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Numerical Inference on the Generalize Gamma Distribution Parameters Using Runge-Kutta Method

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Abstract: - In statistical inference, there are several methods for estimating the distribution parameters in life data analysis. However, most of them are less efficient than the Bayes' method based on the informative prior. Thus, the main objective of this study is to present an optimal numerical iteration method, such as the Runge-Kutta iteration technique, for estimating the distribution parameters. This method has been used for estimating the generalized life distribution parameters and comparing them with the Bayesian estimates based on the informative gamma and kernel priors. A comparison between these methods is provided by using an extensive Monte Carlo simulation study. The simulation results indicated that the Runge-Kutta method is highly favorable and outperforms the Bayesian estimates using different loss functions based on the generalized progressive hybrid censoring scheme. Finally, two real dataset analyses are presented to illustrate the efficiency of the proposed methods.

Key-Words: - Bayesian estimations; informative gamma prior; kernel prior; Runge-Kutta method.

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1 Introduction

In lifetime distributions and reliability analysis, many methods have been used for estimating the distribution parameters. However, the most usable method in reliability analysis is the Bayes' method, despite its subjectivity to prior information other than data. Thus, in this study, we present an optimal estimation method using the Runge-Kutta iteration technique for estimating the distribution parameters. This method is more efficient than the Bayes' method based on the informative and kernel priors using different loss functions. To illustrate that, we employed the proposed methods on one of the employed distributions in lifetime distributions and reliability theory, the generalized life model, with application to the three-parameter generalized gamma distribution (GGD). The generalized life model belongs to the shape-scale family with the cumulative distribution function and the probability density function, which are given respectively, as follows:

$$\mathrm{F(x)} = 1 - [1 + \beta g^{\alpha}(\mathrm{x})]^{-\gamma}, \quad \mathrm{x}, \alpha, \beta, \gamma > 0.$$

$$\mathrm{f(x)} = \alpha\beta\gamma g^{\alpha-1}(x)\mathrm{g}'(\mathrm{x})[1 + \beta g^{\alpha}(\mathrm{x})]^{-\gamma-1}, \quad \mathrm{x}, \alpha, \beta, \gamma > 0.$$

The parameters $\alpha, \gamma > 0$ are shape parameters, and $\beta > 0$ is the scale parameter.

For convenience, we assume $g(x)$ to be differentiable as well as a strictly increasing function of x, $\mathrm{g}(0^+) = 0$ and $g(x) \to \infty \quad as \quad x \to \infty$.
This family includes, among others, the most popular parametric models in lifetime distributions, such as the generalized Weibull distribution, the generalized gamma distribution, and the generalized Pareto distribution. Also, the Pareto distribution, the Burr-XII distribution, and the Lomax distribution are members of this distribution according to the values of $g(x)$. Some important members of this family are given in Table 1.

Table 1: Some important lifetime distributions

No.	$g(x)$	F(x)	Distribution
1	x	$1 - (1 + \beta \mathrm{x}^{\alpha})^{-\gamma}$	3P-Burr-XII
2	x	$1 - (1 + \mathrm{x}^{\alpha})^{-\gamma}$, $\beta = 1$	Burr -Type XII
3	x	$1 - (1 + \beta x)^{-\gamma}$, $\alpha = 1$	Lomax
4	$-x$	$1 - (1 - \beta x)^{-\gamma}$, $\alpha = 1$	Generalized Pareto
5	x	$1 - (1 + x^{\alpha}/\gamma)^{-\gamma}$ $\beta = 1/\gamma$	Generalized Weibull
6	$e^{\alpha x^{\theta}}$	$1 - (1 + e^{\alpha x^{\theta}})^{-\gamma}$, $\beta = 1$	Generalized Gamma

As an application for the generalized shape-scale

family, the GGD, which is the one that has a great part of applications based on the progressive order statistics for its flexibility to describe the life test data. It has been introduced in the literature that has utility in survivorship applications. The cumulative distribution function (cdf) F(x) and the probability density function (pdf) f(x) are given, respectively, as follows:

$$F(x) = 1 - (1 + e^{\beta x^{\alpha}})^{-\gamma}, \quad x, \alpha, \beta, \gamma > 0, \qquad (1)$$

$$f(x) = \alpha\beta\gamma x^{\alpha-1} e^{\beta x^{\alpha}} (1 + e^{\beta x^{\alpha}})^{-\gamma-1}, x, \alpha, \beta, \gamma > 0 \qquad (2)$$

where α and γ are form factors and beta is the scale factor.

The statistical analysis of recent data in applied sciences (medical sciences, environmental sciences, engineering, finance, etc.) has shown the GGD is an appropriate model that allows revealing some important details from these datasets. Thus, several authors have studied the GGD based on complete and censored samples in applied sciences, such as [3], who obtained a new discrete distribution related to the GGD and discussed some of its properties. [7] introduced some concepts of the GGD based on the information theory, and [8] applied the approximate inference for the GGD parameters.

[9] suggested a new re-parameterization of the GGD to maintain the numerical stability for the maximum likelihood estimates (MLE) based on the type-II progressively censored samples. [10] described several difficulties in estimating the parameters and the complexity of the MLE method. [12] derived the asymptotic MLE for the GGD parameters, and [13] derived the moment estimates of the GGD parameters using its characterization. [15, 16] applied the conditional inference for the GGD and log gamma parameters. [17] derived the conditional and the structural confidence intervals for the GGD parameters. [18] derived the structural confidence intervals based on the type-II progressively censored samples from the three-parameter GGD. [26] presented a Bayesian study comparing the GGD with its components, and [27] introduced comparative studies on the modeling of lifetime data using the three-parameter GGD. [28] developed inference procedures with the GGD, especially the MLE, to justify the model in terms of a simpler alternative form, and [29] produced the GGD and derived the parameter estimations.

[32] derived the MLE of the parameters using the numerical root isolation, and [33] derived the MLE for the three-parameter GGD of Stacy and Mirham.

In this study, we introduce an improved estimation method based on the Runge-Kutta iteration technique for estimating the GGD parameters compared with the Bayes' method based on the informative and kernel priors.

In reliability analysis, the progressive Type-II censoring scheme is the most applicable in life test experiments. It is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at different stages before testing is terminated. However, the trial time can be quite long due to some highly reliable units. Thus, [14] recently proposed a censoring scheme called the Type-II progressive hybrid censoring scheme. However, the disadvantage of the progressive hybrid censoring scheme is that very few failures may occur before the time point **T**. To provide a guarantee of the number of failures observed as well as the time to complete the test, [5] and [6] proposed the generalized progressive hybrid censoring scheme (GPHCS) that modifies the progressive hybrid censoring scheme. It allows the experiment to continue beyond time **T** to observe at least **k** failures if the number of failures is less than **m**. The algorithm for the GPHCS has been described in [20 - 25].

Thus, given a generalized progressive hybrid censored sample, the likelihood function can be written in a unified form as follows:

$$L(\bar{X}; \theta) = C \prod_{i=1}^{n} f(x_i) [1 - F(x_i)]^{R_i} [1 - F(T)]^{\delta R_T^*}, \qquad (3)$$

$$n = \begin{cases} m, & \delta = 0, \quad \text{if} \quad X_{k:m:n} \leq X_{m:m:n} < T \\ k, & \delta = 0, \quad \text{if} \quad T < X_{K:m:n} \leq X_{m:m:n} \\ D, & \delta = 0, \quad \text{if} \quad X_{k:m:n} < T < X_{m:mn} \end{cases},$$

where $\bar{X} = (X_1, X_2, \ldots, X_n)$ and R_T^* is the number of surviving units that are removed at the stopping time $T = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$.

The GPHCS has been applied for some distributions such as the Weibull distribution, see [6], the inverse Weibull distribution, see [20]. The exponential distribution, see [5] and [11], and the Rayleigh distribution, see [4]. The shape-scale family, see [21, 22], the generalized Weibull distribution, see [23], and the generalized gamma distribution, see [24].

2 Runge-Kutta Method

The MLE $\hat{\theta} = \hat{\theta}(\bar{x})$ of θ is the solution of the stationary equation, $\frac{\partial H(\bar{x};\theta)}{\partial \theta}|_{\theta=\hat{\theta}} = 0$, which is a function of $\overline{X}$ and $\hat{\theta}(\bar{x})$, where $H(\bar{x}; \bar{\theta})$ is the log-likelihood function that depends on the unknown parameter $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and the data $\overline{X} = (x_1, x_2, \ldots, x_n)$. Applying the implicit function

theorem to the stationary equation by considering all partial derivatives, as well as the total derivatives, are assumed to be evaluated at some known value of $\hat{\theta}(x_0) = \theta_0$, say. Taking the total derivative for the stationary equation with respect to any $x \in \overline{X}$, we obtain.

$$\frac{d}{dx}\left(\frac{\partial H(\bar{x};\theta)}{\partial\theta}\right)|_{\theta=\hat{\theta}} = \frac{\partial^2 H(\bar{x};\theta)}{\partial\theta\,\partial x}|_{\theta=\hat{\theta}} + \frac{\partial^2 H(\bar{x};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}}\frac{d\hat{\theta}}{dx} = 0. \quad (4)$$

Solving (4) we obtain the first derivative with respect to x for $\hat{\theta}$ at $\theta = \hat{\theta}$ as follows:

$$\frac{d\hat{\theta}}{dx} = -\left(\frac{\partial^2 H(\bar{x};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 H(\bar{x};\theta)}{\partial\theta\,\partial x}|_{\theta=\hat{\theta}}. \quad (5)$$

Thus, we can write (5) as the first-order ordinary differential equation for the maximum likelihood estimator $\hat{\theta}(\bar{X})$ as follows:

$$\frac{d\hat{\theta}}{dx} = f(x,\hat{\theta}), \quad \text{with} \quad \hat{\theta}(x_0) = \theta_0, \quad (6)$$

where $f(x,\hat{\theta})$ and $\frac{df(x,\hat{\theta})}{d\hat{\theta}}$ are defined and continuous functions at all points $(\overline{X},\hat{\theta})$, which ensures the existence of a unique solution for (6). Using any numerical technique, such as the fourth-order Runge-Kutta, we can find the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities concerning which the fit is optimized.

Thus, the R-K recurrence solution for (6) can be obtained as

$$\theta^*_{i+1} = \theta^*_i + (K_1 + 2K_2 + 2K_3 + K_4)/6, \quad (7)$$

for $i = 0,1,2,\ldots$, where

$$K_1 = hf(x_i,\theta^*_i),\ K_2 = hf(x_i + h/2, \theta^*_i + K_1/2),$$

$$K_3 = hf(x_i + h/2, \theta^*_i + K_2/2),$$

$$K_4 = hf(x_i + h, \theta^*_i + K_3).$$

Here h is a small known value (say, 1E-02) and $\theta = \theta_0$, is the initial value for θ.
The iterative process is continued using (6), and (7) until two consecutive numerical solutions are almost the same, that is if $|\theta^*_{i+1} - \theta^*_i| < 10^{-5}$,
for $i = 0,1,2,3,\ldots$.

Using α , β and γ instead of θ in (7) we get the corresponding estimator for each parameter, respectively.

For the generalized gamma model (1) and (2) the likelihood function of the GPHCS (3) can be written as follows:

$$L(\bar{X};\theta) = C(\alpha\beta\gamma)^n \exp[(\alpha-1)\sum_{i=1}^{n}\ln(x_i) + \beta\sum_{i=1}^{n}x_i^{\alpha}$$

$$-[\textstyle\sum_{i=1}^{n}[1+\gamma(1+R_i)]ln\left(1+e^{\beta x_i^{\alpha}}\right) + \gamma\delta R_T^{*}\ln\left(1+e^{\beta T^{\alpha}}\right)]$$

The derivatives of the log-likelihood function have been derived in the Appendix A, with a = c= e=1, b = d =f= 0.
Thus, using (7) with the derivatives we can find the point estimates for each α, β and γ separately using the Runge-Kutta method.

3 Bayes' Method

In this section, the Bayes estimators for the parameters will be derived using gamma and kernel prior distributions based on two different loss functions:
Firstly, the squared error loss function (SLF), $L(g(\theta),\hat{g}(\theta)) = (g(\theta) - \hat{g}(\theta))^2$, where $\hat{g}(\theta)$ is the Bayes estimator for the function $g(\theta)$. For this loss function the Bayes estimator that minimizes the risk function is given by $\hat{g}(\theta) = E_\theta(g(\theta)|x)$.

Secondly, the compound LINEX loss function which is defined as follows:

$$L(\Delta) = L_\delta(\Delta) + L_{-\delta}(\Delta) = e^{\delta\Delta} + e^{-\delta\Delta} - 2, \delta > 0.$$

It is named LINEX-based loss function, see [31], where

$$L_\delta(\Delta) = e^{\delta\Delta} - \delta\Delta - 1, \Delta = \hat{g}(\theta) - g(\theta),\ \delta \neq 0,$$

is the LINEX loss function (LLF) that has been introduced in [34]. The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\hat{g}_L(\theta) = \frac{1}{2\delta}\log\left(\frac{E\left(e^{\delta g(\theta)}|X\right)}{E\left(e^{-\delta g(\theta)}|X\right)}\right).$$

3.1 The Informative Gamma Prior

We consider the unknown parameters α, β and γ have independent gamma prior distributions with the joint probability density function, which is given by:

$$h(\alpha,\beta,\gamma) \propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma}, \quad (8)$$

where the hyperparameters a, b, c, d, e and f are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

3.2 The Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function $g(\alpha,\beta,\gamma)$ with support on $(0,\infty)$, which is defined as follows:

$$\hat{g}(\alpha,\beta,\gamma) = \frac{1}{nh_1h_2h_3}\sum_{i=1}^{n} K\left(\frac{\alpha-\alpha_i}{h_1},\frac{\beta-\beta_i}{h_2},\frac{\gamma-\gamma_i}{h_3}\right), \quad (9)$$

$h_i, i = 1,2,3$ are called the bandwidths or smoothing parameters, which chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The optimal choice for h_i that minimizes the mean squared errors is given by $h_i = 1.06\hat{S}_i n^{-0.2}$,where S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the trivariate standard normal distribution for the parameters α, β and γ. The algorithm for deriving the kernel prior estimate can be seen in [1] and [19].

Thus, using the joint priors (8) and (9) with the likelihood function of the GPHCS (3), the posterior density for the parameters α, β and γ can be written in a unified form as follows:

$$f(\alpha,\beta,\gamma|\underline{x}) = Kq(\alpha,\beta,\gamma)L(\bar{X};\alpha,\beta,\gamma), \text{ where}$$

$$q(\alpha,\beta,\gamma) = \hat{g}(\alpha,\beta,\gamma)h(\alpha,\beta,\gamma) =$$

$$\hat{g}_1^{p_1}(\alpha)\hat{g}_2^{p_2}(\beta)\hat{g}_3^{p_3}(\gamma)\,\alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma},$$

is the general prior distribution function with $p_1 = p_2 = p_3 = 0$ for the informative prior (8), and $p_1 = p_2 = p_3 = 1$, $a = c = e = 1$, and $b = d = f = 0$ for the kernel prior (9).

Thus, the log posterior density can be written as

$$H(\bar{X};\theta) = p_1\ln(\hat{g}_1(\alpha)) + p_2\ln(\hat{g}_2(\beta)) + p_3\ln(\hat{g}_3(\gamma)) + (n+a-1)ln\alpha + (n+c-1)ln\beta + (n+e-1)ln\gamma - \alpha b - \gamma f + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) - \beta(d-\sum_{i=1}^{n}x_i^{\alpha}) - [\sum_{i=1}^{n}(1+\gamma(1+R_i))ln\left(1+e^{\beta x_i^{\alpha}}\right)+\gamma\delta R_T^{*}\ln\left(1+e^{\beta T^{\alpha}}\right)]. \quad (10)$$

Thus, based on (10), we can use the Tierney-Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [30] introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter, or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let $w(\alpha,\beta,\gamma)$ be a smooth, positive function on the parameter space. The posterior expectation of $w(\alpha,\beta,\gamma)$ can be obtained as follows:

$$w^* = E(w(\alpha,\beta,\gamma)|\bar{X}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}, \quad (11)$$

where $H = \ln f(\alpha,\beta,\gamma|x)/n$, and

$$H^* = H + \ln w(\alpha,\beta,\gamma)/n.$$

For (α,β,γ) the Bayes estimator using Tierney-Kadane approximation for $w(\alpha,\beta,\gamma)$ can be obtained as

$$w^* = \sqrt{|\Sigma^*|/|\Sigma|}\exp[\,n[H^*(\alpha,\beta,\gamma) - H(\alpha,\beta,\gamma)],$$

Where $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ maximize the $H(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $H^*(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$, respectively. Let

$$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1} \text{ and}$$

$$|\Sigma^*| = \begin{vmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ at $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ respectively. The derivatives of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ have been derived in the Appendix A.

4 Simulation Study

The purpose of the simulation study is to compare the performance of the estimates using the R-K and Bayes methods based on the informative gamma and the informative kernel priors with two different loss functions through two criteria: the average bias (AVB) and the mean squared error (MSE), as given, respectively, by:

$$AVB = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta|, \quad MSE = \sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2/L$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the hyperparameters of α , β and γ as follows:

$a = c = e = 2, b = d = f = 5$ and the values for the parameters are $\alpha = (1,2)$, $\beta = (0.75,\ 1.75)$ and $\gamma = (2,\ 3)$ respectively. Using the parameter values for generating different samples from the GGD with sizes $n = 20$, 40 and 60 to represent small, moderate and large sizes. To assess the performance of these estimates, the average bias (AVB) and the MSEs for each were calculated using 1000 replicates. An algorithm for generating the generalized progressive hybrid censoring scheme has been written, see [20, 21].

From the simulation results in Tables 4, 5, 6, 7, 8 and 9, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

1. It is clear that, generally, for the parameters α, β and γ the estimate values of the AVB and the MSE based on the R-K method outperforms the corresponding values based on the Bayes method for the different loss functions.
2. It is evident that the estimated AVB and MSE values increase with increasing the values of α and γ while decreasing as the values of the parameter β increase.
3. The estimated AVB and MSE values decrease with increasing the termination time of the experiment T, and the sample sizes as expected for all methods.
4. In general, the estimated MSE values for the Bayes method based on the LINEX-based loss function are less than those based on the squared error loss functions.

In conclusion, the simulation results indicated that the R-K method competes and outperforms the Bayes method based on the informative and kernel priors.

5 Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the three-parameter GGD in practice and to illustrate that this distribution can be considered as a good lifetime model for some new areas of applications, comparing with many known distributions such as the Weibull distribution. We have fitted these data sets using some goodness-of-fit tests, such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH2) tests for significance level test equal to 0.05.

5.1 The Reactor Pumps Data

In this section, a real data set for secondary nuclear pumps has been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the recirculating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feedwater pump that takes feedwater pressurized by the booster pump and pushes it into the steam generator through the high-pressure. Accordingly, the main feed pump must be high-temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the times between the failure of the secondary reactor pumps. [27] and [28] have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of the 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the GGD is a good fit for this dataset, as shown in Fig. (1 a). For studying the reliability of these reactor pumps based on this dataset, we find the estimates for the parameters, which represent the shape of the failures between pumps, using our model to determine the behavior of the failure pumps. We noticed that the R-K and Bayes estimates for α lie in the interval [0.5,0.6], for γ lie in the interval [0.9, 1.0] and for β lie in the interval [0.01,0.02] indicating that the above dataset is heavily right-skewed and that means the failure rate decreases with increasing time, see Fig. (1 b), and that means decreasing the reliability of safety mechanism with increasing time.

5.2 The Vinyl Chloride Data

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. Where it is known that a concentration of vinyl chloride in drinking water of 0.5 mg/liter was calculated as being associated with an excess risk of liver and brain tumors for exposure beginning at adulthood, and it would double cancer risk for continuous exposure from birth. Therefore, we consider the dataset used by [2], which represents 34 data points in mg/L from the vinyl chloride

obtained from clean upgrade monitoring wells, as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.
We found the GGD is a very good fit for this dataset, as shown in Fig. (2 a). For studying the concentration of the vinyl chloride in the water of these wells based on this dataset, we find the estimates for the parameters that represent the shape of the concentration using our model to determine the average concentration in the water. We noticed that the R-K and Bayes estimates for α lie in the interval [0.7, 0.8] and for γ lie in the interval [1.2, 1.3], indicating that the above dataset is moderately right skewed, and that means the concentration decreases with increasing time, see Fig. (2 b). Also, the R-K and Bayes estimates for β lie in the interval [0.09, 0.12], which ensures the dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time, therefore, monitoring these wells is very significant.
Based on these data sets, the results in Table 2 ensure the GGD is a very good fit for these data sets since the calculated values for the goodness-of-fit tests are less than the critical values. Moreover, the power of the tests is greater than the significance level of 0.05 of the tests. From the results in Table 3, the estimated values of MSE based on the R-K method are approximately smaller than those for the Bayesian method for these data sets, considering the MLEs are the true values of the parameters.

6 Conclusions

In this study, it is found that, based on the generalized progressive hybrid censored data, the R-K method is strongly unbiased and much more efficient than the Bayesian method based on the informative and kernel priors based on different loss functions. The Bayes estimates based on the nonparametric kernel prior are more efficient than those based on the informative gamma prior and are very close to those based on the R-K method. Thus, the statistical significance of the R-K method is its efficiency compared to most estimation methods, besides, it is reliable and easy to apply, especially for social sciences and psychology researchers.

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Appendix A

$$\mathrm{H}(\bar{\mathrm{X}};\theta) = p_1 \ln\big(\hat{g}_1(\alpha)\big) + p_2 \ln\big(\hat{g}_2(\beta)\big) + p_3 \ln\big(\hat{g}_3(\gamma)\big)$$

$$+(n+a-1)ln\alpha + (n+c-1)ln\beta + (n+e-1)ln\gamma$$
$$-\alpha b - \gamma f + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) - \beta\left(d - \sum_{i=1}^{n} x_i^{\alpha}\right)$$
$$-\left[\sum_{i=1}^{n}(1+\gamma(1+R_i))ln\left(1+e^{\beta x_i^{\alpha}}\right) + \gamma\delta R_T^{*}\ln\left(1+e^{\beta T^{\alpha}}\right)\right]$$

$$\frac{\partial H}{\partial \alpha} = p_1\frac{\hat{g}_1^{l}(\alpha)}{\hat{g}_1(\alpha)} + \frac{n+a-1}{\alpha}$$

$$-b + \sum_{i=1}^{n} lnx_i + \beta\sum_{i=1}^{n} x_i^{\alpha} lnx_i$$

$$-\beta[\sum_{i=1}^{n}\big(1+\gamma(1+R_i)\big)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i}{1+e^{\beta x_i^{\alpha}}} + \delta\gamma R_T^{*}\frac{e^{\beta T^{\alpha}}T^{\alpha}lnT}{1+e^{\beta T^{\alpha}}}]$$

$$\frac{\partial H}{\partial \beta} = p_2\frac{\hat{g}_2^{l}(\beta)}{\hat{g}_2(\beta)} + \frac{N+c-1}{\beta} - d + \sum_{i=1}^{n} x_i^{\alpha}$$

$$-\left[\sum_{i=1}^{n}\big(1+\gamma(1+R_i)\big)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}}{1+e^{\beta x_i^{\alpha}}} + \delta\gamma R_T^{*}\frac{e^{\beta T^{\alpha}}T^{\alpha}}{1+e^{\beta T^{\alpha}}}\right]$$

$$\frac{\partial H}{\partial \gamma} = p_3\frac{\hat{g}_3^{l}(\gamma)}{\hat{g}_3(\gamma)} + \frac{n+e-1}{\gamma} - f$$

$$-\left[\sum_{i=1}^{n}(1+R_i)ln\left(1+e^{\beta x_i^{\alpha}}\right) + \delta R_T^{*}\ln\left(1+e^{\beta T^{\alpha}}\right)\right]$$

$$\frac{\partial^2 H}{\partial \alpha^2} = p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{ll}(\alpha) - \hat{g}_1^{l\,2}(\alpha)}{\hat{g}_1^{2}(\alpha)} - (N+a-1)/\alpha^2 + \beta\sum_{i=1}^{n} x_i^{\alpha}(lnx_i)^2$$

$$-\beta^2[\sum_{i=1}^{n}\frac{\big(1+e^{\beta x_i^{\alpha}}\big)\big[e^{\beta x_i^{\alpha}}x_i^{\alpha}(lnx_i)^2 + e^{\beta x_i^{\alpha}}(x_i^{\alpha}lnx_i)^2\big] - (e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i)^2}{\big(1+e^{\beta x_i^{\alpha}}\big)^2}$$

$$-\beta^2[\sum_{i=1}^{n}R_i\frac{\big(1+e^{\beta x_i^{\alpha}}\big)\big[e^{\beta x_i^{\alpha}}x_i^{\alpha}(lnx_i)^2 + e^{\beta x_i^{\alpha}}(x_i^{\alpha}lnx_i)^2\big] - (e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i)^2}{\big(1+e^{\beta x_i^{\alpha}}\big)^2}$$

$$+\delta\gamma R_T^{*}\frac{\big(1+e^{\beta T^{\alpha}}\big)\big[e^{\beta T^{\alpha}}T^{\alpha}(lnT)^2 + e^{\beta T^{\alpha}}(T^{\alpha}lnT)^2\big] - \big(e^{\beta T^{\alpha}}T^{\alpha}lnT\big)^2}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial^2 \mathrm{H}}{\partial \beta^2} = p_2\frac{\hat{g}_2(\beta)\hat{g}_2^{ll}(\beta) - \hat{g}_2^{l\,2}(\beta)}{\hat{g}_2{}^{2}(\beta)} - (n+c-1)/\beta^2$$

$$-[\sum_{i=1}^{n}(1+\gamma(1+R_i))\frac{(1+e^{\beta x_i^{\alpha}})[e^{\beta x_i^{\alpha}}x_i^{2\alpha}]-(e^{\beta x_i^{\alpha}}x_i^{\alpha})^2}{(1+e^{\beta x_i^{\alpha}})^2}$$

$$+\delta\gamma R_T^{*}\frac{(1+e^{\beta T^{\alpha}})[e^{\beta T^{\alpha}}T^{2\alpha}]-(e^{\beta T^{\alpha}}T^{\alpha})^2}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial^2 H}{\partial\gamma^2}=p_3\frac{\hat{g}_3(\gamma)\hat{g}_3^{ll}(\gamma)-\hat{g}_3^{l\,2}(\gamma)}{\hat{g}_3{}^2(\gamma)}-(n+e-1)/\gamma^2$$

$$\frac{\partial H}{\partial\alpha\,\partial\beta}=\sum_{i=1}^{n}x_i^{\alpha}lnx_i$$

$$-[\sum_{i=1}^{n}(1+\gamma(1+R_i))\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i}{1+e^{\beta x_i^{\alpha}}}+\delta\gamma R_T^{*}\frac{e^{\beta T^{\alpha}}T^{\alpha}lnT}{1+e^{\beta T^{\alpha}}}]$$

$$-\beta[\sum_{i=1}^{n}(1+\gamma(1+R_i))\frac{(1+e^{\beta x_i^{\alpha}})[e^{\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i]-e^{2\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i}{(1+e^{\beta x_i^{\alpha}})^2}$$

$$+\delta\gamma R_T^{*}\frac{(1+e^{\beta T^{\alpha}})[e^{\beta T^{\alpha}}T^{2\alpha}lnT]-e^{2\beta T^{\alpha}}T^{2\alpha}lnT}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\alpha}=-\beta[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^{*}\frac{e^{\beta T^{\alpha}}T^{\alpha}lnT}{1+e^{\beta T^{\alpha}}}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\beta}=-\left[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^{*}\frac{e^{\beta T^{\alpha}}T^{\alpha}}{1+e^{\beta T^{\alpha}}}\right]$$

$$\frac{\partial^2 H}{\partial x\partial\alpha}=\sum_{i=1}^{n}\frac{1}{x_i}+\beta\sum_{i=1}^{n}[\alpha x_i^{\alpha-1}lnx_i+x_i^{\alpha-1}]$$

$$-\beta[\sum_{i=1}^{n}\frac{[\alpha\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha-1}\ln x_i+\alpha x_i^{\alpha-1}e^{\beta x_i^{\alpha}}\ln x_i+e^{\beta x_i^{\alpha}}x_i^{\alpha-1}]}{1+e^{\beta x_i^{\alpha}}}$$

$$-\beta\gamma[\sum_{i=1}^{n}(1+R_i))\frac{[\alpha\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha-1}\ln x_i+\alpha x_i^{\alpha-1}e^{\beta x_i^{\alpha}}\ln x_i+e^{\beta x_i^{\alpha}}x_i^{\alpha-1}]}{1+e^{\beta x_i^{\alpha}}}$$

$$+\alpha\beta^2[\sum_{i=1}^{n}(1+\gamma(1+R_i))\frac{e^{2\beta x_i^{\alpha}}x_i^{2\alpha-1}lnx_i}{(1+e^{\beta x_i^{\alpha}})^2}$$

$$\frac{\partial^2 H}{\partial x\partial\gamma}=-\alpha\beta\sum_{i=1}^{n}(1+R_i)\frac{x_i^{\alpha-1}e^{\beta x_i^{\alpha}}}{1+e^{\beta x_i^{\alpha}}}$$

$$\frac{\partial^2 H}{\partial x\partial\alpha}=\sum_{i=1}^{n}\frac{1}{x_i}+\beta\sum_{i=1}^{n}[\alpha x_i^{\alpha-1}lnx_i+x_i^{\alpha-1}]$$

$$-\beta[\sum_{i=1}^{n}\frac{[\alpha\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha-1}\ln x_i+\alpha x_i^{\alpha-1}e^{\beta x_i^{\alpha}}\ln x_i+e^{\beta x_i^{\alpha}}x_i^{\alpha-1}]}{1+e^{\beta x_i^{\alpha}}}$$

$$-\beta\gamma[\sum_{i=1}^{n}(1+R_i))\frac{[\alpha\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha-1}\ln x_i+\alpha x_i^{\alpha-1}e^{\beta x_i^{\alpha}}\ln x_i+e^{\beta x_i^{\alpha}}x_i^{\alpha-1}]}{1+e^{\beta x_i^{\alpha}}}$$

$$+\alpha\beta^2[\sum_{i=1}^{n}(1+\gamma(1+R_i))\frac{e^{2\beta x_i^{\alpha}}x_i^{2\alpha-1}lnx_i}{(1+e^{\beta x_i^{\alpha}})^2}$$

$$\frac{\partial^2 H}{\partial x\partial\gamma}=-\alpha\beta\sum_{i=1}^{n}(1+R_i)\frac{x_i^{\alpha-1}e^{\beta x_i^{\alpha}}}{1+e^{\beta x_i^{\alpha}}}$$

where the r[th] derivative of the kernel density estimation can be defined as

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r}=\hat{g}_1{}^r(\alpha)=\frac{1}{nh_1^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right),\qquad(12)$$

where r=0,1,2,3,-----.

Using the Gaussian kernel and (12), we have

$$\hat{g}_1(\alpha)=\frac{1}{nh_1\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha)=-\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}(\frac{\alpha-\hat{\alpha}_i}{h_1})e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2}$$

$$\hat{g}_1''(\alpha)=\frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2-1]e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

where r=0,1,2,3,-----.

Using the Gaussian kernel and (A1), we have

$$\hat{g}_1(\alpha)=\frac{1}{nh_1\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha)=-\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}(\frac{\alpha-\hat{\alpha}_i}{h_1})e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2}$$

$$\hat{g}_1''(\alpha)=\frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2-1]e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

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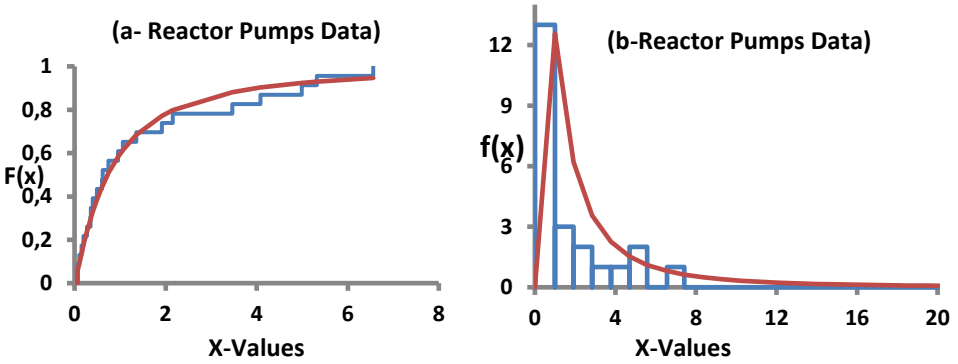

Fig. 1: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

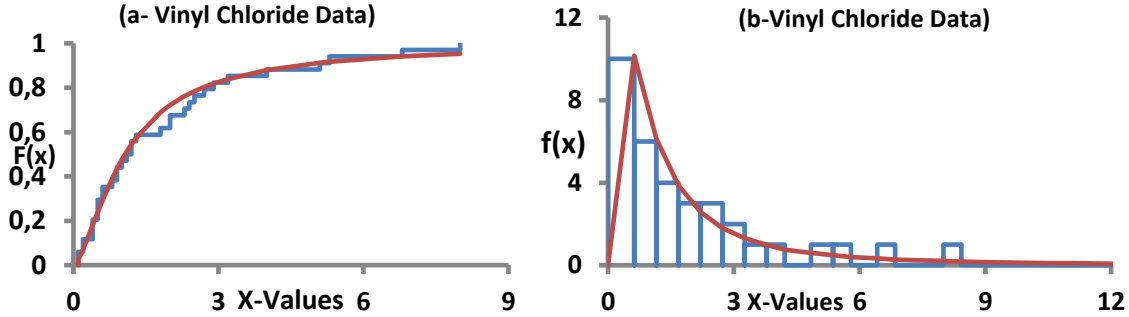

Fig. 2: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

Table 2: The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values)

for the GGD. The MLE for the parameters for these data sets have been calculated.

Data	The Tests	Calculated value	Critical value	p-values	MLES		
					α	β	γ
Reactor pumps Data N=23	K-S	0.4719	0.9001	0.8163	0.7399	3.8584	0.2085
	A-D	0.2466	0.7714	0.7563			
	CH2	9.5741	12.234	0.1355			
Rainfall data N=50	K-S	0.6064	0.9074	0.4872	1.5604	0.3892	1.3286
	A-D	0.3010	0.7711	0.6134			
	CH2	5.6472	15.086	0.3854			

Table 3: The estimate and the MSEs for the parameter α, β and γ based on the R-K and Bayes methods under square error loss function based on the GPHCS: for $m = n/2$, $k = m/2$.

Samples	T	Param	R-K Estamite		Gamma Prior		Kernel prior	
			Estimate	AVB(MSE)	Estimate	AVB(MSE)	Estimate	AVB(MSE)
The reactor Pumps data n=23	0.5	α	0.6474	0.0926(0.0086)	0.6244	0.1155(0.0134)	0.62448	0.1155(0.0133)
		β	3.3431	0.5153(0.2655)	3.1222	0.7361(0.5419)	3.2263	0.6321(0.3995)
		γ	0.1772	0.0276(7.6E-04)	0.1627	0.0421(0.0018)	0.1637	0.0411(0.0017)
	4.5	α	0.6475	0.0925(0.0086)	0.6248	0.1151(0.0133)	0.6249	0.1151(0.0133)
		B	3.3449	0.5135(0.2636)	3.1664	0.6918(0.4788)	3.2348	0.6236(0.3888)
		γ	0.1774	0.0275(7.5E-04)	0.1661	0.0387(0.0015)	0.1661	0.0388(0.0015)
Rainfall data n=50	5	α	1.3264	0.2339(0.0548)	1.2514	0.3089(0.0954)	1.2515	0.3089(0.0954)
		β	0.3307	0.0585(0.0034)	0.3024	0.0869(0.0075)	0.3018	0.0875(0.0076)
		γ	0.0279	0.005(2.5E-05)	0.2534	0.008(5.8E-05)	0.0252	0.0077(5.9E05)
	25	α	1.3264	0.234(0.0548)	1.2492	0.3112(0.0968)	1.2492	0.3112(0.0968)
		β	0.3308	0.0585(0.0034)	0.3061	0.0832(0.0069)	0.3059	0.0833(0.0069)
		γ	0.0279	0.0049(2.4E-05)	0.0258	0.007(5.1E-05)	0.0258	0.0071(5.1E05)

Table 4: The ABS and the MSEs in parentheses for the Parameter α using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1209(0.0150)	0.2059(0.0508)	0.1406(0.0201)	0.1937(0.0388)	0.1418(0.0204)
				2	3	0.2659(0.0713)	0.4664(0.2412)	0.3046(0.0933)	0.3827(0.1501)	0.3027(0.0918)
			1.5	1	2	0.1180(0.0142)	0.1786(0.0325)	0.1381(0.0194)	0.1711(0.0296)	0.1390(0.0196)
				2	3	0.2527(0.0640)	0.4217(0.1809)	0.2927(0.0858)	0.3505(0.1289)	0.2939(0.0865)
		8	0.75	1	2	0.1201(0.0146)	0.1880(0.0410)	0.1400(0.0198)	0.1786(0.0322)	0.1413(0.0201)
				2	3	0.2727(0.0746)	0.4256(0.2081)	0.3110(0.0969)	0.3712(0.1421)	0.3080(0.0949)
			1.5	1	2	0.1167(0.0138)	0.1682(0.0285)	0.1366(0.0189)	0.1647(0.0273)	0.1378(0.0191)
				2	3	0.2600(0.0678)	0.3700(0.1371)	0.2994(0.0898)	0.3461(0.1240)	0.2994(0.0897)
	15	8	0.75	1	2	0.1208(0.0148)	0.1858(0.0352)	0.1406(0.0200)	0.1795(0.0326)	0.1418(0.0203)
				2	3	0.2727(0.0746)	0.4311(0.2275)	0.3110(0.0969)	0.3660(0.1353)	0.3079(0.0949)
			1.5	1	2	0.1167(0.0138)	0.1683(0.0286)	0.1366(0.0189)	0.1643(0.0272)	0.1377(0.0191)
				2	3	0.2612(0.0684)	0.3708(0.1378)	0.3005(0.0905)	0.3469(0.1235)	0.3002(0.0902)
		11	0.75	1	2	0.1231(0.0153)	0.1888(0.0606)	0.1429(0.0206)	0.1770(0.0327)	0.1437(0.0208)
				2	3	0.2725(0.0743)	0.4574(0.5725)	0.3111(0.0968)	0.3799(0.4368)	0.3087(0.0953)
			1.5	1	2	0.1177(0.0140)	0.1626(0.0266)	0.1376(0.0191)	0.1610(0.0261)	0.1387(0.0194)
				2	3	0.2644(0.0701)	0.3482(0.1214)	0.3035(0.0922)	0.3349(0.1123)	0.3028(0.0918)
40	20	10	0.75	1	2	0.1207(0.0148)	0.1685(0.0288)	0.1405(0.0200)	0.1663(0.0282)	0.1414(0.0202)
				2	3	0.2673(0.0718)	0.3806(0.1459)	0.3059(0.0939)	0.3501(0.1249)	0.3042(0.0927)
			1.5	1	2	0.1177(0.0140)	0.1559(0.0245)	0.1377(0.0191)	0.1542(0.0239)	0.1384(0.0193)
				2	3	0.2525(0.0638)	0.3602(0.1303)	0.2926(0.0857)	0.3274(0.1076)	0.2936(0.0863)
		15	0.75	1	2	0.1216(0.0149)	0.1652(0.0277)	0.1414(0.0201)	0.1632(0.0268)	0.1422(0.0203)
				2	3	0.2740(0.0752)	0.3701(0.1505)	0.3121(0.0975)	0.3439(0.1184)	0.3095(0.0958)
			1.5	1	2	0.1181(0.0140)	0.1547(0.0240)	0.1380(0.0191)	0.1536(0.0237)	0.1387(0.0193)
				2	3	0.2589(0.0671)	0.3353(0.1125)	0.2985(0.0892)	0.3243(0.1053)	0.2988(0.0893)
			0.75	1	2	0.1220(0.0150)	0.1656(0.0279)	0.1418(0.0202)	0.1634(0.0269)	0.1426(0.0204)
				2	3	0.2751(0.0758)	0.3762(0.1503)	0.3132(0.0982)	0.3452(0.1193)	0.3103(0.0963)

	30	15	1.5	1	2	0.1178(0.0140)	0.1543(0.0239)	0.1377(0.0191)	0.1533(0.0236)	0.1384(0.0192)
				2	3	0.2591(0.0672)	0.3353(0.1125)	0.2986(0.0893)	0.3256(0.1072)	0.2989(0.0894)
		23	0.75	1	2	0.1255(0.0158)	0.1698(0.0403)	0.1452(0.0211)	0.1630(0.0269)	0.1455(0.0212)
				2	3	0.2715(0.0737)	0.3845(0.2060)	0.3103(0.0963)	0.3436(0.1280)	0.3089(0.0954)
			1.5	1	2	0.1197(0.0144)	0.1508(0.0228)	0.1396(0.0196)	0.1507(0.0228)	0.1402(0.0197)
				2	3	0.2652(0.0704)	0.3232(0.1045)	0.3043(0.0926)	0.3192(0.1019)	0.3038(0.0923)
60	30	15	0.75	1	2	0.1200(0.0145)	0.1577(0.0252)	0.1399(0.0197)	0.1565(0.0246)	0.1406(0.0199)
				2	3	0.2687(0.0725)	0.3568(0.1275)	0.3073(0.0946)	0.3392(0.1156)	0.3054(0.0934)
			1.5	1	2	0.1171(0.0138)	0.1489(0.0223)	0.1371(0.0189)	0.1483(0.0221)	0.1377(0.0191)
				2	3	0.2524(0.0637)	0.3395(0.1154)	0.2925(0.0856)	0.3192(0.1022)	0.2933(0.0861)
		23	0.75	1	2	0.1213(0.0148)	0.1562(0.0245)	0.1411(0.0200)	0.1560(0.0244)	0.1417(0.0202)
				2	3	0.2760(0.0762)	0.3499(0.1233)	0.3140(0.0987)	0.3377(0.1159)	0.3114(0.0970)
			1.5	1	2	0.1177(0.0139)	0.1488(0.0222)	0.1376(0.0190)	0.1484(0.0221)	0.1382(0.0192)
				2	3	0.2591(0.0672)	0.3236(0.1048)	0.2987(0.0893)	0.3177(0.1010)	0.2989(0.0894)
	45	23	0.75	1	2	0.1217(0.0149)	0.1568(0.0247)	0.1415(0.0201)	0.1563(0.0245)	0.1421(0.0203)
				2	3	0.2754(0.0759)	0.3504(0.1239)	0.3135(0.0983)	0.3359(0.1129)	0.3110(0.0968)
			1.5	1	2	0.1174(0.0138)	0.1484(0.0221)	0.1373(0.0189)	0.1480(0.0220)	0.1379(0.0191)
				2	3	0.2594(0.0674)	0.3239(0.1050)	0.2990(0.0894)	0.3180(0.1011)	0.2992(0.0895)
		34	0.75	1	2	0.1253(0.0157)	0.1584(0.0252)	0.1451(0.0211)	0.1568(0.0246)	0.1453(0.0211)
				2	3	0.2719(0.0739)	0.3703(0.1991)	0.3107(0.0966)	0.3352(0.1141)	0.3095(0.0958)
			1.5	1	2	0.1198(0.0144)	0.1475(0.0218)	0.1398(0.0196)	0.1475(0.0218)	0.1402(0.0197)
				2	3	0.2652(0.0704)	0.3183(0.1014)	0.3043(0.0927)	0.3156(0.0996)	0.3039(0.0924)

Table 5: The ABS and the RMSEs in parentheses for the parameter α using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1208(0.0148)	0.1930(0.0380)	0.1406(0.0200)	0.1862(0.0354)	0.1417(0.0203)
				2	3	0.2655(0.0711)	0.4584(0.2149)	0.3042(0.0931)	0.3822(0.1505)	0.3023(0.0916)
			1.5	1	2	0.1185(0.0143)	0.1794(0.0327)	0.1384(0.0194)	0.1718(0.0300)	0.1393(0.0196)
				2	3	0.2525(0.0639)	0.4232(0.1822)	0.2924(0.0857)	0.3498(0.1287)	0.2937(0.0864)
		8	0.75	1	2	0.1203(0.0148)	0.1864(0.0361)	0.1400(0.0199)	0.1787(0.0324)	0.1411(0.0201)
				2	3	0.2731(0.0748)	0.4274(0.2270)	0.3114(0.0971)	0.3717(0.1470)	0.3082(0.0950)
			1.5	1	2	0.1170(0.0139)	0.1685(0.0286)	0.1370(0.0189)	0.1648(0.0273)	0.1380(0.0192)
				2	3	0.2601(0.0678)	0.3704(0.1374)	0.2995(0.0899)	0.3460(0.1251)	0.2995(0.0898)
	15	8	0.75	1	2	0.1210(0.0150)	0.1947(0.0648)	0.1408(0.0201)	0.1802(0.0330)	0.1419(0.0203)
				2	3	0.2728(0.0746)	0.4412(0.2691)	0.3111(0.0970)	0.3685(0.1381)	0.3080(0.0949)
			1.5	1	2	0.1170(0.0139)	0.1687(0.0287)	0.1370(0.0190)	0.1648(0.0274)	0.1381(0.0192)
				2	3	0.2607(0.0682)	0.3706(0.1376)	0.3001(0.0903)	0.3444(0.1200)	0.2999(0.0901)
		11	0.75	1	2	0.1233(0.0153)	0.1879(0.0444)	0.1431(0.0206)	0.1772(0.0337)	0.1439(0.0208)
				2	3	0.2726(0.0743)	0.4620(0.5263)	0.3112(0.0969)	0.3715(0.1834)	0.3088(0.0954)
			1.5	1	2	0.1178(0.0140)	0.1626(0.0266)	0.1378(0.0191)	0.1612(0.0261)	0.1388(0.0194)
				2	3	0.2654(0.0706)	0.3492(0.1221)	0.3045(0.0928)	0.3362(0.1134)	0.3035(0.0922)
40	20	10	0.75	1	2	0.1204(0.0147)	0.1655(0.0276)	0.1403(0.0199)	0.1631(0.0268)	0.1412(0.0201)
				2	3	0.2721(0.0743)	0.3919(0.4049)	0.3104(0.0965)	0.3475(0.1211)	0.3076(0.0947)
			1.5	1	2	0.1169(0.0138)	0.1543(0.0239)	0.1369(0.0188)	0.1532(0.0236)	0.1376(0.0190)
				2	3	0.2551(0.0651)	0.3464(0.1202)	0.2950(0.0871)	0.3258(0.1065)	0.2957(0.0875)
		15	0.75	1	2	0.1208(0.0148)	0.1640(0.0271)	0.1406(0.0199)	0.1622(0.0265)	0.1415(0.0202)
				2	3	0.2741(0.0752)	0.3777(0.1841)	0.3123(0.0976)	0.3454(0.1203)	0.3096(0.0959)
			1.5	1	2	0.1174(0.0139)	0.1537(0.0237)	0.1374(0.0190)	0.1527(0.0234)	0.1381(0.0192)
				2	3	0.2592(0.0673)	0.3356(0.1127)	0.2988(0.0894)	0.3249(0.1056)	0.2990(0.0895)
		15	0.75	1	2	0.1207(0.0148)	0.1641(0.0272)	0.1406(0.0199)	0.1625(0.0270)	0.1415(0.0202)
				2	3	0.2744(0.0754)	0.3692(0.1405)	0.3126(0.0978)	0.3439(0.1186)	0.3098(0.0960)
			1.5	1	2	0.1177(0.0140)	0.1542(0.0239)	0.1376(0.0191)	0.1531(0.0235)	0.1384(0.0192)
				2	3	0.2592(0.0672)	0.3354(0.1126)	0.2987(0.0893)	0.3246(0.1054)	0.2990(0.0894)

	30	23	0.75	1	2	0.1253(0.0157)	0.1657(0.0282)	0.1450(0.0211)	0.1631(0.0273)	0.1453(0.0212)
				2	3	0.2715(0.0737)	0.3831(0.2100)	0.3104(0.0963)	0.3395(0.1174)	0.3090(0.0955)
			1.5	1	2	0.1197(0.0144)	0.1507(0.0228)	0.1396(0.0195)	0.1506(0.0227)	0.1402(0.0197)
				2	3	0.2657(0.0706)	0.3236(0.1047)	0.3048(0.0929)	0.3196(0.1022)	0.3042(0.0926)
60	30	15	0.75	1	2	0.1198(0.0145)	0.1561(0.0245)	0.1397(0.0196)	0.1551(0.0242)	0.1404(0.0198)
				2	3	0.2740(0.0752)	0.3527(0.1257)	0.3120(0.0975)	0.3371(0.1138)	0.3094(0.0958)
			1.5	1	2	0.1176(0.0139)	0.1491(0.0223)	0.1375(0.0190)	0.1485(0.0221)	0.1381(0.0191)
				2	3	0.2557(0.0654)	0.3290(0.1083)	0.2956(0.0874)	0.3189(0.1017)	0.2962(0.0878)
		23	0.75	1	2	0.1213(0.0148)	0.1562(0.0245)	0.1411(0.0200)	0.1557(0.0243)	0.1418(0.0202)
				2	3	0.2752(0.0758)	0.3491(0.1222)	0.3133(0.0982)	0.3353(0.1125)	0.3108(0.0966)
			1.5	1	2	0.1175(0.0139)	0.1486(0.0222)	0.1374(0.0190)	0.1482(0.0220)	0.1380(0.0191)
				2	3	0.2594(0.0673)	0.3239(0.1050)	0.2989(0.0894)	0.3179(0.1011)	0.2992(0.0895)
	45	23	0.75	1	2	0.1208(0.0147)	0.1557(0.0244)	0.1407(0.0199)	0.1554(0.0243)	0.1413(0.0201)
				2	3	0.2752(0.0758)	0.3527(0.1311)	0.3133(0.0982)	0.3357(0.1128)	0.3108(0.0967)
			1.5	1	2	0.1176(0.0139)	0.1485(0.0221)	0.1375(0.0190)	0.1481(0.0220)	0.1380(0.0191)
				2	3	0.2594(0.0674)	0.3240(0.1050)	0.2990(0.0894)	0.3180(0.1012)	0.2992(0.0896)
		34	0.75	1	2	0.1254(0.0158)	0.1592(0.0261)	0.1451(0.0211)	0.1565(0.0246)	0.1454(0.0212)
				2	3	0.2718(0.0739)	0.3619(0.1757)	0.3107(0.0965)	0.3340(0.1142)	0.3095(0.0958)
			1.5	1	2	0.1196(0.0144)	0.1473(0.0218)	0.1395(0.0195)	0.1474(0.0218)	0.1400(0.0196)
				2	3	0.2655(0.0705)	0.3185(0.1015)	0.3046(0.0928)	0.3158(0.0997)	0.3041(0.0925)

Table 6:The ABS and the RMSEs in parentheses for the parameter β using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1036(0.0107)	0.1710(0.1574)	0.1183(0.0140)	0.2302(0.4802)	0.1183(0.0140)
				2	3	0.1102(0.0121)	0.1375(0.0423)	0.1244(0.0155)	0.2074(0.5048)	0.1245(0.0155)
			1.5	1	2	0.2209(0.0488)	0.3186(0.8009)	0.2497(0.0624)	0.3734(0.4815)	0.2451(0.0601)
				2	3	0.2327(0.0542)	0.3006(0.1307)	0.2606(0.0679)	0.3840(0.5830)	0.2549(0.0650)
		8	0.75	1	2	0.1014(0.0103)	0.1482(0.0585)	0.1162(0.0135)	0.1962(0.1037)	0.1163(0.0135)
				2	3	0.1075(0.0116)	0.1264(0.0362)	0.1218(0.0148)	0.1410(0.0390)	0.1220(0.0149)
			1.5	1	2	0.2117(0.0448)	0.3232(0.6725)	0.2407(0.0580)	0.3762(0.5313)	0.2386(0.0570)
				2	3	0.2247(0.0505)	0.2478(0.0635)	0.2529(0.0640)	0.2921(0.2588)	0.2500(0.0625)
	15	8	0.75	1	2	0.1014(0.0103)	0.1587(0.1120)	0.1162(0.0135)	0.1878(0.0644)	0.1163(0.0135)
				2	3	0.1075(0.0116)	0.1188(0.0172)	0.1218(0.0148)	0.1378(0.0586)	0.1220(0.0149)
			1.5	1	2	0.2118(0.0448)	0.3666(0.9862)	0.2408(0.0580)	0.3889(0.5462)	0.2387(0.0570)
				2	3	0.2247(0.0505)	0.2489(0.0663)	0.2529(0.0640)	0.2827(0.1505)	0.2500(0.0625)
		11	0.75	1	2	0.1003(0.0101)	0.1535(0.1299)	0.1151(0.0132)	0.1751(0.0808)	0.1152(0.0133)
				2	3	0.1058(0.0112)	0.1209(0.0217)	0.1201(0.0144)	0.1368(0.0894)	0.1204(0.0145)
			1.5	1	2	0.2071(0.0429)	0.2562(0.2884)	0.2363(0.0558)	0.3546(0.6870)	0.2352(0.0553)
				2	3	0.2198(0.0483)	0.2396(0.0578)	0.2481(0.0616)	0.2571(0.1329)	0.2464(0.0607)
40	20	10	0.75	1	2	0.1036(0.0107)	0.1215(0.0308)	0.1182(0.0140)	0.1658(0.1098)	0.1184(0.0140)
				2	3	0.1107(0.0123)	0.1219(0.0156)	0.1248(0.0156)	0.1442(0.1972)	0.1252(0.0157)
			1.5	1	2	0.2204(0.0486)	0.2631(0.2578)	0.2493(0.0621)	0.3508(0.9285)	0.2467(0.0609)
				2	3	0.2338(0.0547)	0.2706(0.0752)	0.2617(0.0685)	0.2531(0.0866)	0.2585(0.0668)
		15	0.75	1	2	0.1016(0.0103)	0.1112(0.0175)	0.1163(0.0135)	0.1414(0.0425)	0.1165(0.0136)
				2	3	0.1079(0.0116)	0.1197(0.0145)	0.1221(0.0149)	0.1138(0.0152)	0.1225(0.0150)
			1.5	1	2	0.2123(0.0451)	0.2661(0.8963)	0.2413(0.0582)	0.2886(0.3608)	0.2402(0.0577)
				2	3	0.2258(0.0510)	0.2523(0.0637)	0.2539(0.0645)	0.2451(0.1074)	0.2524(0.0637)
	30	15	0.75	1	2	0.1015(0.0103)	0.1166(0.0256)	0.1163(0.0135)	0.1631(0.2282)	0.1165(0.0136)
				2	3	0.1079(0.0116)	0.1205(0.0146)	0.1222(0.0149)	0.1126(0.0136)	0.1225(0.0150)
			1.5	1	2	0.2123(0.0451)	0.2390(0.0888)	0.2413(0.0582)	0.2920(0.3762)	0.2402(0.0577)
				2	3	0.2258(0.0510)	0.2525(0.0638)	0.2539(0.0645)	0.2366(0.0597)	0.2523(0.0637)
		23	0.75	1	2	0.1001(0.0100)	0.1062(0.0137)	0.1149(0.0132)	0.1369(0.0817)	0.1151(0.0132)
				2	3	0.1055(0.0111)	0.1207(0.0154)	0.1199(0.0144)	0.1132(0.0136)	0.1202(0.0144)
			1.5	1	2	0.2063(0.0426)	0.2205(0.0524)	0.2356(0.0555)	0.2539(0.1454)	0.2352(0.0553)

				2	3	0.2190(0.0480)	0.2444(0.0598)	0.2474(0.0612)	0.3057(0.5463)	0.2468(0.0609)
60	30	15	0.75	1	2	0.1036(0.0107)	0.1211(0.0437)	0.1183(0.0140)	0.1690(0.2261)	0.1185(0.0141)
				2	3	0.1109(0.0123)	0.1236(0.0161)	0.1250(0.0156)	0.1184(0.0157)	0.1255(0.0157)
			1.5	1	2	0.2205(0.0486)	0.2428(0.0756)	0.2494(0.0622)	0.2947(0.4493)	0.2477(0.0614)
				2	3	0.2343(0.0549)	0.2684(0.0721)	0.2621(0.0687)	0.2570(0.1147)	0.2600(0.0676)
		23	0.75	1	2	0.1015(0.0103)	0.1080(0.0135)	0.1162(0.0135)	0.1222(0.0635)	0.1164(0.0136)
				2	3	0.1078(0.0116)	0.1210(0.0147)	0.1221(0.0149)	0.1167(0.0138)	0.1225(0.0150)
			1.5	1	2	0.2120(0.0449)	0.2275(0.0530)	0.2409(0.0581)	0.2335(0.0832)	0.2404(0.0578)
				2	3	0.2254(0.0508)	0.2535(0.0643)	0.2536(0.0643)	0.2442(0.0597)	0.2528(0.0639)
	45	23	0.75	1	2	0.1015(0.0103)	0.1079(0.0127)	0.1162(0.0135)	0.1190(0.0336)	0.1164(0.0136)
				2	3	0.1078(0.0116)	0.1211(0.0147)	0.1221(0.0149)	0.1159(0.0135)	0.1225(0.0150)
			1.5	1	2	0.2120(0.0449)	0.2283(0.0552)	0.2410(0.0581)	0.2533(0.2004)	0.2404(0.0578)
				2	3	0.2254(0.0508)	0.2535(0.0643)	0.2536(0.0643)	0.2440(0.0596)	0.2528(0.0639)
		34	0.75	1	2	0.1001(0.0100)	0.1069(0.0120)	0.1150(0.0132)	0.1226(0.0462)	0.1151(0.0133)
				2	3	0.1056(0.0111)	0.1224(0.0157)	0.1199(0.0144)	0.1169(0.0137)	0.1203(0.0145)
			1.5	1	2	0.2065(0.0426)	0.2255(0.0519)	0.2357(0.0556)	0.2327(0.1588)	0.2356(0.0555)
				2	3	0.2192(0.0481)	0.2465(0.0608)	0.2476(0.0613)	0.2413(0.0582)	0.2474(0.0612)

Table 7:The AVB and the RMSEs in parentheses for the parameter β using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1028(0.0106)	0.1727(0.1486)	0.1174(0.0138)	0.2064(0.1244)	0.1175(0.0138)
				2	3	0.1102(0.0122)	0.1332(0.0290)	0.1244(0.0155)	0.1937(0.5361)	0.1245(0.0155)
			1.5	1	2	0.2210(0.0489)	0.4402(0.5636)	0.2498(0.0624)	0.4638(0.4526)	0.2452(0.0601)
				2	3	0.2326(0.0541)	0.4949(0.4637)	0.2605(0.0679)	0.3492(0.2149)	0.2548(0.0649)
		8	0.75	1	2	0.1014(0.0103)	0.1576(0.0750)	0.1162(0.0135)	0.2465(0.4536)	0.1163(0.0135)
				2	3	0.1075(0.0116)	0.1174(0.0157)	0.1218(0.0148)	0.1648(0.3425)	0.1220(0.0149)
			1.5	1	2	0.2117(0.0448)	0.2791(0.2017)	0.2407(0.0579)	0.3770(0.5440)	0.2386(0.0569)
				2	3	0.2248(0.0505)	0.2501(0.0682)	0.2529(0.0640)	0.2917(0.2871)	0.2500(0.0625)
	15	8	0.75	1	2	0.1014(0.0103)	0.1684(0.1643)	0.1161(0.0135)	0.2408(0.9503)	0.1162(0.0135)
				2	3	0.1075(0.0116)	0.1199(0.0197)	0.1218(0.0148)	0.1482(0.0553)	0.1220(0.0149)
			1.5	1	2	0.2117(0.0448)	0.2991(0.3629)	0.2407(0.0579)	0.5168(0.4536*)	0.2386(0.0569)
				2	3	0.2248(0.0505)	0.2482(0.0685)	0.2529(0.0640)	0.2794(0.2004)	0.2501(0.0625)
		11	0.75	1	2	0.1003(0.0101)	0.1378(0.0700)	0.1151(0.0132)	0.1817(0.0720)	0.1152(0.0133)
				2	3	0.1058(0.0112)	0.1248(0.0311)	0.1202(0.0144)	0.1242(0.0380)	0.1204(0.0145)
			1.5	1	2	0.2071(0.0429)	0.2608(0.2010)	0.2363(0.0558)	0.3675(0.8276)	0.2352(0.0553)
				2	3	0.2198(0.0483)	0.2389(0.0573)	0.2482(0.0616)	0.2540(0.1478)	0.2465(0.0607)
40	20	10	0.75	1	2	0.1026(0.0105)	0.1278(0.1068)	0.1173(0.0138)	0.1581(0.0921)	0.1175(0.0138)
				2	3	0.1094(0.0120)	0.1196(0.0144)	0.1236(0.0153)	0.1325(0.0908)	0.1240(0.0154)
			1.5	1	2	0.2166(0.0469)	0.3411(0.6758)	0.2455(0.0603)	0.3195(0.3545)	0.2437(0.0594)
				2	3	0.2302(0.0530)	0.2609(0.0687)	0.2582(0.0666)	0.2465(0.0954)	0.2558(0.0654)
		15	0.75	1	2	0.1015(0.0103)	0.1103(0.0175)	0.1163(0.0135)	0.1626(0.1190)	0.1165(0.0136)
				2	3	0.1079(0.0116)	0.1200(0.0145)	0.1222(0.0149)	0.1200(0.0327)	0.1225(0.0150)
			1.5	1	2	0.2124(0.0451)	0.2288(0.0697)	0.2413(0.0582)	0.2918(0.2899)	0.2402(0.0577)
				2	3	0.2258(0.0510)	0.2524(0.0637)	0.2539(0.0645)	0.2375(0.0588)	0.2524(0.0637)
	30	15	0.75	1	2	0.1015(0.0103)	0.1142(0.0229)	0.1163(0.0135)	0.1526(0.1168)	0.1165(0.0136)
				2	3	0.1079(0.0116)	0.1204(0.0148)	0.1222(0.0149)	0.1132(0.0154)	0.1225(0.0150)
			1.5	1	2	0.2123(0.0451)	0.2277(0.0648)	0.2413(0.0582)	0.2824(0.2214)	0.2402(0.0577)
				2	3	0.2258(0.0510)	0.2523(0.0637)	0.2539(0.0645)	0.2359(0.0632)	0.2524(0.0637)
		23	0.75	1	2	0.1001(0.0100)	0.1212(0.0755)	0.1149(0.0132)	0.1395(0.0670)	0.1151(0.0132)
				2	3	0.1055(0.0111)	0.1279(0.0405)	0.1199(0.0144)	0.1133(0.0133)	0.1202(0.0145)
			1.5	1	2	0.2064(0.0426)	0.2191(0.0532)	0.2356(0.0555)	0.2555(0.1341)	0.2353(0.0553)
				2	3	0.2190(0.0480)	0.2445(0.0598)	0.2474(0.0612)	0.2343(0.0551)	0.2468(0.0609)
			0.75	1	2	0.1024(0.0105)	0.1482(0.9024)	0.1170(0.0137)	0.1279(0.0305)	0.1173(0.0138)

60	30	15		2	3	0.1091(0.0119)	0.1216(0.0148)	0.1233(0.0152)	0.1164(0.0140)	0.1238(0.0153)
			1.5	1	2	0.2155(0.0465)	0.2817(0.7857)	0.2444(0.0597)	0.2669(0.1628)	0.2434(0.0592)
				2	3	0.2293(0.0526)	0.2590(0.0671)	0.2573(0.0662)	0.2480(0.0709)	0.2560(0.0655)
		23	0.75	1	2	0.1014(0.0103)	0.1114(0.0222)	0.1162(0.0135)	0.1170(0.0216)	0.1164(0.0135)
				2	3	0.1078(0.0116)	0.1209(0.0146)	0.1221(0.0149)	0.1169(0.0139)	0.1225(0.0150)
			1.5	1	2	0.2120(0.0449)	0.2304(0.0619)	0.2410(0.0581)	0.2657(0.2934)	0.2404(0.0578)
				2	3	0.2254(0.0508)	0.2535(0.0643)	0.2536(0.0643)	0.2452(0.0616)	0.2528(0.0639)
	45	23	0.75	1	2	0.1015(0.0103)	0.1116(0.0163)	0.1162(0.0135)	0.1165(0.0262)	0.1164(0.0136)
				2	3	0.1078(0.0116)	0.1211(0.0148)	0.1221(0.0149)	0.1164(0.0137)	0.1225(0.0150)
			1.5	1	2	0.2120(0.0449)	0.2290(0.0559)	0.2410(0.0581)	0.2365(0.0800)	0.2404(0.0578)
				2	3	0.2254(0.0508)	0.2535(0.0643)	0.2536(0.0643)	0.2440(0.0597)	0.2528(0.0639)
		34	0.75	1	2	0.1002(0.0100)	0.1082(0.0129)	0.1150(0.0132)	0.1147(0.0231)	0.1151(0.0133)
				2	3	0.1056(0.0111)	0.1223(0.0159)	0.1199(0.0144)	0.1168(0.0137)	0.1203(0.0145)
			1.5	1	2	0.2065(0.0426)	0.2251(0.0518)	0.2357(0.0556)	0.2286(0.0729)	0.2356(0.0555)
				2	3	0.2193(0.0481)	0.2465(0.0608)	0.2476(0.0613)	0.2415(0.0583)	0.2474(0.0612)

Table 8:The AVB and the RMSEs in parentheses for the parameter γ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.2824(0.0798)	0.5550(0.5645)	0.3209(0.1030)	0.4867(0.8745)	0.3107(0.0965)
				2	3	0.4661(0.2176)	0.5644(0.7222)	0.5198(0.2705)	0.5619(0.5388)	0.4725(0.2233)
			1.5	1	2	0.3086(0.0953)	0.4768(0.6758)	0.3459(0.1197)	0.4626(0.4995)	0.3238(0.1049)
				2	3	0.5073(0.2583)	0.6507(0.5463)	0.5595(0.3138)	0.5714(0.5823)	0.4860(0.2363)
		8	0.75	1	2	0.2741(0.0751)	0.4514(0.6252)	0.3131(0.0980)	0.5200(0.5637)	0.3076(0.0946)
				2	3	0.4526(0.2050)	0.5162(0.6769)	0.5068(0.2569)	0.5526(0.8631)	0.4719(0.2227)
			1.5	1	2	0.2909(0.0846)	0.4693(0.5463)	0.3289(0.1082)	0.5489(0,4573)	0.3169(0.1004)
				2	3	0.4871(0.2375)	0.5182(0.3809)	0.5396(0.2913)	0.5094(0.5327)	0.4850(0.2352)
	15	8	0.75	1	2	0.2741(0.0751)	0.4417(0.7275)	0.3130(0.0980)	0.4870(0.7255)	0.3075(0.0946)
				2	3	0.4523(0.2047)	0.4750(0.2771)	0.5065(0.2567)	0.5415(0.4365)	0.4718(0.2226)
			1.5	1	2	0.2909(0.0846)	0.4070(0.5955)	0.3289(0.1082)	0.5092(0.8796)	0.3169(0.1004)
				2	3	0.4884(0.2388)	0.5083(0.2750)	0.5407(0.2926)	0.5155(0.4949)	0.4854(0.2357)
		11	0.75	1	2	0.2704(0.0731)	0.4336(0.6547)	0.3096(0.0958)	0.4768(0.8063)	0.3060(0.0936)
				2	3	0.4425(0.1958)	0.4908(0.3235)	0.4973(0.2473)	0.5484(0.4536)	0.4703(0.2212)
			1.5	1	2	0.2831(0.0801)	0.3538(0.7174)	0.3214(0.1033)	0.5294(0.5735)	0.3134(0.0982)
				2	3	0.4715(0.2224)	0.4955(0.2811)	0.5247(0.2754)	0.5220(0.9904)	0.4824(0.2328)
40	20	10	0.75	1	2	0.2826(0.0799)	0.3493(0.4581)	0.3211(0.1031)	0.4019(0.3677)	0.3129(0.0979)
				2	3	0.4692(0.2203)	0.5104(0.3942)	0.5227(0.2733)	0.4663(0.2678)	0.4805(0.2309)
			1.5	1	2	0.3089(0.0954)	0.3656(0.4113)	0.3462(0.1199)	0.4487(0.8546)	0.3287(0.1080)
				2	3	0.5127(0.2633)	0.5708(0.3817)	0.5644(0.3188)	0.5198(0.5977)	0.4989(0.2489)
		15	0.75	1	2	0.2750(0.0756)	0.3060(0.1916)	0.3139(0.0985)	0.3680(0.2710)	0.3093(0.0956)
				2	3	0.4551(0.2071)	0.4907(0.2443)	0.5091(0.2592)	0.4530(0.2243)	0.4782(0.2287)
			1.5	1	2	0.2930(0.0859)	0.3378(0.5120)	0.3308(0.1094)	0.3681(0.3043)	0.3207(0.1028)
				2	3	0.4920(0.2422)	0.5303(0.2817)	0.5441(0.2962)	0.4711(0.2353)	0.4956(0.2456)
	30	15	0.75	1	2	0.2750(0.0756)	0.3228(0.3331)	0.3139(0.0985)	0.3990(0.6041)	0.3093(0.0956)
				2	3	0.4558(0.2079)	0.4953(0.2482)	0.5098(0.2599)	0.4552(0.2270)	0.4786(0.2290)
			1.5	1	2	0.2930(0.0859)	0.3070(0.1274)	0.3308(0.1094)	0.3652(0.4614)	0.3207(0.1028)
				2	3	0.4921(0.2423)	0.5312(0.2828)	0.5442(0.2963)	0.4817(0.3355)	0.4956(0.2457)
		23	0.75	1	2	0.2700(0.0729)	0.3186(0.5017)	0.3092(0.0956)	0.3766(0.7168)	0.3066(0.0940)
				2	3	0.4405(0.1940)	0.4939(0.2552)	0.4954(0.2454)	0.4886(0.6338)	0.4745(0.2252)
			1.5	1	2	0.2823(0.0797)	0.2958(0.1314)	0.3206(0.1028)	0.3666(0.8393)	0.3149(0.0992)
				2	3	0.4692(0.2202)	0.5095(0.2597)	0.5225(0.2730)	0.4712(0.2429)	0.4895(0.2396)
		15	0.75	1	2	0.2828(0.0800)	0.3427(0.8574)	0.3213(0.1032)	0.4684(0.6574)	0.3143(0.0988)
				2	3	0.4699(0.2209)	0.5075(0.2651)	0.5233(0.2739)	0.4618(0.2257)	0.4852(0.2354)
			1.5	1	2	0.3090(0.0955)	0.3402(0.2487)	0.3464(0.1200)	0.3962(0.5479)	0.3314(0.1098)
				2	3	0.5132(0.2637)	0.5668(0.3222)	0.5649(0.3193)	0.5019(0.2685)	0.5061(0.2562)

60	30	23	0.75	1	2	0.2747(0.0755)	0.2883(0.1517)	0.3136(0.0983)	0.3164(0.2351)	0.3098(0.0960)
				2	3	0.4552(0.2072)	0.4999(0.2503)	0.5091(0.2592)	0.4661(0.2207)	0.4821(0.2324)
			1.5	1	2	0.2923(0.0855)	0.3222(0.6439)	0.3301(0.1090)	0.3280(0.4089)	0.3219(0.1036)
				2	3	0.4913(0.2415)	0.5368(0.2883)	0.5435(0.2954)	0.4876(0.2379)	0.5013(0.2513)
	45	23	0.75	1	2	0.2747(0.0755)	0.3733(0.4573)	0.3136(0.0983)	0.3186(0.2883)	0.3098(0.0960)
				2	3	0.4550(0.2070)	0.4999(0.2504)	0.5089(0.2590)	0.4629(0.2152)	0.4820(0.2323)
			1.5	1	2	0.2923(0.0855)	0.3042(0.1029)	0.3301(0.1090)	0.3389(0.4265)	0.3219(0.1036)
				2	3	0.4915(0.2417)	0.5370(0.2885)	0.5436(0.2956)	0.4875(0.2378)	0.5014(0.2514)
		34	0.75	1	2	0.2702(0.0730)	0.2798(0.0875)	0.3093(0.0957)	0.3311(0.3866)	0.3071(0.0943)
				2	3	0.4411(0.1946)	0.5143(0.3862)	0.4960(0.2460)	0.4758(0.2293)	0.4775(0.2280)
			1.5	1	2	0.2826(0.0799)	0.2969(0.0903)	0.3209(0.1030)	0.2983(0.1295)	0.3161(0.0999)
				2	3	0.4699(0.2209)	0.5163(0.2666)	0.5232(0.2738)	0.4822(0.2326)	0.4941(0.2441)

Table 9:The AVB and the RMSEs in parentheses for the Parameter γ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.75 and $\delta = 2$ for LINEX loss function.

n	m	k	β	α	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.2788(0.0777)	0.5218(0.76843)	0.3174(0.1008)	0.5582(0,7684)	0.3094(0.0957)
				2	3	0.4655(0.2170)	0.5556(0.6571)	0.5193(0.2700)	0.5544(0.6748)	0.4724(0.2232)
			1.5	1	2	0.3086(0.0953)	0.4991(0.8817)	0.3459(0.1197)	0.5363(0.8289)	0.3238(0.1049)
				2	3	0.5064(0.2574)	0.6699(0.6836)	0.5587(0.3129)	0.6301(0.8494)	0.4857(0.2360)
		8	0.75	1	2	0.2740(0.0751)	0.4846(0.9658)	0.3130(0.0980)	0.5254(0.8794)	0.3075(0.0946)
				2	3	0.4527(0.2051)	0.4911(0.6579)	0.5069(0.2570)	0.4898(0.3579)	0.4719(0.2227)
			1.5	1	2	0.2910(0.0847)	0.5967(0.7694)	0.3290(0.1082)	0.5203(0.6574)	0.3169(0.1004)
				2	3	0.4875(0.2379)	0.5315(0.5640)	0.5400(0.2918)	0.5199(0.8545)	0.4851(0.2354)
	15	8	0.75	1	2	0.2740(0.0751)	0.5075(0.6743)	0.3130(0.0979)	0.5571(0.8795)	0.3075(0.0946)
				2	3	0.4526(0.2049)	0.4898(0.5461)	0.5068(0.2569)	0.5322(0.6645)	0.4719(0.2227)
			1.5	1	2	0.2909(0.0847)	0.3893(0.4545)	0.3289(0.1082)	0.4741(0.5207)	0.3169(0.1004)
				2	3	0.4882(0.2386)	0.5073(0.2715)	0.5406(0.2924)	0.5437(0.8796)	0.4854(0.2356)
		11	0.75	1	2	0.2704(0.0731)	0.3800(0.5436)	0.3096(0.0958)	0.5287(0.6587)	0.3060(0.0936)
				2	3	0.4428(0.1962)	0.5430(0.8794)	0.4976(0.2476)	0.4934(0.3722)	0.4704(0.2213)
			1.5	1	2	0.2831(0.0801)	0.3668(0.3687)	0.3214(0.1033)	0.5226(0.6754)	0.3134(0.0982)
				2	3	0.4723(0.2232)	0.4895(0.2415)	0.5254(0.2761)	0.5068(0.9188)	0.4827(0.2331)
40	20	10	0.75	1	2	0.2788(0.0777)	0.3463(0.7865)	0.3174(0.1008)	0.4013(0.4696)	0.3111(0.0968)
				2	3	0.4630(0.2144)	0.4889(0.2419)	0.5166(0.2670)	0.4561(0.2397)	0.4797(0.2301)
			1.5	1	2	0.3009(0.0906)	0.3246(0.1992)	0.3385(0.1146)	0.4370(0.8975)	0.3248(0.1055)
				2	3	0.5034(0.2537)	0.5461(0.3006)	0.5552(0.3085)	0.4848(0.2655)	0.4976(0.2477)
		15	0.75	1	2	0.2750(0.0756)	0.2934(0.1408)	0.3139(0.0985)	0.4257(0.8762)	0.3093(0.0956)
				2	3	0.4554(0.2075)	0.4928(0.2461)	0.5094(0.2595)	0.4712(0.4155)	0.4784(0.2289)
			1.5	1	2	0.2930(0.0858)	0.3002(0.1438)	0.3308(0.1094)	0.4249(0.7865)	0.3207(0.1028)
				2	3	0.4922(0.2424)	0.5309(0.2822)	0.5444(0.2964)	0.4719(0.2286)	0.4957(0.2457)
	30	15	0.75	1	2	0.2749(0.0756)	0.2942(0.1423)	0.3138(0.0985)	0.3920(0.4614)	0.3092(0.0956)
				2	3	0.4554(0.2074)	0.4924(0.2473)	0.5093(0.2595)	0.4520(0.2237)	0.4784(0.2288)
			1.5	1	2	0.2930(0.0859)	0.2991(0.1132)	0.3308(0.1094)	0.3908(0.5456)	0.3207(0.1028)
				2	3	0.4921(0.2423)	0.5306(0.2819)	0.5442(0.2963)	0.4747(0.2616)	0.4956(0.2457)
		23	0.75	1	2	0.2700(0.0729)	0.2950(0.1664)	0.3092(0.0956)	0.3848(0.5643)	0.3066(0.0940)
				2	3	0.4405(0.1941)	0.5057(0.3111)	0.4955(0.2455)	0.4669(0.2254)	0.4745(0.2252)
			1.5	1	2	0.2823(0.0797)	0.2952(0.1368)	0.3206(0.1028)	0.4208(0.7869)	0.3149(0.0992)
				2	3	0.4695(0.2205)	0.5101(0.2603)	0.5228(0.2734)	0.4651(0.2171)	0.4896(0.2398)
	30	15	0.75	1	2	0.2778(0.0772)	0.3321(0.8386)	0.3164(0.1001)	0.3373(0.2073)	0.3115(0.0970)
				2	3	0.4616(0.2132)	0.5017(0.2526)	0.5153(0.2656)	0.4617(0.2172)	0.4836(0.2339)
			1.5	1	2	0.2986(0.0892)	0.3138(0.2073)	0.3362(0.1131)	0.3841(0.7015)	0.3255(0.1060)
				2	3	0.5017(0.2518)	0.5492(0.3018)	0.5535(0.3065)	0.4901(0.2414)	0.5040(0.2540)
		23	0.75	1	2	0.2746(0.0754)	0.2841(0.1001)	0.3135(0.0983)	0.3104(0.1498)	0.3097(0.0959)
				2	3	0.4547(0.2068)	0.4990(0.2492)	0.5087(0.2588)	0.4641(0.2161)	0.4818(0.2322)
			1.5	1	2	0.2923(0.0855)	0.3046(0.1223)	0.3301(0.1090)	0.3230(0.1807)	0.3219(0.1036)

60				2	3	0.4913(0.2414)	0.5366(0.2881)	0.5434(0.2954)	0.4879(0.2386)	0.5012(0.2513)
	45	23	0.75	1	2	0.2747(0.0754)	0.2951(0.1402)	0.3135(0.0983)	0.3085(0.1741)	0.3098(0.0960)
				2	3	0.4548(0.2069)	0.4998(0.2520)	0.5088(0.2589)	0.4644(0.2178)	0.4819(0.2322)
			1.5	1	2	0.2923(0.0855)	0.3093(0.1266)	0.3301(0.1090)	0.3240(0.2106)	0.3219(0.1036)
				2	3	0.4913(0.2415)	0.5367(0.2882)	0.5435(0.2955)	0.4868(0.2375)	0.5013(0.2513)
		34	0.75	1	2	0.2701(0.0730)	0.2900(0.1324)	0.3093(0.0957)	0.3130(0.2472)	0.3071(0.0943)
				2	3	0.4410(0.1945)	0.5068(0.3161)	0.4959(0.2459)	0.4752(0.2299)	0.4774(0.2279)
			1.5	1	2	0.2826(0.0799)	0.2977(0.0916)	0.3209(0.1030)	0.3309(0.8795)	0.3161(0.0999)
				2	3	0.4702(0.2211)	0.5166(0.2669)	0.5234(0.2740)	0.4828(0.2332)	0.4942(0.2443)