Decenralized Robust Servomechanism Problem for Descriptor-type Discrete-time-delay Systems

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Abstract: - Decentralized stabilizing controller design for a system to achieve robust tracking of certain reference signals despite certain disturbances and modeling uncertainties is an important problem, which is known as the decentralized robust servomechanism problem. In this work, this problem is considered for linear time-invariant descriptor-type discrete-time-delay systems. The necessary and sufficient conditions for the solvability of this problem are presented. The structure of the controller which solves this problem, when a solution exists, is also given.

Key-Words: - Servomechanism problem, Decentralized control, Robust tracking, Disturbance rejection, Descriptor-type systems, Time-delay systems

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1 Introduction

Many practical systems involve time delays, [1]. Dynamics of such systems, known as time-delay systems, [2], can be described by delay-differential equations, [3]. However, there are also many examples of time-delay systems (e.g., telerobotic systems, [4]), whose dynamics can be described only by delay-differential equations coupled by delay-algebraic equations. Such systems are known as descriptor-type time-delay systems, [5]. Time delays in a system may be discrete or distributed, [6]. Although distributed delay is more challenging, discrete delays occur more naturally, [7].

An important problem in control engineering is the so-called *servomechanism problem*, [8], which is to design a controller for a plant such that the controlled system is asymptotically stable and the plant output asymptotically tracks a given reference despite certain disturbances. Furthermore, since it is not in general possible to exactly model any practical system, [9], it is important that such tracking takes place despite certain uncertainties in the plant model. The problem of designing a controller for a plant such that the controlled system is asymptotically stable and the plant output asymptotically tracks a given reference despite certain disturbances and uncertainties in the plant model is called the *robust servomechanism problem*, [10].

For many large-scale systems, it is not possible or practical to collect all the measurements in a centralized place, process them there and dispatch the control signals to all the input ports, [11]. Such systems require *decentralized control*, [12], where each local control agent applies local controls, based on local measurements. Large-scale systems are especially prone to time-delays, [13], [14]. Decentralized controller design for such systems has been considered in many works, such as [15], [16], [17], [18], among others.

Robust servomechanism problem under a decentralized control stucture was first considered in [19]. A number of approaches to solve this problem, which is commonly known as the *decentralized robust servomechanism problem*, have also been suggested, [20], [21].

Although, earlier consideration of robust servomechanism and decentralized robust servomechanism problems were restricted to delay-free systems, recently they have also been considered for time-delay systems, [22].

The robust servomechanism problem for centralized descriptor-type time-delay systems was considered in [23]. Decenralized robust servomechanism problem for non-descriptor-type distributed-time-delay systems was then considered in [24]. In the present work, we consider the decentralized robust servomechanism problem for linear time-invariant (LTI) descriptor-type discrete-time-delay systems. We present the necessary and sufficient conditions for the solvability of this problem and also present the structure of the controller which solves this problem when a solution exists.

After explaining the notation in the next paragraph, we formally state our problem in the next section. We present some necessary preliminaries in Section 3. Our main results are given in Section 4. Finally, Section 5 includes some concluding remarks.

Throughout, $\mathbf{R}$ and $\mathbf{C}$ denote the sets of real and complex numbers, respectively. $\mathbf{R}^k$ denotes the space of k-dimensional real vectors. $\mathrm{Re}(\cdot)$ denotes the real part of $(\cdot)$. I_k and I respectively denote the $k \times k$-dimensional identity matrix and an identity matrix of appropriate dimensions. 0_k, $0_{k\times l}$, and 0, respectively denote the $k \times k$-dimensional zero matrix, $k \times l$-dimensional zero matrix, and a zero matrix of appropriate dimensions. $\det(\cdot)$ and $\mathrm{rank}(\cdot)$ respectively denote the determinant and the rank of $(\cdot)$. $\mathrm{bdiag}(\cdots)$ denotes a block diagonal matrix with $(\cdots)$ on the main diagonal. For a (vector) function g, $\dot{g}$ denotes the derivative of g. $\otimes$ denotes the Kronecker product. Finally, p denotes the differentiation operator and d_θ denotes the delay operator by θ; i.e., for any positive integer k and any at least k-times differentiable (vector) function f,

$$p^k f(t) = \frac{d^k}{dt^k} f(t) \tag{1}$$

and, for any (vector) function f and any non-negative number θ,

$$d_\theta f(t) = f(t-\theta) \; . \tag{2}$$

Furthermore, as a natural extension of (1), $p^0 f(t) = f(t)$, for any (vector) function f.

2 Problem Statement

We consider an LTI decentralized descriptor-type time-delay system with μ discrete time-delays and ν control agents. Such a system contains a *delay-differential part* and a *delay-algebraic part*, dynamics of which can be described respectively as

$$\dot{\xi}_1(t) = \sum_{i=0}^{\mu} \Big[A_i^{11}\xi_1(t-\theta_i) + A_i^{12}\xi_2(t-\theta_i) + B_i^1\omega(t-\theta_i) + \textstyle\sum_{j=1}^{\nu} C_{ij}^1 \upsilon_j(t-\theta_i) \Big] \tag{3}$$

and

$$0 = \sum_{i=0}^{\mu} \Big[A_i^{21}\xi_1(t-\theta_i) + A_i^{22}\xi_2(t-\theta_i) + B_i^2\omega(t-\theta_i) + \textstyle\sum_{j=1}^{\nu} C_{ij}^2 \upsilon_j(t-\theta_i) \Big] \; , \tag{4}$$

where $\xi_1(t) \in \mathbf{R}^{n_1}$ and $\xi_2(t) \in \mathbf{R}^{n_2}$ are the state vectors for the delay-differential and delay-algebraic parts, respectively, at time t. Furthermore, $\omega(t) \in \mathbf{R}^{n^\omega}$ is the *disturbance input* and $\upsilon_j(t) \in \mathbf{R}^{n_j^\upsilon}$ is the *control input* of the $j^{\rm th}$ control agent ($j = 1, \ldots, \nu$) at time t. Moreover, $\theta_i > 0$, $i = 1, \ldots, \mu$, are the time-delays and $\theta_0 := 0$ (i.e., $i = 0$ indicates the delay-free part of the system). All the matrices shown in (3)–(4) are appropriately dimensioned constant real matrices. To assure the existence and uniqueness of solutions to (3)–(4), it is assumed that $\det\left(A_0^{22}\right) \neq 0$, [5]. Although the system (3)–(4) in general has infinitely many modes, it is known that it has only finitely many modes with real part greater than λ_f, [3], where

$$\lambda_f := \sup \left\{ \mathrm{Re}(s) \mid \det\left(\sum_{i=0}^{\mu} A_i^{22} e^{-s\theta_i} \right) = 0 \right\} \; . \tag{5}$$

Since it is not in general possible to robustly stabilize an LTI time-delay system by a proper LTI controler if it has infinitely many modes with non-negative real part, [25], here we also assume that $\lambda_f < 0$.

The *measurement* $\eta_j(t) \in \mathbf{R}^{n_j^\eta}$, which is given by

$$\eta_j(t) = \sum_{i=0}^{\mu} \Big[D_{ij}^1\xi_1(t-\theta_i) + D_{ij}^2\xi_2(t-\theta_i) + E_{ij}\omega(t-\theta_i) + \textstyle\sum_{k=1}^{\nu} F_{ijk} \upsilon_k(t-\theta_i) \Big] \; , \tag{6}$$

is assumed to be available to the $j^{\rm th}$ control agent ($j = 1, \ldots, \nu$) at time t. Furthermore, the $j^{\rm th}$ control agent is suppose to *regulate* the *output* $\psi_j(t) \in \mathbf{R}^{n_j^\psi}$, which is given by

$$\psi_j(t) = \sum_{i=0}^{\mu} \Big[G_{ij}^1 \xi_1(t-\theta_i) + G_{ij}^2 \xi_2(t-\theta_i) + H_{ij}\omega(t-\theta_i) + \textstyle\sum_{k=1}^{\nu} K_{ijk} \upsilon_k(t-\theta_i)\Big] \ . \quad (7)$$

All the matrices shown in (6)–(7) are appropriately dimensioned constant real matrices.

Here, the j^{th} output, $\psi_j(t)$, is required to asymptotically track the j^{th} *reference*, $\rho_j(t) \in \mathbf{R}^{n_j^\psi}$, i.e., it is required that the j^{th} *tracking error*, $\epsilon_j(t) \in \mathbf{R}^{n_j^\psi}$, given by

$$\epsilon_j(t) := \psi_j(t) - \rho_j(t) \ , \quad (8)$$

satisfies

$$\lim_{t\to\infty} \epsilon_j(t) = 0 \ , \quad j = 1, \ldots, \nu \ . \quad (9)$$

It is assumed that the j^{th} reference, $\rho_j(t)$, is available to the j^{th} control agent on-line, not necessarily known in advance, but satisfies

$$\mathcal{D}_j \rho_j(t) = 0 \ , \quad j = 1, \ldots, \nu \ , \quad (10)$$

where $\mathcal{D}_j$ is a linear delay-differential operator.

On the other hand, the disturbance, $\omega(t)$, is not available to any one of the control agents, not necessarily known in advance, but satisfies

$$\mathcal{D}_\omega \omega(t) = 0 \ , \quad (11)$$

where $\mathcal{D}_\omega$ is another linear delay-differential operator.

Let $\mathcal{D}$ be the least common multiple of $\mathcal{D}_1$, …, $\mathcal{D}_\nu$, $\mathcal{D}_\omega$, which is given by

$$\mathcal{D} := p^m + \sum_{k=0}^{m-1} \sum_{i=0}^{\mu} \alpha_{ki} p^k d_{\theta_i} \ , \quad (12)$$

where m is the differential degree of $\mathcal{D}$ and α_{ki}'s are real constant coefficients. Operators p and d_θ are defined at the end of Section 1. We note that when $m = 1$ (e.g., when all the references and the disturbance include only step-like signals), (12) reduces to $\mathcal{D} := p + \sum_{i=0}^{\mu} \alpha_{0i} d_{\theta_i}$. Here, we used the same time-delays both in the system definition (3)–(4) & (6)–(7) and in the delay-differential operator $\mathcal{D}$. This, however, can be done without loss of generality, since any time-delay present in the system but not in $\mathcal{D}$, can be included in $\mathcal{D}$ by multiplying them by zero coefficients, and any time-delay present in $\mathcal{D}$ but not in the system, can be included in the system definition (3)–(4) & (6)–(7) by multiplying them by zero matrices.

We can now formally state our problem.

Decenralized Robust Servomechanism Problem (DRSP): Design ν decentralized LTI (possibly time-delay) feedback controllers (each from η_j to υ_j, $j = 1, \ldots, \nu$) for the system described by (3)–(4) & (6)–(7), such that the closed-loop system is asymptotically stable and, for all initial condtions of the system (3)–(4), for all references $\rho_j(t)$, satisfying (10), for all disturbances $\omega(t)$ satisfying (11), and for all non-destabilizing perturbations in the system matrices in (3)–(4) & (6)–(7), (9) is satisfied.

Since DRSP is concerned with asymptotic stability, in the remainder of this paper, by “stable” we will mean *asymptotically stable* and by “stabilization” we will mean *asymptotic stabilization*.

3 Preliminaries

Defining $\xi(t) := \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \in \mathbf{R}^{n^\xi}$, where $n^\xi := n_1 + n_2$, system (3)–(4) & (6)–(7) can be compactly written as

$$L\dot{\xi}(t) = \sum_{i=0}^{\mu} \Big[A_i \xi(t-\theta_i) + B_i \omega(t-\theta_i) + \textstyle\sum_{j=1}^{\nu} C_{ij} \upsilon_j(t-\theta_i)\Big] \quad (13)$$

$$\eta_j(t) = \sum_{i=0}^{\mu} \Big[D_{ij} \xi(t-\theta_i) + E_{ij} \omega(t-\theta_i) + \textstyle\sum_{k=1}^{\nu} F_{ijk} \upsilon_k(t-\theta_i)\Big] \quad (14)$$

$$\psi_j(t) = \sum_{i=0}^{\mu} \Big[G_{ij} \xi(t-\theta_i) + H_{ij} \omega(t-\theta_i) + \textstyle\sum_{k=1}^{\nu} K_{ijk} \upsilon_k(t-\theta_i)\Big] \ , \quad (15)$$

$j = 1, \ldots, \nu$, where $L := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0_{n_2} \end{bmatrix}$, $A_i := \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix}$, $B_i := \begin{bmatrix} B_i^1 \\ B_i^2 \end{bmatrix}$, $C_{ij} := \begin{bmatrix} C_{ij}^1 \\ C_{ij}^2 \end{bmatrix}$, $D_{ij} := \begin{bmatrix} D_{ij}^1 & D_{ij}^2 \end{bmatrix}$, and $G_{ij} := \begin{bmatrix} G_{ij}^1 & G_{ij}^2 \end{bmatrix}$. Any $\lambda \in \mathbf{C}$, satisfying $\det\left(\lambda L - \sum_{i=0}^{\mu} A_i e^{-\lambda\theta_i}\right) = 0$ is said to be a *mode* of the system (13) (equivalenly of the system (3)–(4)). A mode with a nonnegative real part is said to be an *unstable mode*. Although system (13) has in general infinitely many modes, there are only finitely many modes λ with $\mathrm{Re}(\lambda) \geq \lambda_f$, given by (5). By the assumption made in Section 2, we have $\lambda_f < 0$. Hence, system (13) has only finitely many unstable modes. A descriptor-type LTI time-delay system of the form (13) is stable if and only if it has no unstable modes and $\lambda_f < 0$, [26].

Consider the decentralized LTI static feedback law for the system (13)–(14):

$$\upsilon_j(t) = J_j \eta_j(t) \; , \quad j = 1, \ldots, \nu \; , \qquad (16)$$

where J_j, $j = 1, \ldots, \nu$, are appropriately dimensioned constant real matrices satisfying $\det(I - JF_0) \neq 0$, where $J := \mathrm{bdiag}(J_1, \ldots, J_\nu)$ and

$$F_0 := \begin{bmatrix} F_{011} & \cdots & F_{01\nu} \\ \vdots & & \vdots \\ F_{0\nu 1} & \cdots & F_{0\nu\nu} \end{bmatrix} .$$

This condition is required for the well-posedness of the closed-loop system, [27].

A mode of (13) which remains a mode of the closed-loop system under all controls of the form (16) is known as a *decentralized fixed mode (DFM)* of (13)–(14), [28]. A necessary and sufficient condition for $\lambda \in \mathbf{C}$ to be a DFM of (13)–(14) is that for some $\kappa \in \{0, \ldots, \nu\}$ and $\{k_1, \ldots, k_\kappa\} \subset \{1, \ldots, \nu\}$, where $k_1, \ldots, k_\kappa$ are distinct ($\{k_1, \ldots, k_\kappa\} = \emptyset$ if $\kappa = 0$),

$$\mathrm{rank} \begin{bmatrix} \bar{A}(\lambda) - \lambda L & \bar{C}_{k_1}(\lambda) & \cdots & \bar{C}_{k_\kappa}(\lambda) \\ \bar{D}_{k_{\kappa+1}}(\lambda) & \bar{F}_{k_{\kappa+1},k_1}(\lambda) & \cdots & \bar{F}_{k_{\kappa+1},k_\kappa}(\lambda) \\ \vdots & \vdots & & \vdots \\ \bar{D}_{k_\nu}(\lambda) & \bar{F}_{k_\nu,k_1}(\lambda) & \cdots & \bar{F}_{k_\nu,k_\kappa}(\lambda) \end{bmatrix} < n^\xi \; , \qquad (17)$$

where $\{k_{\kappa+1}, \ldots, k_\nu\} := \{1, \ldots, \nu\} \setminus \{k_1, \ldots, k_\kappa\}$, $\bar{A}(s) := \sum_{i=0}^{\mu} A_i e^{-s\theta_i}$, $\bar{C}_j(s) := \sum_{i=0}^{\mu} C_{ij} e^{-s\theta_i}$, $\bar{D}_j(s) := \sum_{i=0}^{\mu} D_{ij} e^{-s\theta_i}$, and $\bar{F}_{jk}(s) := \sum_{i=0}^{\mu} F_{ijk} e^{-s\theta_i}$, [27]. Given that $\lambda_f < 0$, a necessary and sufficient condition for the existence of a (possibly dynamic) decentralized LTI feedback controller which stabilizes (13)–(14) is that (13)–(14) must not have any unstable DFMs, [27]. We note that, all the modes of the system (13), which has real part greater than λ_f (hence all the unstable modes, since $\lambda_f < 0$) can be found using the spectral discretization method of [29]. Once these modes have been found, the test (17) can be used to determine all unstable DFMs, if any.

4 Main Results

In this section, the necessary and sufficient conditions for the solvability of DRSP and (when a solution exists) the structure of the controller which solves this problem will be presented. For this purpose, let us first define the fictitious system:

$$\dot{\phi}(t) = \sum_{i=0}^{\mu} M_i \phi(t - \theta_i) \; , \qquad (18)$$

where $\phi(t) \in \mathbf{R}^m$ is the state vector at time t,

$$M_0 := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_{00} \\ 1 & 0 & & & -\alpha_{10} \\ 0 & 1 & & & -\alpha_{20} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -\alpha_{m-1,0} \end{bmatrix} ,$$

and, for $i = 1, \ldots, \mu$,

$$M_i := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_{0i} \\ 0 & 0 & \cdots & 0 & -\alpha_{1i} \\ 0 & 0 & \cdots & 0 & -\alpha_{2i} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_{m-1,i} \end{bmatrix}$$

(which reduce to $M_i := -\alpha_{0i}$, $i = 0, \ldots, \mu$, when $m = 1$), where α_{ki}'s are the coefficients in (12). Then, for some arbitrary matrices N_1, …, N_ν, N_ω (these matrices, as well as the initial condition of (18) are arbitrary, since $\rho_1(t)$, …, $\rho_\nu(t)$, and $\omega(t)$ are unknown apart from the fact that they satisfy (10) and (11)), $\rho_1(t)$, …, $\rho_\nu(t)$, and $\omega(t)$ can be expressed as

$$\rho_j(t) = N_j \phi(t) \; , \quad j = 1, \ldots, \nu \; , \qquad (19)$$

and

$$\omega(t) = N_\omega \phi(t) \; . \qquad (20)$$

To avoid triviality, it is assumed that the system (18) is totally unstable, i.e., $\lim_{t\to\infty} \phi(t) = 0$ only if $\phi(\theta) = 0$, for all $\theta \in [-\theta_{\max}, 0]$, where $\theta_{\max}$ is the maximum time-delay in (12). If this assumption does not hold, then $\lim_{t\to\infty} \phi(t) = 0$, for any initial condition of (18), which in turn implies $\lim_{t\to\infty} \rho_j(t) = 0$, $j = 1, \ldots, \nu$, and $\lim_{t\to\infty} \omega(t) = 0$. Thus, DRSP reduces to a decentralized stabilization problem and any decentralized controller which stabilizes (13)–(14) (which exists if and only if (13)–(14) does not have any unstable DFMs, [27]) also solves DRSP. Note that, this assumption means that there exists at least one $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) \geq 0$, such that $\det\left(\lambda I - \sum_{i=0}^{\mu} M_i e^{-\lambda\theta_i}\right) = 0$. In fact, typically, (18) has some modes with zero real parts (i.e., $\det\left(\lambda I - \sum_{i=0}^{\mu} M_i e^{-\lambda\theta_i}\right) = 0$, for some $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) = 0$) so that the references and/or the disturbance contain some undamped oscillations and/or some polynomials (e.g., step, ramp, etc.).

Furthermore, again to avoid triviality, we also assume that the system (18) with output

$$\begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_\nu(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} N_1 \\ \vdots \\ N_\nu \\ N_\omega \end{bmatrix} \phi(t) \,, \tag{21}$$

is spectrally observable, [2]. If this assumption does not hold, it means that the system (18) has some dynamics which affect neither any one of the references, nor the disturbance.

We note that both of the above two assumptions are necessary only for the necessity part of our main result (Theorem 1 below). Even if any one of these two assumptions do not hold, the sufficiency part of Theorem 1 continues to hold and the controller described below (which is depicted in Fig. 1 for the case $\nu = 2$) still solves DRSP.

Next, we note that, in order to achieve robust tracking under the above assumptions, i.e., in order to achieve (9), for all uncertainties in the matrices appearing in (6) and (7), by decenralized feedback, the output $\psi_j(t)$ must be included in the measurement $\eta_j(t)$, for each $j = 1, \ldots, \nu$, [19]. Therefore, here we also assume that for some non-negative integers $\hat{\mu}_j$, some positive delays $\hat{\theta}_{ij}$, $i = 1, \ldots, \hat{\mu}_j$, and some appropriately dimensioned matrices P_{ij}, $i = 0, \ldots, \hat{\mu}_j$,

$$\psi_j(t) = \sum_{i=0}^{\hat{\mu}_j} P_{ij}\eta_j(t - \hat{\theta}_{ij}) \,, \quad j = 1, \ldots, \nu \,, \tag{22}$$

where $\hat{\theta}_{0j} := 0$, $j = 1, \ldots, \nu$. We note that in many cases we may have $\hat{\mu}_j = 0$, for some or all j, in which case, (22) will reduce to $\psi_j(t) = P_{0j}\eta_j(t)$.

Now, to present our main result, let us define the following decentralized control system with state vector $\hat{\xi}(t) \in \mathbf{R}^{n^\xi + mn^\psi}$, where $n^\psi := \sum_{j=1}^{\nu} n_j^\psi$, inputs $\hat{\upsilon}_j(t) \in \mathbf{R}^{n_j^\upsilon}$, and measurements $\hat{\eta}_j(t) \in \mathbf{R}^{n_j^\eta + mn_j^\psi}$, $j = 1, \ldots, \nu$:

$$\hat{L}\dot{\hat{\xi}}(t) = \sum_{i=0}^{\mu} \left[\hat{A}_i\hat{\xi}(t - \theta_i) + \textstyle\sum_{j=1}^{\nu} \hat{C}_{ij}\hat{\upsilon}_j(t - \theta_i)\right] \tag{23}$$

$$\hat{\eta}_j(t) = \sum_{i=0}^{\mu} \left[\hat{D}_{ij}\hat{\xi}(t - \theta_i) + \textstyle\sum_{k=1}^{\nu} \hat{F}_{ijk}\hat{\upsilon}_k(t - \theta_i)\right] \tag{24}$$

$j = 1, \ldots, \nu$, where $\hat{L} := \begin{bmatrix} L & 0 \\ 0 & I_{mn^\psi} \end{bmatrix}$, $\hat{A}_i := \begin{bmatrix} A_i & 0 \\ QG_i & \hat{M}_i \end{bmatrix}$, $\hat{C}_{ij} := \begin{bmatrix} C_{ij} \\ QK_{ij} \end{bmatrix}$, $\hat{D}_{ij} := \begin{bmatrix} D_{ij} & 0 \\ 0 & \hat{R}_{ij} \end{bmatrix}$, and $\hat{F}_{ijk} := \begin{bmatrix} F_{ijk} \\ 0_{mn_j^\psi \times n_k^\upsilon} \end{bmatrix}$, where $Q := \begin{bmatrix} I_{n^\psi} \\ 0_{(m-1)n^\psi \times n^\psi} \end{bmatrix}$ (which reduces to $Q := I_{n^\psi}$ if $m = 1$), $G_i := \begin{bmatrix} G_{i1} \\ \vdots \\ G_{i\nu} \end{bmatrix}$, $\hat{M}_i := M_i \otimes I_{n^\psi}$, $K_{ij} := \begin{bmatrix} K_{i1j} \\ \vdots \\ K_{i\nu j} \end{bmatrix}$, for $i = 0, \ldots, \mu$ and $j, k = 1, \ldots, \nu$. Finally, $\hat{R}_{0j} := I_m \otimes R_j$, and, for $i = 1, \ldots, \mu$, $\hat{R}_{ij} := 0_{mn_j^\psi \times mn^\psi}$, where $R_j := \begin{bmatrix} 0_{n_j^\psi \times \sum_{k=1}^{j-1} n_k^\psi} & I_{n_j^\psi} & 0_{n_j^\psi \times \sum_{k=j+1}^{\nu} n_k^\psi} \end{bmatrix}$, $j = 1, \ldots, \nu$, where $\sum_{k=1}^{0}(\cdot)$ and $\sum_{k=\nu+1}^{\nu}(\cdot)$ are to be interpreted as zero (i.e., the first zero block does not appear in R_1 and the last zero block does not appear in R_ν).

We can now state our main result:

Theorem 1: There exists a solution to DRSP if and only if the decentralized control system (23)–(24) does not have any unstable DFMs.

Proof: First, to prove the necessity, let us define

$$\epsilon(t) := \begin{bmatrix} \epsilon_1(t) \\ \vdots \\ \epsilon_\nu(t) \end{bmatrix}$$

and

$$\tilde{\epsilon}(t) := \begin{cases} \epsilon(t) \,, & \text{if } m = 1 \\ \begin{bmatrix} \epsilon^{m-1}(t) \\ \vdots \\ \epsilon^1(t) \\ \epsilon(t) \end{bmatrix} \,, & \text{if } m \geq 2 \end{cases} \,,$$

where (for $m \geq 2$)

$$\epsilon^1(t) := \dot{\epsilon}(t) + \sum_{i=0}^{\mu} \alpha_{m-1,i}\epsilon(t - \theta_i) \,,$$

and (for $m \geq 3$)

$$\epsilon^l(t) := \frac{d}{dt}\epsilon^{l-1}(t) + \sum_{i=0}^{\mu} \alpha_{m-l,i}\epsilon(t - \theta_i) \,,$$

for $l = 2, \ldots, m-1$. Note that, these relations give

$$\frac{d}{dt}\epsilon^{m-1}(t) = \mathcal{D}\epsilon(t) - \sum_{i=0}^{\mu} \alpha_{0i}\epsilon(t-\theta_i) \ ,$$

for $m \geq 2$, and

$$\dot{\epsilon}(t) = \mathcal{D}\epsilon(t) - \sum_{i=0}^{\mu} \alpha_{0i}\epsilon(t-\theta_i) \ ,$$

for $m = 1$.

Next, let us also define $\tilde{\xi}(t) := \mathcal{D}\xi(t)$, $\tilde{v}_j(t) := \mathcal{D}v_j(t)$, $j = 1, \ldots, \nu$, and $\bar{\xi}(t) := \begin{bmatrix} \tilde{\xi}(t) \\ \tilde{\epsilon}(t) \end{bmatrix}$. Then, we obtain

$$\hat{L}\dot{\bar{\xi}}(t) = \sum_{i=0}^{\mu} \left[\hat{A}_i\bar{\xi}(t-\theta_i) + \textstyle\sum_{j=1}^{\nu}\hat{C}_{ij}\tilde{v}_j(t-\theta_i)\right] \tag{25}$$

which is dynamically equivalent to (23). Note that

$$\epsilon(t) = \tilde{R}\tilde{\epsilon}(t) = \begin{bmatrix} 0_{n^\psi \times n^\xi} & \tilde{R} \end{bmatrix} \bar{\xi}(t) \ , \tag{26}$$

where $\tilde{R} := \begin{bmatrix} 0_{n^\psi \times (m-1)n^\psi} & I_{n^\psi} \end{bmatrix}$, and

$$\text{rank} \begin{bmatrix} \tilde{R} \\ \tilde{R}\hat{M}_0 \\ \vdots \\ \tilde{R}\hat{M}_0^{m-1} \end{bmatrix} = mn^\psi$$

(which reduce to $\tilde{R} := I_{n^\psi}$ and $\text{rank}(\tilde{R}) = n^\psi$, for $m = 1$). This implies that, in order to have (9), i.e., in order to have $\lim_{t\to\infty}\epsilon(t) = 0$, we must have $\lim_{t\to\infty}\tilde{\epsilon}(t) = 0$, [2]. This means that the part of the system (25) which corresponds to $\tilde{\epsilon}(t)$ must be stabilized. On the other hand, the remaining part, i.e., the part which corresponds to $\tilde{\xi}(t)$, must also be stabilized, since this part is dynamically equivalent to the given system (13), which must be stabilized as a problem requirement. Thus, the system (25) must be stabilized. Furthermore, according to the problem statement, this stabilization must be achieved by decentralized feedback, where $\tilde{v}_j(t)$ must be applied by the j^{th} control agent, $j = 1, \ldots, \nu$. Recall that the j^{th} control agent can access $\eta_j(t)$ and $\rho_j(t)$. However, from $\eta_j(t)$, one can obtain $\tilde{\eta}_j(t) := \mathcal{D}\eta_j(t)$. Furthermore, using (22), one can also obtain $\psi_j(t)$ and, thus, $\epsilon_j(t) := \psi_j(t) - \rho_j(t)$, from which (for the case $m \geq 2$)

$$\epsilon_j^1(t) := \dot{\epsilon}_j(t) + \sum_{i=0}^{\mu} \alpha_{m-1,i}\epsilon_j(t-\theta_i)$$

and (for the case $m \geq 3$)

$$\epsilon_j^l(t) := \frac{d}{dt}\epsilon_j^{l-1}(t) + \sum_{i=0}^{\mu} \alpha_{m-l,i}\epsilon_j(t-\theta_i) \ ,$$

for $l = 2, \ldots, m-1$, can also be obtained. Thus,

$$\tilde{\epsilon}_j(t) := \begin{cases} \epsilon_j(t) \ , & \text{if } m = 1 \\ \begin{bmatrix} \epsilon_j^{m-1}(t) \\ \vdots \\ \epsilon_j^1(t) \\ \epsilon_j(t) \end{bmatrix} , & \text{if } m \geq 2 \end{cases}$$

can also be obtained. Note that $\tilde{\epsilon}_j(t)$ can also be expressed as $\tilde{\epsilon}_j(t) = \hat{R}_{0j}\tilde{\epsilon}(t) = \sum_{i=0}^{\mu}\hat{R}_{ij}\tilde{\epsilon}(t-\theta_i)$. Therefore, to stabilize (25), the j^{th} control agent can use

$$\bar{\eta}_j(t) := \begin{bmatrix} \tilde{\eta}_j(t) \\ \tilde{\epsilon}_j(t) \end{bmatrix} = \sum_{i=0}^{\mu} \Big[\hat{D}_{ij}\bar{\xi}(t-\theta_i) + \textstyle\sum_{k=1}^{\nu}\hat{F}_{ijk}\tilde{v}_k(t-\theta_i)\Big] \ , \tag{27}$$

$j = 1, \ldots, \nu$. However, a necessary condition for the decentralized stabilization of (25) using the measurements (27) is that, the decentralized control system (25) & (27) should not have any unstable DFMs, [27]. This proves the necessity, since (25) & (27) is equivalent to (23)–(24).

Next, we will provide a constructive proof for sufficiency. As a part of the j^{th} decentralized controller, let the j^{th} control agent build the following system, to be called the j^{th} *servocompensator*,

$$\dot{\sigma}_j(t) = \sum_{i=0}^{\mu} \tilde{M}_{ij}\sigma_j(t-\theta_i) + \tilde{Q}_j\epsilon_j(t) \ , \tag{28}$$

where $\sigma_j(t) \in \mathbf{R}^{mn_j^\psi}$ is the state vector, $\tilde{M}_{ij} := M_i \otimes I_{n_j^\psi}$, $i = 0, \ldots, \mu$, $\tilde{Q}_j := \begin{bmatrix} I_{n_j^\psi} \\ 0_{(m-1)n_j^\psi \times n_j^\psi} \end{bmatrix}$ (which reduces to $\tilde{Q}_j := I_{n_j^\psi}$ if $m = 1$), and $\epsilon_j(t)$ is obtained using (22) and (8).

Let $\sigma(t) := \begin{bmatrix} \sigma_1(t) \\ \vdots \\ \sigma_\nu(t) \end{bmatrix}$ and $\rho(t) := \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_\nu(t) \end{bmatrix}$. Then, the given system (13) augmented by the ν

servocompensators (28) can be described as

$$\hat{L}\begin{bmatrix} \dot{\xi}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \sum_{i=0}^{\mu}\left(\begin{bmatrix} A_i & 0 \\ \tilde{Q}G_i & \tilde{M}_i \end{bmatrix}\begin{bmatrix} \xi(t-\theta_i) \\ \sigma(t-\theta_i) \end{bmatrix} + \begin{bmatrix} B_i \\ \tilde{Q}H_i \end{bmatrix}\omega(t-\theta_i) + \sum_{j=1}^{\nu}\begin{bmatrix} C_{ij} \\ \tilde{Q}K_{ij} \end{bmatrix}\upsilon_j(t-\theta_i)\right) - \begin{bmatrix} 0_{n^\xi \times n^\psi} \\ \tilde{Q} \end{bmatrix}\rho(t) \ , \qquad (29)$$

where $\tilde{Q} := \text{bdiag}(\tilde{Q}_1, \dots, \tilde{Q}_\nu)$, $\tilde{M}_i := \text{bdiag}(\tilde{M}_{i1}, \dots, \tilde{M}_{i\nu})$, and $H_i := \begin{bmatrix} H_{i1} \\ \vdots \\ H_{i\nu} \end{bmatrix}$, $i = 0, \dots, \mu$. From this system, the measurement available to the j^{th} control agent is

$$\begin{bmatrix} \eta_j(t) \\ \sigma_j(t) \end{bmatrix} = \sum_{i=0}^{\mu}\left(\begin{bmatrix} D_{ij} & 0 \\ 0 & \tilde{R}_{ij} \end{bmatrix}\begin{bmatrix} \xi(t-\theta_i) \\ \sigma(t-\theta_i) \end{bmatrix} + \begin{bmatrix} E_{ij} \\ 0 \end{bmatrix}\omega(t-\theta_i) + \sum_{k=1}^{\nu}\begin{bmatrix} F_{ijk} \\ 0 \end{bmatrix}\upsilon_k(t-\theta_i)\right) \ , \qquad (30)$$

where $\tilde{R}_{0j} := R_j \otimes I_m$, and, for $i = 1, \dots, \mu$, $\tilde{R}_{ij} := 0_{mn_j^\psi \times mn^\psi}$. Note that $\hat{L} = T\hat{L}T^{-1}$,

$$\begin{bmatrix} A_i & 0 \\ \tilde{Q}G_i & \tilde{M}_i \end{bmatrix} = T\hat{A}_i T^{-1} \ ,$$

$$\begin{bmatrix} C_{ij} \\ \tilde{Q}K_{ij} \end{bmatrix} = T\hat{C}_{ij} \ ,$$

$$\begin{bmatrix} D_{ij} & 0 \\ 0 & \tilde{R}_{ij} \end{bmatrix} = \hat{D}_{ij}T^{-1} \ ,$$

and

$$\begin{bmatrix} F_{ijk} \\ 0 \end{bmatrix} = \hat{F}_{ijk} \ ,$$

$i = 0, \dots, \mu$, $j, k = 1, \dots, \nu$, where

$$T := \begin{bmatrix} I_{n^\xi} & 0 \\ 0 & I_m \otimes R_1 \\ \vdots & \vdots \\ 0 & I_m \otimes R_\nu \end{bmatrix} .$$

This implies that, apart from the existence of external signals $\omega(t)$ and $\rho(t)$ in (29)–(30), (29)–(30) is equivalent to (23)–(24), [30]. Therefore, by the hypothesis of the theorem, (29)–(30) does not have any unstable DFMs. This implies that there exist ν decentralized controllers, to be called *stabilizing compensators* (each from $\begin{bmatrix} \eta_j(t) \\ \sigma_j(t) \end{bmatrix}$ to $\upsilon_j(t)$, $j = 1, \dots \nu$), which stabilize the system (29)–(30), [27]. Thus, these controllers also stabilize the given system (3)–(4), since (29) includes (3)–(4). Furthermore, the same controllers also stabilize (23)–(24), since (23)–(24) is equivalent to (29)–(30). Therefore, when these controllers are applied, we have $\lim_{t\to\infty} \hat{\xi}(t) = 0$. Thus, we also have $\lim_{t\to\infty} \bar{\xi}(t) = 0$, since (23) is equivalent to (25). Thus, by (26), $\lim_{t\to\infty} \epsilon(t) = 0$, which implies (9). This completes the proof. □

The above proof implies that, as in the case of delay-free systems, [19], the solution to DRSP involves ν decentralized controllers, each of which is composed of two parts: a servocompensator, given by (28), and a stabilizing compensator, which is designed to stabilize the augmented system (29)–(30). The implementation of these controllers is depicted in Fig. 1 for the case $\nu = 2$. Each servocompensator is fixed and is determined by the dynamics of the fictitious system (18) producing the references and the disturbance. Stabilizing compensators, on the other hand, can be designed using any decentralized stabilizing controller design method developed for descriptor-type time-delay systems (e.g., see [31] and references therein; a software package developed in [32] may also be used for this purpose).

5 Conclusions

Decentralized robust servomechanism problem for LTI descriptor-type discrete-time-delay systems has been considered. It has been shown that the necessary and sufficient condition for the solvability of this problem is that the decentralized control system (23)–(24), which in fact is an augmented system of the given plant and a fictitious system producing the disturbance and the references, should not have any unstable DFMs. It has further been shown that, when this condition is satisfied, the solution involves ν decentralized controllers, each of which is composed of two parts: a servocompensator, given by (28), and a stabilizing compensator, which is designed to stabilize the augmented system (29)–(30). Each servocompensator is fixed and is determined by the dynamics of the fictitious system (18) producing the references and the disturbance. Stabilizing

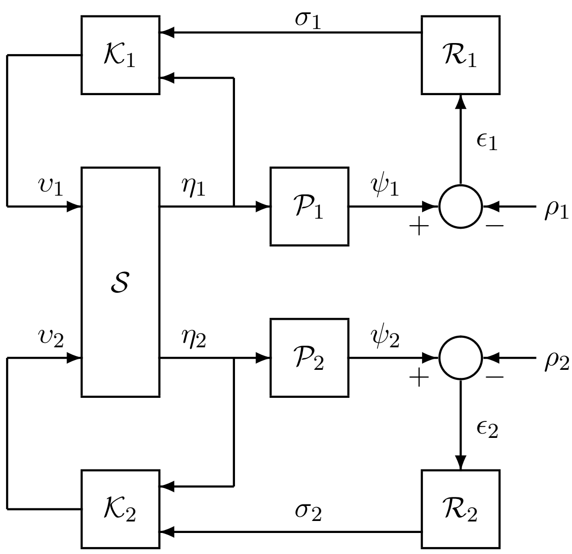

Fig. 1. Implementation of the decentralized controllers for the case $\nu = 2$, where $\mathcal{S}$ is the given system (13)–(15), $\mathcal{P}_j$ is the transformation (22), $\mathcal{R}_j$ is the j^{th} servocompensator (28), and $\mathcal{K}_j$ is the j^{th} stabilizing compensator, $j = 1, 2$.

compensators, on the other hand, can be designed using any decentralized stabilizing controller design method developed for descriptor-type time-delay systems.

In this work, the decentralized controllers to be designed have been restricted to be LTI controllers. Although LTI controllers may be desired for many practical reasons, [33], it may be possible to relax the necessary and sufficient conditions for the existence of a solution if nonlinear and/or time-varying controllers are allowed. To research how these conditions may be relaxed might be a direction for future work.

Furthermore, in (22), we assumed that the outputs depend only on the present and past values of the measurements. It may, however, be possible that the measurements may lag the outputs, i.e., the outputs may also depend on the future values of the measurements. In such a case, in order to obtain the outputs, and hence the tracking error, a predictor would be needed. The design of such a predictor is another subject for future research.

Another direction for future research is to consider descriptor type systems with distributed time delay. Yet another direction is to consider non-linear and/or time-varying time-delay systems.

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