An Optimal Landing Problem for a Bessel Process

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Abstract: - A homing problem for a one-dimensional Bessel diffusion process is considered. The aim is to bring the controlled process to a value representing ground level as quickly as possible while taking the control costs into account. The cost function includes a parameter that takes the risk sensitivity of the optimizer into account. An explicit solution is found for both the value function and the optimal control in a particular problem.

Key-Words: - Stochastic control, homing problem, first-passage time, Wiener process, dynamic programming, risk parameter.

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1 Introduction

We consider the one-dimensional controlled diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation:

$$\mathrm{d}X(t) = b_0 \theta u[X(t)]\mathrm{d}t + \frac{(\alpha - 1)}{2X(t)}\mathrm{d}t + \sigma \mathrm{d}B(t), \quad (1)$$

where b_0, θ, α and σ are non-negative constants, the continuous function $u(\cdot)$ is the control variable and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The uncontrolled process $\{X_0(t), t \geq 0\}$ is a Bessel process of dimension α (if $\sigma = 1$).

Let $T(x)$ be the *first-passage time* defined by:

$$T(x) = \inf\{t > 0 : X(t) = d \mid X(0) = x\}, \quad (2)$$

where $x > d \geq 0$. The aim is to find the control $u^*[X(t)]$ that minimizes the expected value of the cost function:

$$J(x) = \int_0^{T(x)} \left\{\frac{1}{2} q_0 g(\theta) u^2[X(t)] X^2(t) + \lambda\right\} \mathrm{d}t, \quad (3)$$

where q_0 and λ are positive constants and $g(\theta)$ is a non-negative function.

This type of problem, in which the optimizer controls a stochastic process until a certain event occurs, is known as a *homing problem*; [1], [2],3]. Other papers on homing problems are [4], [5], [6] and [7]. The above problem can be interpreted as an optimal landing problem, with d representing ground level. Because the parameter λ is positive, the optimizer tries to reach d as quickly as possible, while taking the control costs into account. Therefore, the optimal control $u^*[X(t)]$ should in general be negative. Moreover, θ is a risk parameter. If $\theta < 1$ (respectively, $\theta > 1$) the optimizer is risk-averse (resp., risk-seeking) and does not want to land too rapidly (resp., wants to land rapidly). The case when $\theta = 1$ is the risk-neutral case.

Homing problems are generally very difficult to solve explicitly. In the next section, using dynamic programming, the equation satisfied by the *value function*:

$$F(x) := \inf_{\substack{u[X(t)] \\ 0 \leq t < T(x)}} E[J(x)] \quad (4)$$

will be derived. This equation is a non-linear second-order ordinary differential equation (ODE). From the value function, the optimal control is obtained explicitly. An exact solution to the ODE will be found in a particular problem. Moreover, the effect of the risk parameter θ on the optimal control will be presented.

2 Dynamic Programming

Let

$$h(t) := \frac{1}{2} q_0 g(\theta) u^2[X(t)] X^2(t) + \lambda. \quad (5)$$

We can write that

$$F(x) := \inf_{\substack{u[X(t)] \\ 0 \le t < T(x)}} E\left[\int_0^{T(x)} h(t)\,\mathrm{d}t\right]. \qquad (6)$$

Next, we divide the integral into two parts:

$$F(x) = \inf_{\substack{u[X(t)] \\ 0 \le t < T(x)}} E\left[\int_0^{)t} h(t)\,\mathrm{d}t + \int_{)t}^{T(x)} h(t)\,\mathrm{d}t\right]. \quad (7)$$

The first integral above can be approximated as follows:

$$\int_0^{)t} h(t)\,\mathrm{d}t = h(0))t + o()t). \qquad (8)$$

Moreover, from Bellman's principle of optimality, [8], we deduce that:

$$\inf_{\substack{u[X(t)] \\ )t \le t < T(x)}} E\left[\int_{)t}^{T(x)} h(t)\,dt\right] = E[F(x +)t)] + o()t). \qquad (9)$$

We have:

$$F(x +)t) = F\left(x + b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t + \sigma B()t)\right) + o()t). \qquad (10)$$

Using Taylor's formula, we obtain that:

$$F\left(x + b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t + \sigma B()t)\right) = F(x) + \left(b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t + \sigma B()t)\right)F'(x) + \tfrac{1}{2}\left(b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t + \sigma B()t)\right)^2 F''(x) + o()t). \qquad (11)$$

Now, we have $E[B()t)] = 0$ and $E[B^2()t)] = VAR[B()t)] =)t$. It follows that:

$$E\left[F\left(x + b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t + \sigma B()t)\right)\right] = F(x) + \left(b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t\right)F'(x) + \tfrac{1}{2}\sigma^2)t\, F''(x) + o()t). \qquad (12)$$

From what precedes, we find that the value function satisfies the equation:

$$0 = \inf_{u(x)}\left\{\begin{matrix} h(0))t + \left(b_0\theta u(x))t + \frac{(\alpha-1)}{2x})t\right)F'(x) \\ +\tfrac{1}{2}\sigma^2)t\, F''(x) + o()t) \end{matrix}\right\}. \qquad (13)$$

Finally, dividing both sides of the above equation by $)t$, and taking the limit as $)t$ decreases to zero, we obtain the following proposition.

Proposition 2.1. *The value function F(x) satisfies the dynamic programming equation (DPE)*

$$0 = \inf_{u(x)}\left\{h(0) + \left(b_0\theta u(x) + \frac{(\alpha-1)}{2x}\right)F'(x) + \tfrac{1}{2}\sigma^2 F''(x)\right\}. \qquad (14)$$

Moreover, we have the boundary condition $F(d) = 0$.

Differentiating the DPE with respect to $u(x)$, we obtain an explicit expression for the optimal control in terms of the value function.

Corollary 2.1. *The optimal control is given by*

$$u^*(x) = -\frac{b_0 2}{q_0 g(2) x^2}F'(x) \qquad (15)$$

for $x > d \ge 0$.

Next, substituting the expression for $u^*(x)$ into (14), we find that we must solve the second-order non-linear ODE:

$$\lambda - \frac{1}{2}\frac{b_0^2}{q_0 x^2}\frac{\theta^2}{g(2)}[F'(x)]^2 + \frac{\alpha-1}{2x}F'(x) + \frac{\sigma^2}{2}F''(x) = 0. \quad (16)$$

Remark. Since (16) is a second-order equation, we need two boundary conditions to obtain a unique solution. In addition to the condition $F(d) = 0$ mentioned in Proposition 2.1, we may state that:

$$\lim_{x\to\infty} F(x) = \infty. \qquad (17)$$

Indeed, as x tends to infinity, so will $T(x)$. Equation (17) then follows from the fact that all the terms in the cost function $J(x)$ are non-negative (and 8 is assumed to be positive).

In the next section, a particular problem will be solved explicitly.

3 A Particular Problem

Assume that $b_0 = q_0 = \sigma = 1$. Furthermore, we take $g(2) = 2$. Equation (16) then reduces to:

$$\lambda - \frac{1}{2}\frac{2}{x^2}[F'(x)]^2 + \frac{\alpha-1}{2x}F'(x) + \frac{1}{2}F''(x) = 0. \quad (18)$$

It is not easy to obtain an explicit solution to the above equation (without arbitrary constants).

However, if we try a solution of the form:

$$F(x) = k(x-d)^2, \qquad (19)$$

where k is a constant to be determined, we find that this function does indeed satisfy (18) if and only if:

$$k = \frac{\alpha}{4\mathit{2}} \pm \frac{\sqrt{\alpha^2 + 8\mathit{2}\lambda}}{4\mathit{2}}. \qquad (20)$$

In order to respect the condition in (17), we must choose the "+" sign. It follows that:

$$F(x) = \frac{1}{4\mathit{2}}\left(\alpha + \sqrt{\alpha^2 + 8\mathit{2}\lambda}\right)(x-d)^2 \qquad (21)$$

for $x \geq d \geq 0$. Moreover, the optimal control is given by:

$$u^*(x) = -\frac{1}{2\mathit{2}}\left(\alpha + \sqrt{\alpha^2 + 8\mathit{2}\lambda}\right)\frac{(x-d)}{x^2}. \qquad (22)$$

Let us take $\alpha = 2$, $\lambda = 1$ and $d = 0$. In Figure 1, we show the optimal control for three different values of the risk parameter $\mathit{2}$ when x is in the interval [0,10].

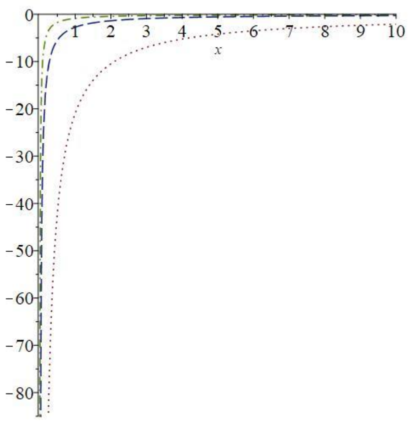

Fig. 1: Optimal control for $x \in [0,10]$ when $\mathit{2}$ = 1/10 (dotted line), $\mathit{2}$ = 1 (dashed line) and $\mathit{2}$ = 5 (dash-dotted line)

Let us define:

$$\gamma = 1 + \sqrt{1 + 2\mathit{2}}. \qquad (23)$$

The optimally controlled process $\{X^*(t), t \geq 0\}$ satisfies the following stochastic differential equation:

$$\mathrm{d}X^*(t) = -\frac{\gamma}{X^*(t)}\mathrm{d}t + \frac{1}{2X^*(t)}\mathrm{d}t + \sigma \mathrm{d}B(t). \qquad (24)$$

That is,

$$\mathrm{d}X^*(t) = \frac{1-2\gamma}{2X^*(t)}\mathrm{d}t + \sigma \mathrm{d}B(t). \qquad (25)$$

Therefore, it is also a (generalized) Bessel process.

Remark. For a Bessel process of dimension $\alpha \exists 0$, the origin is an *exit* boundary if $\alpha = 0$, a *regular* boundary if $0 < \alpha < 2$, and an *entrance* boundary if $\alpha \exists 2$; see, [9]. Here, the dimension of the optimally controlled process is equal to $2(1-\gamma) < 0$.

We see in Figure 1 that when $\mathit{2}$ = 1/10, the optimizer uses much more control (in absolute value) than when $\mathit{2}$ = 1 or $\mathit{2}$ = 5. However, as can be seen in (25), when $\mathit{2}$ is small, $X^*(t)$ will decrease less rapidly than when $\mathit{2}$ is large.

In Figure 2, we present the optimal control multiplied by $\mathit{2}$ (as in (1)) for the three values of the risk parameter $\mathit{2}$ used in Figure 1.

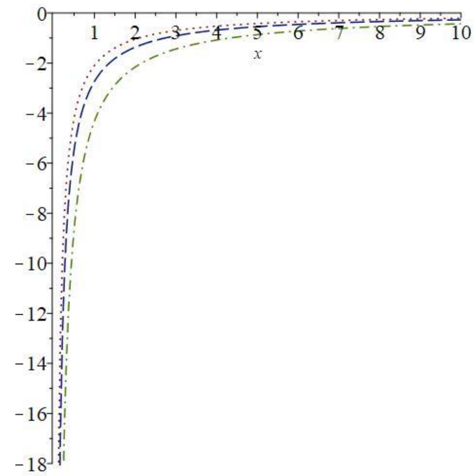

Fig. 2: Optimal control multiplied by $\mathit{2}$, for $x \in [0,10]$, when $\mathit{2}$ = 1/10 (dotted line), $\mathit{2}$ = 1 (dashed line) and $\mathit{2}$ = 5 (dash-dotted line)

Notice that the position of the curves is reversed compared to Figure 1.

4 Conclusion

In this paper, we have considered a homing problem for a Bessel diffusion process. The problem formulation involved a risk parameter in the definition of the controlled process and the cost function.

In Section 2, we derived the dynamic programming equation satisfied by the value function. Moreover, in Section 3, we found an exact and explicit solution to the ODE (16) also satisfied by the value function in a particular problem. From the value function, the optimal control is deduced at once. In this particular problem, we also saw the effect of the risk parameter on the optimal control.

We could try to find other explicit solutions to the ODE (16). We could also try to use numerical methods to solve this second-order non-linear ordinary differential equation.

Finally, we could compare our solution to the one obtained in the risk-sensitive formulation of the homing problem proposed in [3] and in [10].

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Conflict of Interest

The author has no conflicts of interest to declare.