Application of Splines of the Fifth Order Approximation for Solving Integral Equations of the Second Kind with a Weak Singularity

I. G. BUROVA, G. O. ALCYBEEV
The Department of Computational Mathematics,
St. Petersburg State University,
7-9 Universitetskaya Embankment, St.Petersburg,
RUSSIA

Abstract: - Previously, the authors showed that local splines of the different order of approximation give good results on both uniform and non-uniform grids. In this paper, we investigate the stability of the numerical method based on the splines of the fifth order of approximation and the use of these splines for solving weak singular Fredholm and Volterra integral equations of the second kind. The solution method consists of replacing the unknown function under the integral sign with a spline approximation. We compare the errors of the solutions of integral equations obtained using splines of the second, fifth, and seventh orders with the results which were received in recent papers by using other methods. The results of the numerical experiments are presented in this paper.

Key-Words: - Fredholm integral equation, Volterra integral equation, weak singularity, polynomial spline approximations, local basis splines, stability.

Received: April 9, 2024. Revised: September 11, 2024. Accepted: December 17, 2024. Published: March 26, 2025.

1 Introduction

Integral equations are used in problems of diffraction and acoustics, electrodynamics and electrical engineering, elasticity theory, and aerohydromechanics. They are also encountered in problems such as mathematical apparatus for the analysis of economic systems and the theory of economic-mathematical modeling and in a number of other sections of physics, mechanics, mathematical physics, and economics.

As is known, equations with a weak singularity include an equation, whose kernel has the form:

$$K(x,s)=\frac{H(x,s)}{|x-s|^{\alpha}},$$

where $0<\alpha<1$ and the function $H(x,s)$ is bounded.

There are many classical algorithms for obtaining numerical solutions to integral equations of the second kind, such as the iterated kernel method, the method of quadratures, and the collocation method,.

To solve problems, the use of classical methods is insufficient. Therefore, the authors need to develop new algorithms and investigate their properties, [1], [2], [3], [4], [5].

Authors often use the Galerkin method with orthogonal polynomials or spline approximations on uniform and non-uniform grids, [5], [6].

In [7], the authors apply Burton's method to Hammerstein-type integral equations.

An algorithm, based on exponential transformations is proposed in paper [8].

An algorithm, based on the piecewise polynomial collocation method is discussed in paper [9].

Paper [10] discusses algorithms based on the explicit Adams methods.

Special spline interpolants are used in [11], [12], [13].

In this paper, we propose to modify existing numerical methods using local spline approximations, [14].

2 Problem Formulation

Consider the Fredholm integral equation of the second kind:

$$u(x)-\int_0^1 K(x,s)u(s)ds=f(x)\,.$$

Here the kernel $K(x,s)$ of the integral equation has a weak singularity of the form:

$$K(x,s) = H(x,s)p(x,s), \qquad 0 < \alpha < 1,$$
$$p(x,s) = \frac{1}{|x-s|^{\alpha}}.$$

Suppose a grid of ordered nodes s_j is given on the segment $[a,b]$:

$$a = s_0 < s_1 < \cdots < s_n = b.$$

Let us recall how the local spline of the fifth order of approximation can be constructed.

Suppose, $u_i \approx u(s_i)$. When $i = 0, 1,$ we apply the approximation with the right local basis splines using the following formula:

$$U_R^i(s) = \sum_{j=i}^{i+4} u_j w_j^R(s),\ s \in [s_i, s_{i+1}].$$

The basis splines $w_j^R(s)$ are described by the following formulas:

$$w_i^R(s) = \prod_{k=1}^{4} \frac{s - s_{i+k}}{s_i - s_{i+k}},$$
$$w_{i+1}^R(s) = \prod_{k\neq 1, k=0}^{4} \frac{s - s_{i+k}}{s_{i+1} - s_{i+k}},$$
$$w_{i+2}^R(s) = \prod_{k\neq 2, k=0}^{4} \frac{s - s_{i+k}}{s_{i+2} - s_{i+k}},$$
$$w_{i+3}^R(s) = \prod_{k\neq 3, k=0}^{4} \frac{s - s_{i+k}}{s_{i+3} - s_{i+k}},$$
$$w_{i+4}^R(s) = \prod_{k=0}^{3} \frac{s - s_{i+k}}{s_{i+4} - s_{i+k}}.$$

When $i = 2, 4, \dots, n-3,$ we apply the approximation with the middle splines using the following formula:

$$U_M^i(x) = \sum_{j=i-2}^{i+2} u_j w_j^M(x),\ x \in [s_i, s_{i+1}],$$

where basis splines w_j^M are described by the following formulas:

$$w_{i-2}^M(s) = \prod_{k=-1}^{2} \frac{s - s_{i+k}}{s_{i-2} - s_{i+k}},$$
$$w_{i-1}^M(s) = \prod_{k\neq -1, k=-2}^{2} \frac{s - s_{i+k}}{s_{i-1} - s_{i+k}},$$
$$w_i^M(s) = \prod_{k\neq 0, k=-2}^{2} \frac{s - s_{i+k}}{s_i - s_{i+k}},$$
$$w_{i+1}^M(s) = \prod_{k\neq 1, k=-2}^{2} \frac{s - s_{i+k}}{s_{i+1} - s_{i+k}},$$
$$w_{i-2}^M(s) = \prod_{k=-2}^{1} \frac{s - s_{i+k}}{s_{i+2} - s_{i+k}}.$$

When $i = n-2, n-1,$ we apply the approximation with the left splines using the following formula:

$$U_L^i(s) = \sum_{j=i-3}^{i+1} u_j w_j^L(s),\ \ s \in [s_i, s_{i+1}],$$

where the basis splines w_j^L are described by the following formulas:

$$w_{i-3}^L(s) = \prod_{k=-2}^{1} \frac{s - s_{i+k}}{s_{i-3} - s_{i+k}},$$
$$w_{i-2}^L(s) = \prod_{k\neq -2, k=-3}^{1} \frac{s - s_{i+k}}{s_{i-2} - s_{i+k}},$$
$$w_{i-1}^L(s) = \prod_{k\neq -1, k=-3}^{1} \frac{s - s_{i+k}}{s_{i-1} - s_{i+k}},$$
$$w_i^L(s) = \prod_{k\neq 0, k=-3}^{1} \frac{s - s_{i+k}}{s_i - s_{i+k}},$$
$$w_{i+1}^L(s) = \prod_{k=-3}^{0} \frac{s - s_{i+k}}{s_{i+1} - s_{i+k}}.$$

Splines $U_L^i(s), U_R^i(s), U_M^i(s), s \in [s_i, s_{i+1}],$ are called fifth-order splines, since the following inequalities hold:

$$\left|u(s) - U_L^i(s)\right| \le h^5 K_1 \left\|u^{(5)}\right\|_{[s_{i-3}, s_{i+1}]},$$
$$K_1 = 0.03027;$$
$$\left|u(s) - U_R^i(s)\right| \le h^5 K_1 \left\|u^{(5)}\right\|_{[s_i, s_{i+4}]},$$
$$K_1 = 0.03027;$$
$$\left|u(s) - U_M^i(s)\right| \le h^5 K \left\|u^{(5)}\right\|_{[s_{i-2}, s_{i+2}]},$$
$$K_1 = 0.01185.$$

In the next section, we will consider the use of fifth-order approximation splines to solve an integral equation.

3 About the Stability of Calculation

In this section we apply the fifth order splines to construct a numerical method for solving the integral equation:

$$u(x) - \int_0^1 K(x,s)u(s)ds = f(x),$$

and we investigate the stability of this numerical method. We assume that the kernel of the integral equation $K(x,s)$ and the right-hand side $f(x)$ are continuous functions. Suppose $|K(x,s)| < \rho < 1$. We take an integer n, $n \geq 7$, and calculate $h = \frac{1}{n}$, thus, $h = x_{k+1} - x_k, k = 0,1,\dots,n-1$. On each grid interval $[x_k, x_{k+1}]$ we replace the unknown function u with a fifth-order spline $U_L^i(s)$, or $U_R^i(s)$, or $U_M^i(s)$. Now we have:

$$\int_0^1 K(x,s)\,u(s)ds \approx$$
$$\sum_{k=0}^{1} \int_{s_k}^{s_{k+1}} K(x,s)\left(\sum_{i=k}^{k+4} u_i w_i^R(s)\right)ds +$$
$$\sum_{k=2}^{n-3} \int_{s_k}^{s_{k+1}} K(x,s)\left(\sum_{i=k-2}^{k+2} u_i w_i^M(s)\right)ds$$
$$+ \sum_{k=n-2}^{n-1} \int_{s_k}^{s_{k+1}} K(x,s)\left(\sum_{i=k-3}^{k+1} u_i w_i^L(s)\right)ds.$$

Now we put $x = s_j$ and reduce the integral equation to a system of algebraic equations with the approximate solution $U = (u_0, \dots, u_n)$.

Assume that u_j is the largest component among the components: $|u_j| = \max_k |u_k|$.

Thus, we have the equation:

$$u_j -$$
$$\sum_{k=0}^{1} \int_{s_k}^{s_{k+1}} K(x_j,s)\left(\sum_{i=k}^{k+4} u_i w_i^R(s)\right)ds -$$
$$\sum_{k=2}^{n-3} \int_{s_k}^{s_{k+1}} K(x_j,s)\left(\sum_{i=k-2}^{k+2} u_i w_i^M(s)\right)ds -$$
$$\sum_{k=n-2}^{n-1} \int_{s_k}^{s_{k+1}} K(x_j,s)\left(\sum_{i=k-3}^{k+1} u_i w_i^L(s)\right)ds = f(x_j).$$

For further transformations we need to simplify the last equation above. It is easy to calculate these integrals:

$$\int_{x_k}^{x_{k+1}} w_k^R(s)ds \approx 0.349\,h, \quad \int_{x_k}^{x_{k+1}} w_{k+1}^R(s)ds \approx 0.897\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k+2}^R(s)ds \approx -\,0.367\,h, \int_{x_k}^{x_{k+1}} w_{k+3}^R(s)ds \approx 0.147\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k+4}^R(s)ds \approx -\,0.0264\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k-2}^M(s)ds \approx 0.0153\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k-1}^M(s)ds \approx -\,0.103\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k+2}^M(s)ds \approx -\,0.0264\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k+1}^M(s)ds \approx 0.481\,h, \int_{x_k}^{x_{k+1}} w_k^M(s)ds \approx 0.633\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k-3}^L(s)ds \approx -\,0.0264\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k-2}^L(s)ds \approx 0.147\,h, \int_{x_k}^{x_{k+1}} w_{k-1}^L(s)ds \approx -\,0.367\,h,$$

$$\int_{x_k}^{x_{k+1}} w_{k+1}^L(s)ds \approx 0.349\,h, \int_{x_k}^{x_{k+1}} w_k^L(s)ds \approx 0.897\,h.$$

Now we can calculate the coefficients at u_0. On the first grid interval $[x_0, x_1]$ we obtain the coefficient at u_0:

$$u_0 \int_{x_0}^{x_1} K(x_j,s)w_0^R(s)ds + u_0 \int_{x_0}^{x_1} K(x_j,s)w_0^M(s)ds$$
$$\approx 0.576\,K(x_j,\eta_k)h\,u_0\,.$$

This result follows from the application of the mean value theorem. Using the mean value theorem of integral calculus, we obtain the relation:

$$\int_{x_k}^{x_{k+1}} K(x_j,s)w_k^R(s)ds = K(x_j,\eta_k)\int_{x_k}^{x_{k+1}} w_k^R(s)ds$$
$$\approx K(x_j,\eta_k)\,0.349\,h,$$

where $\eta_k \in [x_k, x_{k+1}]$.

Similarly, we get the other coefficients. From the intervals with numbers from $i = 7$ to $i = n - 7$ we can calculate that the coefficient at u_i is $1.258\,h$.

Next, we use the relation:

$$|f(x_j)| = |u_j - \int_0^1 K(x_j,s)\,u(s)ds| .$$

Finally, we obtain the inequality:

$$|f(x_j)| \geq (1-\rho\, c)|u_j| .$$

Thus, we have the estimation

$$|u_j| \leq \frac{1}{1-\rho\, c}\ |f(x_j)|,$$

where $c \approx 0.731$.

Assume $\tilde{f} = (f(x_0),\ f(x_1), \dots,\ f(x_n))$ and $f(x_i) \in F$, where F is a linear normed space, $\| \tilde{f} \|_F = \max_i |f(x_i)|$.

Assume $U = (u_0, u_1, \dots, u_n)$, $U \in \widetilde{U}$, where $\widetilde{U}$ is a linear normed space, $\| U \|_{\widetilde{U}} = \max_k |u_k|$.

Now we have:

$$\| U \|_{\widetilde{U}} = \max_k |u_k| = |u_j| \leq \frac{1}{1-\rho\, c}\ |f(x_j)| \leq \frac{1}{1-\rho\, c} \| f \|_F .$$

This inequality means the stability of the numerical method with the constant $C = \frac{1}{1-\rho\, c}$

4 Application of the Splines to Solve an Integral Equation with a Weak Singularity

Let $n \geq 7$. We represent the integral in the form:

$$\int_a^b K(x,s)u(s)ds = \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} K(x,s)u(s)ds,$$

where $K(x,s) = H(x,s)p(x,s)$, $\alpha \in (0,1)$, $p(x,s) = \frac{1}{|x-s|^{\alpha}}$.

The function $g(x,s) = H(x,s)u(s), s \in [s_k, s_{k+1}]$ can be approximated with the expression: $g(s) \approx \tilde{g}(x,s) = H(x,s)\tilde{u}(s)$. Next, we have to solve the system of equations:

$$u_i - \sum_{k=0}^{1}\sum_{j=k}^{k+4} u_j \int_{s_k}^{s_{k+1}} K(x_i,s) w_j^R(s)ds + \sum_{k=2}^{n-3}\sum_{j=k-2}^{k+2} u \int_{s_k}^{s_{k+1}} K(x_i,s) w_j^M(s)ds + \sum_{k=n-2}^{n-1}\sum_{j=k-3}^{k+1} u_j \int_{s_k}^{s_{k+1}} K(x_i,s) w_j^L(s)ds = f(x_i),$$

$$i = 0, 1, \dots, n-1.$$

Thus, the problem is reduced to calculating integrals:

$$\int_{s_k}^{s_{k+1}} p(x_i,s)\, w_j^M(s)ds,$$

$$\int_{s_k}^{s_{k+1}} p(x_i,s) w_j^R(s)ds,$$

$$\int_{s_k}^{s_{k+1}} p(x_i,s) w_j^L(s)ds.$$

using the Gaussian type of quadrature formulas.

5 Numerical Experiments

Using the method described above, we reduce the solution of a linear integral equation to the solution of a system of linear algebraic equations. As a result of solving this system, we obtain an approximate solution of the original integral equation in the form of a framework for constructing approximate solution, i.e. the values of the approximate solution at the nodes of the grid: $U = (u_0, u_1, \dots, u_n)$. Suppose $V = (v_0, v_1, \dots, v_n)$ is the vector of the exact solution, of the integral equation. Further, we use the following vector norms to calculate the solution errors:

$$\| U - V \|_1 = \max_i |\, u_i - v_i \,| ,$$

$$\| U - V \|_2 = \sqrt{\sum_{i=0}^{n} |u_i - v_i|^2} .$$

Example 1. Consider the following integral equation, [11]:

$u(t) - \int_0^1 |t-s|^{-\frac{1}{2}} u(s)ds = f(t),\ t \in [0,1]$,

with the exact solution $u(t) = \exp(t)$. The function $f(t)$ is continues function.

In paper [11], an integration method was used for calculating the numerical solution.

We construct the system of equations using the splines of the fifth, and second order of approximation and a uniform grid of nodes. Calculations were provided in MAPLE with Digits = 15. Figure 1, Figure 2, Figure 3, Figure 4 and Figure 5 show the errors of the solution in absolute value.

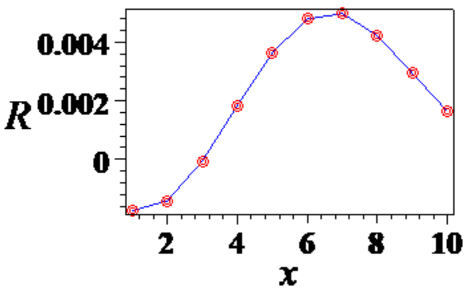

Fig. 1: The errors of the solution when second order spline approximation was used ($n = 10$)

Note, that the numbers of the grid nodes are marked along the abscissa axis. Figure 1 and Figure 2 show the errors when the second order spline approximation was used ($n = 10, 55$). Figure 3, Figure 4 and Figure 5 shows the errors when splines of the fifth order approximation were used ($n = 10, 90$).

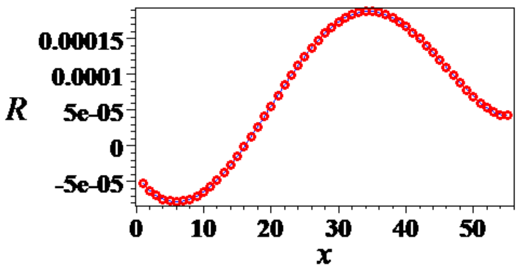

Fig. 2: The errors of the solution when second order spline approximation was used ($n = 55$)

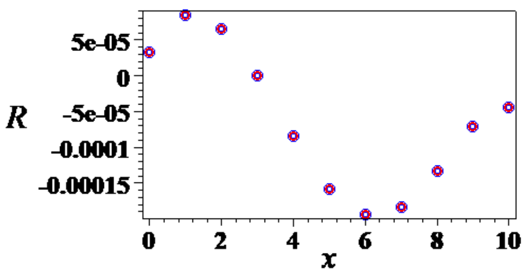

Fig. 3: The errors of the solution when we use the fifth order spline approximation ($n =\ 10$)

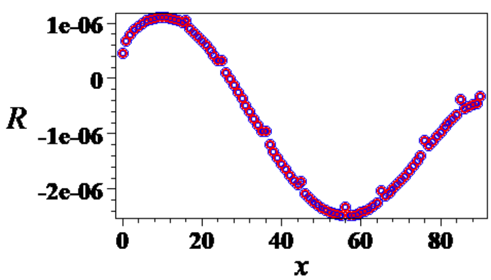

Fig. 4: The errors of the solution when we use the fifth order spline approximation ($n =\ 90, \text{Digits} = 15$)

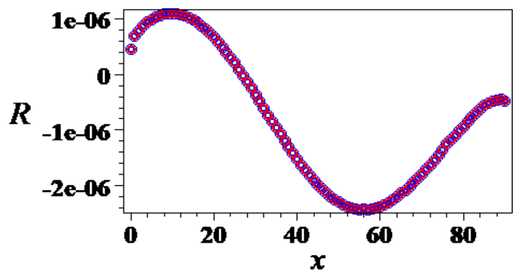

Fig. 5: The errors of the solution when we use the fifth order spline approximation ($n =\ 90, \text{Digits} = 17$)

Comparison of the results presented in Figure 2 and Figure 3 shows that the use of fifth-order approximation splines for $n = 10$ allows us to obtain a solution with approximately the same error as when using second-order approximation splines for $n = 55$.

If the number of digits in the mantissa is insufficient, instability in calculations may occur. Increasing the number of digits in the mantissa allows them to eliminate instability in calculations. Comparison of the results presented in Figure 4 and Figure 5 shows that instability occurs at n = 90, Digits = 15). With an increase in the Digits parameter, Digits = 17, instability disappears.

Table 1 shows the maximum absolute errors of solution S_5, shown in the middle column, obtained by applying fifth-order splines of approximation with a number of grid nodes of $n = 16, 32$.

Table 1 also shows the maximum absolute errors (using $\| U - V \|_1$) of solution Q_3 from paper [11], which is shown in the third column.

Table 1. The maximum absolute errors of the solution

n	Errors of solution S_5	Errors of solution Q_3
8	$0.348 \cdot 10^{-4}$	$0.182 \cdot 10^{-3}$
16	$0.678 \cdot 10^{-6}$	$0.115 \cdot 10^{-4}$
32	$0.167 \cdot 10^{-7}$	$0.939 \cdot 10^{-6}$

Example 2. Consider the following integral equation, [11]:

$$u(t) - \int_0^{\pi} \sin(s - t)|t - s|^{-\frac{1}{2}} u(s)ds = f(t),$$

$t \in [0, \pi]$, with the exact solution $u(t) = \cos(t)$.

The function $f(t)$ is continues function. We use a grid of nodes $x_i = ih$, when $h = \frac{\pi}{n}$. Using splines of the fifth order of approximation, we construct the system of equations. Figure 6, Figure 7 and Figure 8 show the result of application splines of the second order approximation ($n = 32$, $n = 64$ and $n = 16$).

Figure 9 shows the errors when the fifth order approximation splines were applied ($n = 32$).

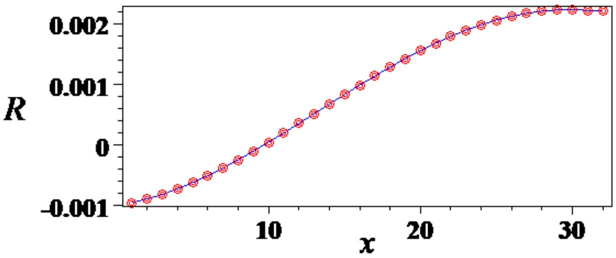

Fig. 6: The plot of the errors of the solution obtained with the polynomial splines of the second order approximation (32 nodes)

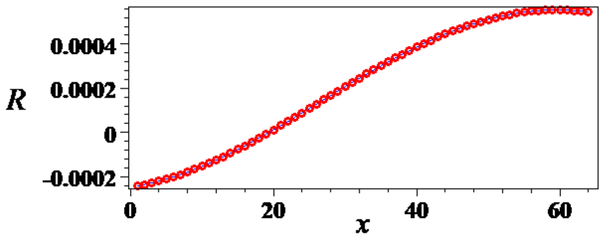

Fig. 7: The plot of the errors of the solution obtained with the polynomial splines of the second order approximation (64 nodes)

Note that in numerical calculations of integrals with a weak singularity, we can use Gaussian-type formulas with three or four nodes over the grid interval $[s_k, s_{k+1}]$, or composite Gaussian-type quadrature formulas.

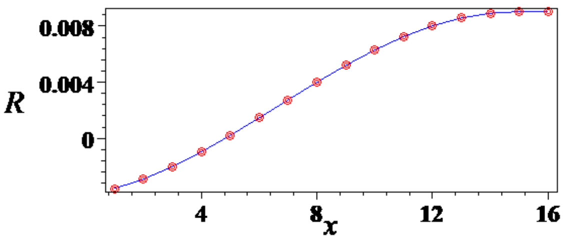

Fig. 8: The plot of the errors of the solution obtained with the polynomial splines of the second order (16 nodes)

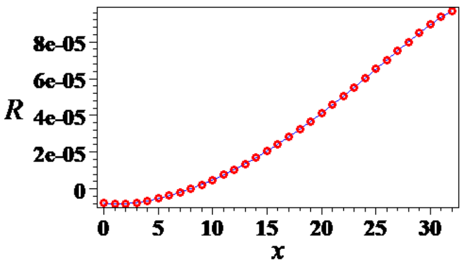

Fig. 9: The plot of the errors obtained with the polynomial splines of the fifth order (32 nodes)

Table 2 presents the maximum absolute errors of the solution (S_5), middle column, obtained by applying fifth-order splines of approximation with a number of grid nodes of $n = 16, 32$. Table 2 also presents the maximum absolute errors Q_3, third column, of the solution from paper [11].

Table 2. The maximum absolute errors of the solution

n	Errors of solution S_5	Errors of solution Q_3
8	$0.903 \cdot 10^{-3}$	$0.507 \cdot 10^{-2}$
16	$0.144 \cdot 10^{-4}$	$0.442 \cdot 10^{-3}$
32	$0.323 \cdot 10^{-6}$	$0.425 \cdot 10^{-4}$

Example 3. Consider the following Volterra integral equation, [13]:

$$u(x) + \int_0^x \frac{u(s)}{\sqrt{x-s}} ds = \frac{\pi x}{2} + \sqrt{x}, \quad x \in [0, b],$$

where the exact solution is $u(x) = \sqrt{x}$. Let $b = 0.6$.

In paper [13], the authors modified some vector-matrix barycentric Lagrange interpolation formulas in order to interpolate the kernel. The authors also introduced some new ideas for selecting interpolation nodes that ensure isolation of the singularity of the kernel. The errors of the solutions are given in Table 3.

We use the local splines to obtain the solution of the integral equation. On the interval $[0, 0.6]$ we construct a uniform grid with a step $h = 0.1$. We find an approximate solution of the integral equation at the grid nodes using fifth-order splines. Next, we perform the interpolation using the fifth-order splines, as was discussed in Section 2. Table 3 shows the values of the approximate solution found using fifth-order splines approximation and the results obtained by the author of paper [13].

Table 3. The errors of the solution obtained with the splines of the fifth order of approximation and the errors from paper [13]

x_i	Errors given in paper [13]	Errors obtained using splines of the fifth order
0	-	0
0.06	$0.107 \cdot 10^{0}$	$0.125 \cdot 10^{-1}$
0.12	$0.977 \cdot 10^{-1}$	$0.498 \cdot 10^{-2}$
0.18	$0.941 \cdot 10^{-1}$	$0.300 \cdot 10^{-2}$
0.24	$0.853 \cdot 10^{-1}$	$0.655 \cdot 10^{-3}$
0.3	$0.748 \cdot 10^{-1}$	$0.256 \cdot 10^{-2}$
0.36	$0.717 \cdot 10^{-1}$	$0.192 \cdot 10^{-2}$
0.42	$0.836 \cdot 10^{-1}$	$0.126 \cdot 10^{-2}$
0.48	$0.110 \cdot 10^{-1}$	$0.105 \cdot 10^{-2}$
0.54	$0.136 \cdot 10^{0}$	$0.118 \cdot 10^{-2}$
0.6	$0.112 \cdot 10^{-1}$	$0.819 \cdot 10^{-3}$

Example 4. Consider the following Volterra - Fredholm integral equation, [15]:

$$u(x) = q(x) + \int_0^x \frac{u(t)dt}{\sqrt{|x-t|}} + \int_0^1 \frac{u(t)dt}{\sqrt{|x-t|}},$$

where the exact solution is $u(x) = x^2(1-x)$.

In paper [15] the authors obtain the solution using the quasiaffine biorthogonal collocation type method. Here we use splines of the fifth order of approximation and splines of the second order of approximation. We construct an equidistant grid on the interval [0,1] with step h, $h = 1/n$. The errors of the solution are given in Table 4.

Table 4. The errors of the solution obtained with the splines of the fifth order of approximation, splines of the second order of approximation and the errors from paper [15]

n	Errors given in paper [15]	Errors obtained using splines of the fifth order	Errors obtained using splines of the second order
8	$0.544 \cdot 10^{-8}$	$0.21 \cdot 10^{-14}$	$0.29 \cdot 10^{-2}$
10	$0.131 \cdot 10^{-10}$	$0.23\ 10^{-14}$	$0.18 \cdot 10^{-2}$

Note that the small error of the solution obtained using fifth-order splines is due to the fact that these splines provide accuracy of approximation on fourth-degree polynomials. The calculations were performed with 15 Digits in the Maple package.

Example 5. As noted in paper [16], the property of viscoelasticity implies that, starting from a non-zero moment of time, the body has pronounced elastic properties over short time intervals and pronounced viscous properties over longer time intervals. Over such long-time intervals T, viscoelastic media can be considered changing slowly. The fundamental law of nonlinear deformation is often written either in differential form through the rheological constants of the material, or in integral form. Let us consider the equation:

$$E\varepsilon(x) = \sigma(x) + \int_0^x K(x,s) f\big(\sigma(s)\big) ds,$$
$$x \in [0,T],$$

here E represents the instantaneous modulus of elasticity, and the kernel has the following form

$$K(x,s) = Ae^{-\beta(x-s)}(x-s)^{\alpha-1},$$
$$0 < \alpha < 1,\ A > 0,\ \beta > 0.$$

Paper [16] is devoted to the numerical treatment of rheological models in the context of nonlinear heritable creep theory. An approximate method for nonlinear weak singular Volterra integral equations with Rzhanitsyn's kernel used in rheological models of viscoelastic continuum is shown in paper [16].

Consider the following integral equation, [16]:

$$\sigma(x) + A\int_0^x \frac{a\sigma(\tau) + b\sigma^2(\tau)}{\sqrt{x-\tau}} d\tau = F(x), x \in [0,1].$$

The exact solution of the integral equation is the function $\sigma(x) = 1 - \sqrt{x}$. We take $A = 1, a = 0.015, b = 0.399$.

We use the local fifth-order spline approximations to obtain the solution of the integral equation.

Figure 10 shows the plots of the exact and approximate solutions. The red plot is the exact solution, the blue plot is the approximate solution.

Figure 11 shows the plot of the errors of the solution of the integral equation ($n = 7$).

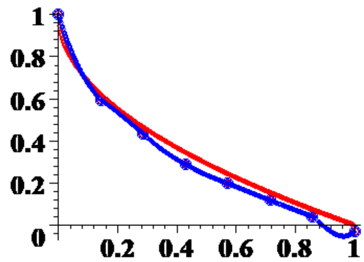

Fig. 10: The exact solution (red) and the approximate solution (blue), ($n = 7$)

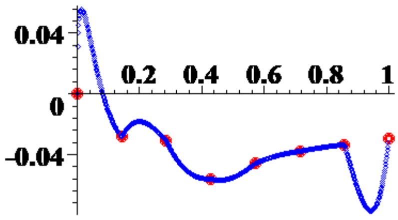

Fig. 11: The plot of the errors of the solution ($n = 7$)

The following example illustrates the dependence of the quality of an approximate solution on the number of digits retained in the mantissa of the numbers with which we perform calculations.

Example 6. Consider the following integral equation:

$$u(x) + \int_0^1 \exp(x+s)\, u(s) ds = f(x),$$

The exact solution of this integral equation is $u(x) = \exp(-x)$.

To solve this integral equation we used splines of the fifth and seventh order of approximation with the following values of the parameter Digits: Digits=10 and Digits=20. Table 5 shows that for Digits=10 we obtain an approximate solution of the integral equation with only two significant digits after the decimal point only when the splines of the fifth order of approximation were used and the splines of the seventh order of approximation can't be used. But for Digits=20 splines of the fifth order of approximation and splines of the seventh order of approximation can be used. The splines of the seventh order of approximation gives the lesser error of the solution.

Table 5. The errors of the solution obtained with the splines of the fifth and the seventh order of approximation, ($n = 16$)

Digits	$\parallel U - V \parallel_1$		$\parallel U - V \parallel_2$	
	Splines of the fifth order	Splines of the seventh order	Splines of the fifth order	Splines of the seventh order
10	0.0019	7.82	0.0029	13.14
20	$0.85\cdot 10^{-8}$	$0.29\cdot 10^{-9}$	$0.23\ 10^{-7}$	$0.47\cdot 10^{-9}$

Here vector $U = (u_0, u_1, \dots, u_n)$, is the vector of the approximate solution, and $V = (v_0, v_1, \dots, v_n)$ is the vector of the exact solution.

Figure 12 shows the errors when the fifth order approximation splines were applied (Digits = 10, $n = 16$).

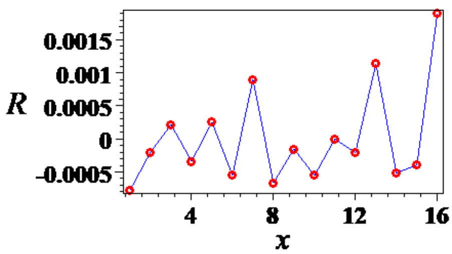

Fig. 12: The plot of the errors obtained with the polynomial splines of the fifth order (Digits $= 10, n = 16$)

6 Conclusion

This paper discusses a special method for solving weak singular integral equations based on the use of local interpolation splines. Our spline approximation method is based on replacing the unknown function under the integral sign with a spline approximation with unknown coefficients. We compared the errors of the solutions of integral equations obtained using splines of the second, fifth and seventh orders of approximation with the results obtained by solving the same integral equations with other methods. The methods based on the use of spline approximations, as a rule, give a smaller error than the use of other methods for solving integral equations. A comparison of the stability of numerical methods based on spline approximations of the fifth and seventh order of approximation shows that the method based on the fifth order of approximation has an advantage when the number of nodes is large. When using a method based on the splines of the seventh order of approximation, it is necessary to keep more digits in the mantissa of numbers when calculating. With a small number of grid nodes (10-20), the use of methods based on splines of the seventh order produces a smaller solution error.

In the future, it is proposed to pay more attention to the construction of non-uniform grids and solving various problems on adaptive grids.

Acknowledgement:
The authors are gratefully indebted to a resource center of St. Petersburg University for providing the Maple package.

References:

[1] A. Pedas, M. Vikerpuur, On the regularity of solutions to a class of nonlinear Volterra integral equations with singularities, *Applied Numerical Mathematics*, Vol. 204, 2024, pp. 176-187, DOI: 10.1016/j.apnum.2024.06.008.

[2] Sumaira Rehman, Arvet Pedas, Gennadi Vainikko, Fast Solvers of Weakly Singular Integral Equations of the Second Kind, *Mathematical Modelling and Analysis*, Vol. 23, Iss. 4, pp. 639–664, 2018, DOI: 10.3846/mma.2018.039.

[3] Sumaira Rehman, Arvet Pedas, Gennadi Vainikko, A Quasi-Fast Solver for Weakly Singular Integral Equations of the Second Kind, *Numerical Functional Analysis and Optimization,* Vol. 41, Iss. 7, 2020, pp. 850-870, DOI: 10.1080/01630563.2019.1704781.

[4] Esmaeil Najafi, Nyström-quasilinearization method and smoothing transformation for the numerical solution of nonlinear weakly singular Fredholm integral equations, *Journal of Computational and Applied Mathematics*, Vol. 368, 2020, pp.1-13, DOI: 10.1016/j.cam.2019.112538.

[5] H. Adibi, P. Assari, On the numerical solution of weakly singular Fredholm integral

equations of the second kind using Legendre wavelets, *J. Vib. Control,* Vol.17, No. 5, 2011, pp. 689-698, DOI: 10.1177/1077546310366865.

[6] A. Pedas, G. Vainikko, Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations, *Computing*, Vol.73, 2004, pp. 271-293, DOI: 10.1007/s00607-004-0088-9.

[7] C. Tunç, F.S. Alshammari, F.T. Akyildiz, Existence and Uniqueness of Solutions of Hammerstein-Type Functional Integral Equations, *Symmetry,* Vol.15, 2023, pp.1-16, https://doi.org/10.3390/sym15122205.

[8] K. Maleknejad, R. Mollapourasl, A. Ostadi, Convergence analysis of Sinc-collocation methods for nonlinear Fredholm integral equations with a weakly singular kernel, *J. Comput. Appl. Math.*, Vol. 278, 2015, pp. 1-11, DOI: 10.1016/j.cam.2014.09.018.

[9] H.Brunner, A. Pedas, G.Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations, *Math. of Comp*., Vol. 68, No. 227, 1999, pp. 1079-1095, DOI:10.1090/S0025-5718-99-01073-X.

[10] M. V. Bulatov, O. S. Budnikova, Numerical Solution of Integral-Algebraic Equations for multistep methods. *Comput. Math. and Math. Phys.* Vol.**52**, 2012, pp. 691–701, https://doi.org/10.1134/S0965542512050041.

[11] C. Allouch, P. Sablonnière, D. Sbibiha, M. Tahrichi, Product Integration Methods based on Discrete Spline Quasi-interpolants and Application to Weakly Singular Integral Equations, *Journal of Computational and Applied Mathematics,* Vol.233, Iss.11, 2010, pp. 2855–2866, DOI: 10.1016/j.cam.2009.11.031.

[12] M. Mohammad, C. Cattani, A Collocation Method via the Quasi-Affine Biorthogonal Systems for Solving Weakly Singular Type of Volterra-Fredholm Integral Equations. *Alexandria Engineering Journal*, Vol.59, No.4, 2020, pp. 2181-2191, DOI: 10.1016/j.aej.2020.01.046.

[13] Emil Sobhy Shoukralla, Interpolation Method for Solving Weakly Singular Integral Equations of the Second Kind, *Applied and Computational Mathematics*, Vol.10, No.3, 2021, pp.76-85, DOI: 10.11648/j.acm.20211003.14.

[14] I. G. Burova, G. O. Alcybeev, The Application of Splines of the Seventh Order Approximation to the Solution of Integral Fredholm Equations, *WSEAS Transactions on Mathematics*, Vol. 22, pp. 409-418, 2023, DOI: 10.37394/23206.2023.22.48

[15] Mutaz Mohammad, Carlo Cattani, A collocation method via the quasi-affine biorthogonal systems for solving weakly singular type of Volterra-Fredholm integral equations, *Alexandria Engineering Journal,* Vol. 59, Iss. 4, 2020, pp. 2181-2191, DOI: 10.1016/j.aej.2020.01.046.

[16] A.N. Tynda, A.E. Romanov, Numerical solution of nonlinear Volterra integral equations with fractionally-exponential kernels of rheological models of viscoelastic continuum, *News of Irkutsk State University, Series "Mathematics*", Vol. 5. No.3, 2012. pp. 69–80.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- G. O. Alcybeev executed the numerical experiments.
- I. G. Burova developed the theoretical part.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.