\Omega)$ *we have*

$$\|u_0\|_2^{m_2} = \lambda^{m_2} \| \phi \|_2^{m_2} > 0. \tag{71}$$

For this fixed ϕ and $m_2 < p_1$, we pick $\lambda^{p_1 - m_2} < \frac{p_1 C_3^{m_2}}{m_2}\frac{\|\phi\|_2^{m_2}}{\|\phi\|_{p_1}^{p_1}}$, using (11) to ensure that:

$$\begin{aligned}
\mathrm{J}(u_0) &= \int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}\mathrm{d}x \\
&\ge \frac{C_3^{m_2}}{m_2}\|u_0\|_2^{m_2} - \frac{1}{p_1}\|u_0\|_{p_1}^{p_1} \\
&= C_3^{m_2}\frac{\lambda^{m_2}}{m_2}\| \phi \|_2^{m_2} - \frac{\lambda^{p_1}}{p_1}\| \phi \|_{p_1}^{p_1} \\
&= \lambda^{m_2}\left(\frac{C_3^{m_2}}{m_2}\| \phi \|_2^{m_2} - \frac{\lambda^{p_1 - m_2}}{p_1}\| \phi \|_{p_1}^{p_1}\right) > 0.
\end{aligned} \tag{72}$$

Next, we verify the condition (52). By comparing (35) and (36), we only need to verify:

$$\| \phi \|_2^{m_2} > \frac{\frac{C_3^{m_2}}{m_2}\| \phi \|_2^{m_2} - \frac{\lambda^{p_1 - m_2}}{p_1}\| \phi \|_{p_1}^{p_1}}{\mathrm{A}}.$$

A simple calculation shows that we need:

$$\lambda^{p_2 - m_2} > \frac{C_3^{m_2} p_1 \| \phi \|_2^{m_2}}{m_2 \| \phi \|_{p_1}^{p_1}} - \frac{\mathrm{A}p_1 \| \phi \|_2^{m_2}}{\| \phi \|_{p_1}^{p_1}},$$

also

$$\lambda^{p_2 - m_2} \in \left] \frac{C_3^{m_2} p_1 \| \phi \|_2^{m_2}}{m_2 \| \phi \|_{p_1}^{p_1}} - \frac{\mathrm{A}p_1 \| \phi \|_2^{m_2}}{\| \phi \|_{p_1}^{p_1}}, \frac{C_3^{m_2} p_1 \| \phi \|_2^{m_2}}{m_2 \| \phi \|_{p_1}^{p_1}} \right[$$

$$\text{if } \frac{C_3^{m_2} p_1 \|\phi\|_2^{m_2}}{m_2\|\phi\|_{p_1}^{p_1}} - \frac{\mathrm{A}p_1\|\phi\|_2^{m_2}}{\|\phi\|_{p_1}^{p_1}} > 0,$$

$$\lambda^{p_2 - m_2} \in \left] 0, \frac{C_3^{m_2} p_1 \| \phi \|_2^{m_2}}{m_2 \| \phi \|_{p_1}^{p_1}} \right[$$

$$\text{if } \frac{C_3^{m_2} p_1 \|\phi\|_2^{m_2}}{m_2\|\phi\|_{p_1}^{p_1}} - \frac{\mathrm{A}p_1\|\phi\|_2^{m_2}}{\|\phi\|_{p_1}^{p_1}} < 0.$$

Hence, there exists an initial value $u_0 = \lambda\phi$ to satisfy $J(u_0) > 0$, then the last condition in Theorem 17 follows.

6 Conclusion

The findings related to diffusion problems play a crucial role in engineering applications by offering valuable insights into the behavior of materials, systems, and processes in real-world situations. These insights are essential across various engineering fields, including materials science, chemical engineering, and biomedical engineering, as they help engineers design, optimize, and predict the behavior of complex systems.

By applying the results from diffusion problems, engineers can create more efficient, sustainable, and innovative solutions to real-world challenges, ultimately enhancing the quality of life and promoting economic growth.

Blow-up phenomena refer to the rapid growth or explosion of a solution to a differential equation within a finite time. In the context of reaction-diffusion equations, blow-up can occur when the strength of the reaction term surpasses that of the diffusion term, leading to unbounded growth in the solution. Analyzing reaction-diffusion equations that involve a non-autonomous m(.)-Laplacian and variable exponent nonlinearities can complicate the analysis due to these nonlinearities. This complexity may result in challenging mathematical issues and could potentially lead to blow-up phenomena in certain cases. To determine blow-up outcomes for a specific reaction-diffusion equation, a thorough analysis of the equation's properties and behavior is generally required. This may involve techniques such as energy methods, maximum principles, or numerical simulations.

In this paper, we consider a reaction-diffusion equation with variable exponent sources. We examine three initial energy levels: sub-critical, critical, and supercritical. For the sub-critical initial energy, we present the blow-up result and estimate both the lower and upper bounds of the blow-up time. In the case of critical initial energy, we demonstrate the global existence, asymptotic behavior, finite-time blow-up, and the lower bound of the blow-up time. Finally, for the supercritical initial energy, we establish the occurrence of finite-time blow-up and estimate the lower and upper bounds of the blow-up time.

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