Estimation of the Burr-III Distribution Parameters Based on the Dual Generalized Progressive Hybrid Censoring Scheme

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Abstract: - In reliability theory, the Bayes' method based on the informative prior is the most applicable and more efficient than the other estimation methods in reliability analysis. Therefore, the main objective of this work is to introduce optimal numerical iteration techniques, such as the Picard and Runge- Kutta methods, which are more efficient and compatible with the Bayes' method. The proposed methods have been used for estimating the Burr-III distribution parameters and compared to Bayes' method via an extensive Monte Carlo simulation. The simulation results indicated that the Picard and Runge-Kutta methods provide better estimates and outperform the Bayes' method estimates based on the dual generalized progressive hybrid censoring scheme. Finally, two real datasets have been analyzed for illustrations and comparison of the proposed methods.

Key-Words: - Bayesian estimation; Characteristic prior; Informative prior; Kernel prior; Runge- Kutta method; Picard's method.

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1 Introduction

In statistical inference, the Bayesian method is the most applicable estimation method in reliability theory, despite its subjectivity to prior information other than data. Therefore, the main objective of this work is to introduce optimal numerical iteration techniques, such as the Picard and Runge-Kutta (R-K) methods, as an efficient and compatible method with the Bayes method for estimating the distribution parameters. These methods have been used for estimating the Burr-III distribution parameters, which have the cumulative distribution function (cdf) and the probability density function (pdf) as follows:

$$F(x) = (1 + x^{-\alpha})^{-\gamma}, \qquad x, \alpha, \gamma > 0, \qquad (1)$$

$$f(x) = \alpha\gamma x^{-\alpha-1}(1 + x^{-\alpha})^{-\gamma-1}, x, \alpha, \gamma > 0. (2)$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are shape parameters.

The Burr type-III distribution offers effective analysis capabilities for diverse fields such as materials engineering, forestry, and meteorology, see [23], and fracture roughness, see [22]. Moreover, the Burr type-III distribution can also be used as an alternate model to other distributions such as gamma, Weibull, and log-normal distributions for fitting a variety of lifetime data, see [12]. This applicability led several authors to estimate the parameters of this distribution using censored samples. For example, one can see [2-4] and [20-22]. However, its application under the generalized progressive hybrid censoring scheme remains relatively unexplored.

[6] defined the dual generalized order statistics (DGOS), sometimes called the lower generalized order statistics (LGOS), which is a combined mechanism for studying random variables arranged in descending order. This technique was introduced as the inverse image of the generalized order statistics (GOS) that were introduced by [11]. The DGOS has been used by many authors who considered different estimation procedures for lifetime distributions based on complete and progressive data, see [2-4] and [14].

Recently, [7-9] have suggested the generalized progressive hybrid censoring scheme (GPHCS), which has a great part of applications with different lifetime distributions, see [15-18]. Therefore, in this work, we introduce a new scheme called the dual generalized progressive hybrid censoring scheme (DGPHCS), which is analogous to the dual generalized order statistics. This scheme can be described as follows:

Consider a life-testing experiment in which n identical units $X_1, X_2, \ldots\ldots, X_n$ are placed on test. For $\mathrm{T} \in (0, \infty)$ and integers k and m are pre-fixed such that $k < m$ with $R_1, R_2, \ldots, R_m$ are the random removal units that are fixed at the beginning of the experiment such that $n = m + \sum_{i=1}^{m} R_i$. Generally, at the time of the i_th failure, R_i units are randomly removed from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i = 1(1)m$. This process continues until, immediately following the terminated time $T^* = \min\{X_{k:m:n}, \max\{X_{m:n:n}, T\}\}$, where at this time all the remaining surviving units are removed from the experiment according to the following cases. Let D denote the number of observed failures up to the time T. Thus, we have one of the following types of observations, see Fig. 1:

Case I: $X_{1:m:n} \geq \ldots \geq X_{k:m:n} > X_{k+1:m:n} > \ldots > X_{D:m:n}$

If $X_{m:m:n} \leq T < X_{k:m:n}$.

Case II: $X_{1:m:n} \geq \ldots \geq X_{k:m:n} > X_{k+1:m:n} > \ldots > X_{m:m:n}$
If $T < X_{m:m:n} < X_{k:m:n}$.

Case III: $X_{1:m:n} \geq \ldots \geq X_{D:m:n} > X_{D+1:m:n} > \ldots > X_{k:m:n}$
If $X_{m:m:n} < X_{k:m:n} < T$.

Note that for **Case I,** $X_{D:m:n} > T > X_{D+1:m:n}$ and $X_{D+1:m:n}, \ldots, X_{m:m:n}$ are not observed.

For **Case III,** $T > X_{k:m:n} > X_{m:m:n}$ and $X_{k+1:m:n}, \ldots, X_{m:m:n}$ are not observed.

Thus, given a dual generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\theta; x) = C \prod_{i=1}^{n} f(x_{i,m,n}) \, [F(x_{i,m,n})]^{R_i} [F(T)]^{\delta R_T^*}, \qquad (3)$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$\boldsymbol{n} = \begin{cases} D, & \delta = 1, \ \text{if} \ X_{m:m:n} \leq T < X_{k:m:n} \\ m, & \delta = 0, \ \text{if} \ T < X_{m:m:n} \leq X_{k:m:n} \\ k, & \delta = 0, \ \text{if} \ X_{m:m:n} < X_{k:m:n} < T \end{cases}$$

and

$$\underline{X} = \begin{cases} (X_{1:m:n}, \ldots, X_{D:m:n}), & \text{if} \ X_{m:m:n} \leq T < X_{k:m:n} \\ (X_{1:m:n}, \ldots, X_{m:m:n}), & \text{if} \ T < X_{m:m:n} \leq X_{k:m:n} \\ (X_{1:m:n}, \ldots, X_{D:m:n}, X_{D+1:m:n} \ldots, X_{k:m:n}), & \text{if} \ X_{m:m:n} < X_{k:m:n} < T \end{cases},$$

$F^{-1}(0) > x_{1:m:n} > x_{2:m:n} > \ldots > x_{n:m:n} > F^{-1}(1)$ of R^n.

The estimation of the inverse Weibull distribution parameters based on the DGPHCS have been studied, see [19].

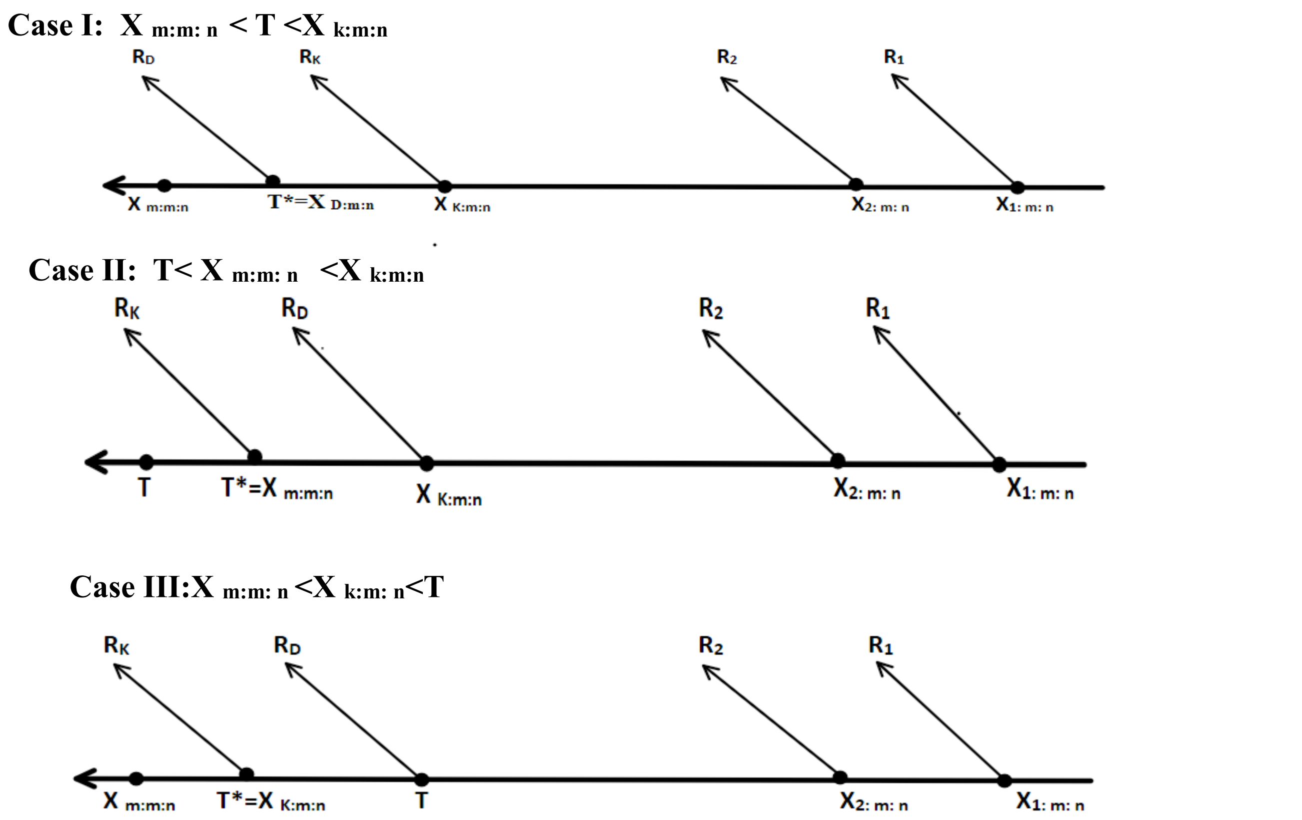


Fig. 1: Schematic representation of the DGPHCS.

2 Estimation Methods

2.1 Runge-Kutta Method

Theoretically, it is known that the traditional log-likelihood function, $H(\theta; x)$, depends on the unknown parameter $\theta = (\alpha, \beta)$ and the sample data $\underline{X} = (x_1, x_2, \ldots, x_n)$, which can be used to derive the maximum likelihood estimator (MLE) $\hat{\theta}(x)$ of θ, by solving the stationary equation $\frac{\partial H(\theta;x)}{\partial\theta}|_{\hat{\theta}(x)} = 0$. Thus, based on the dependence of the MLE on the sample data, we can apply the implicit function theorem to the stationary equation by taking the total derivative with respect to any value $x \in \underline{X}$, with all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x_0) = \theta_0$, we obtain the following equation:

$$\frac{d}{dx}\left(\frac{\partial H(\theta|X)}{\partial\theta}\right) = \frac{\partial^2 H(\theta|X)}{\partial\theta\,\partial x}|_{\theta=\hat{\theta}} + \frac{\partial^2 H(\theta|X)}{\partial\theta^2}|_{\theta=\hat{\theta}}\frac{d\hat{\theta}}{dx} = 0. \tag{4}$$

Solving (4) we obtain the first derivative of $\hat{\theta}$ with respect to any $x \in \underline{X}$ at $\theta = \hat{\theta}$, as follows:

$$\frac{d\hat{\theta}(x)}{dx} = -\left(\frac{\partial^2 H(\theta|X)}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 H(\theta|X)}{\partial\theta\,\partial x}|_{\theta=\hat{\theta}}. \tag{5}$$

Thus, we can write (5) in the following form:

$$\frac{d\hat{\theta}(x)}{dx} = f(x, \hat{\theta}), \quad \text{at } \hat{\theta}(x_0) = \theta_0 , \tag{6}$$

where $f(x, \hat{\theta}) = -\left(\frac{\partial^2 H(\theta|X)}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 H(\theta|X)}{\partial\theta\,\partial x}|_{\theta=\hat{\theta}}$.

The equation (6) is a first- order ordinary differential equation in $\hat{\theta}$. Using any numerical technique, such as the fourth order Runge-Kutta, and Picard methods, we can find the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities that fit the optimized solution. Thus, the recurrence solution for (6) can be written as follows:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + (K_1 + 2K_2 + 2K_3 + K_4)/6, \tag{7}$$

for $i = 0,1,2,\ldots$,

where

$K_1 = hf(x_i, \hat{\theta}_i)$, $\quad K_2 = hf(x_i + h/2, \hat{\theta}_i + K_1/2)$,

$K_3 = hf(x_i + h/2, \hat{\theta}_i + K_2/2)$,

and $K_4 = hf(x_i + h, \hat{\theta}_i + K_3)$.

Here h is the step height with a small value say (10^{-2}) and $\hat{\theta}(x_0) = \theta_0$, is the initial value for $\hat{\theta}$.
It is important to notice that the Runge-Kutta method has been used for estimating several distribution parameters, see [15-17].

2.2 Picard's Method

The general iteration rule for Picard's method for the parameter α (say) for any $x \in \underline{X}$, can be derived by integrating (6) with respect to x from x_0 to x^* as follows:

$$\hat{\alpha}(x^*) = \hat{\alpha}(x_0) + \int_{x_0}^{x^*} f(x, \hat{\alpha}_0, \hat{\gamma})dx$$

$$= \alpha_0 + \int_{\alpha_0}^{\alpha^*} f(x, \hat{\alpha}_0, \hat{\gamma})\frac{dx}{d\hat{\gamma}}d\hat{\gamma}$$

Using the differential equation for the parameter γ , $\frac{d\hat{\gamma}}{dx} = g(x, \hat{\alpha}_0, \hat{\gamma}_0)$, we get the recurrence relation for the parameter α as follows:

$$\alpha^*_{i+1} = \alpha_0 + \int_{\alpha_0}^{\alpha^*}\left[\frac{f(x,\hat{\alpha}_i,\hat{\gamma})}{g(x,\hat{\alpha}_i\hat{\gamma})}\right]d\hat{\gamma} , \tag{8}$$

for $i = 0,1,2,3, ..$
where α_0 is the initial point and α^* is the value for which the desired solution should be optimized.
Similarly, the Picard estimator for γ can be derived from the following recurrence relation:

$$\gamma^*_{i+1}(x) = \gamma_0 + \int_{\beta_0}^{\beta^*}\left[\frac{f(x,\hat{\alpha},\hat{\gamma}_i)}{f(x,\hat{\alpha},\hat{\gamma}_i)}\right]d\hat{\alpha}, \tag{9}$$

for $i = 0,1,2,3, ..$
The iterative process in (7), (8) and (9) are continued until two consecutive numerical solutions are almost the same, that is if $|\theta^*_{i+1} - \theta^*_i| < 10^{-5}$, for $i = 0,1,2,3, ..$.
It is worthwhile to notice that the Picard's method has been used for estimating the three-parameter Burr-XII, see [18].

For the Burr-III distribution, using (1) and (2) the log-likelihood function of the DGPHCS (3) can be derived as follows:

$$L(\underline{x}; \alpha, \gamma) = C\prod_{i=1}^{n}\alpha\gamma x^{-\alpha-1}(1 + x^{-\alpha})^{-\gamma-1}$$

$$\times [(1 + x^{-\alpha})^{-\gamma}]^{R_i}[(1 + T^{-\alpha})^{-\gamma}]^{\delta R^*_T}. \tag{10}$$

The log-likelihood $H = \ln L(\underline{x}; \alpha, \gamma)$ can be derived as follows:

$$H = n\ln(\alpha\gamma) - (\alpha+1)\sum_{i=1}^{n} \ln x - \sum_{i=1}^{n} \ln(1+x^{-\alpha})$$

$$-\gamma[\sum_{i=1}^{n}(1+R_i)\ln(1+x^{-\alpha}) + \delta R_T^* ln(1+T^{-\alpha})].$$

The derivatives of H can be derived from the corresponding derivatives in Appendix A, with a = b =1, and c = d = 0.

2.3 Bayes Method

In this section, the Bayes estimations will be derived based on the informative gamma prior, the kernel prior, and the characteristic prior.

2.3.1 The Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with a joint probability density function defined as follows:

$$h(\alpha,\gamma) \propto \alpha^{a-1}\gamma^{c-1}e^{b\alpha-d\gamma}. \qquad (11)$$

The hyper-parameter a, b, c and d are assumed to be known and positives to reflect the prior belief about the unknown parameters.

2.3.2 The Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha,\gamma)$ with support on $(0,\infty)$, which is defined as follows:

$$\hat{g}(\alpha,\gamma) = \frac{1}{nh_1h_2}\sum_{i=1}^{n} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1}, \frac{\gamma-\hat{\gamma}_i}{h_2}\right), \qquad (12)$$

$h_i, i = 1,2$ are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The optimal choice for h_i, which minimizes the mean squared errors is given by $h_i = 1.06S_i n^{-0.2}$, where S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the bivariate standard normal distribution for the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. It is worthwhile to mention that this kernel prior has been explained in detail for some distributions, see [1] and [13].

2.3.3 The Characteristic Prior

The characteristic function (CF) is the Fourier transform of the cumulative distribution function (CDF), and hence there is a one-to-one correspondence between the CF and the CDF. Thus, the CF is fully characterizing the distribution of the underlying random variable. Since the CF can be estimated using the empirical characteristic function (ECF), which retains all the information in the sample, it plays an increasing and important role in econometrics and finance, see [10]. Thus, based on the CF for two random variables and its inversion formula for the probability density function, the characteristic prior for the parameters α and γ can be derived as follows:

$$\hat{q}(\alpha,\gamma) = \frac{1}{4n\pi^2}\sum_{i=1}^{n}\frac{1}{|(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}_i)(\hat{\boldsymbol{\gamma}}_i-\boldsymbol{\gamma})|}, \qquad (13)$$

see [19].

Thus, using the priors (11), (12) and (13) we can write the general prior density function as follows:

$$Q(\alpha,\gamma) = h(\alpha,\gamma)g(\alpha,\gamma)q(\alpha,\gamma) =$$

$$\alpha^{a-1}\gamma^{c-1}\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\gamma)\hat{q}_1^{s_1}(\alpha)\,\hat{q}_2^{s_2}(\gamma)\,exp(-b\alpha - d\gamma),$$

with the following cases:

i) For the informative prior (11): $p_1 = p_2 = 0$, $s_1 = s_2 = 0$.
ii) For the kernel prior (12): $p_1 = p_2 = 1$, $s_1 = s_2 = 0$, and $a = c = 1$, $b = d = 0$.
iii) For the characteristic prior (13): $p_1 = p_2 = 0$, $s_1 = s_2 =$ $a = c = 1, b = d = 0$.

Thus, the posterior density for the unknown parameters α and γ can be written using (10) in a unified form as follows:

$$f(\alpha,\gamma|\underline{X}) = KQ(\alpha,\gamma)L(X;\alpha,\gamma)$$

$$= K\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\gamma)\hat{q}_1^{s_1}(\alpha)\,\hat{q}_2^{s_2}(\gamma)\alpha^{n+a-1}\gamma^{n+c-1}$$

$$\times exp[-\sum_{i=1}^{n}\ln(1+x^{-\alpha}) - \alpha b - (1+\alpha)\sum_{i=1}^{n} lnx_i)]$$

$$\times \exp[-\gamma(d + (\sum_{i=1}^{n}(1+R_i)\ln(1+x^{-\alpha})$$

$$+\delta R_T^*\ln(1+T^{-\alpha}))]. \qquad (14)$$

Based on (14) we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [14] introduced an easily computable approximation for the posterior mean of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let $q(\alpha,\gamma)$ be a smooth, positive function in the parameter space. The posterior expectation of $q(\alpha,\gamma)$ can be obtained as:

$$q^* = E(q(\alpha,\gamma)|x) = \frac{\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\gamma)}d\alpha d\gamma}{\int_0^\infty\int_0^\infty e^{nH(\alpha,\gamma)}d\alpha d\gamma}, \qquad (15)$$

where $H(\alpha,\gamma) = \ln f(\alpha,\gamma|x)/n$,
and $H^*(\alpha,\gamma) = H(\alpha,\gamma) + \ln q(\alpha,\gamma)/n$.

For (α,γ) the Bayes estimator using Tierney and Kadane approximation for $q(\alpha,\gamma)$ can be obtained as follows:

$$q^* = \sqrt{|\Sigma^*|/|\Sigma|}\exp[n[H^*(\alpha,\gamma) - H(\alpha,\gamma)],$$

where $(\hat{\alpha},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\gamma}^*)$ maximize the $H(\hat{\alpha},\hat{\beta})$ and $H^*(\hat{\alpha}^*,\hat{\gamma}^*)$, respectively.

Let $|\Sigma| = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix}^{-1}$ and $|\Sigma*| = \begin{vmatrix} H_{11}^* & H_{12}^* \\ H_{21}^* & H_{22}^* \end{vmatrix}^{-1}$

denote the minus of inverse of Hessians of $H(\alpha,\gamma)$ and $H^*(\alpha,\gamma)$ at $(\hat{\alpha},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\gamma}^*)$ respectively.
Using (15) we can define the log posterior density $H = H(\alpha,\gamma)$ as follows:

$$H(\alpha,\gamma|\underline{x}) = [p_1\ln\hat{g}_1(\alpha) + p_2\ln\hat{g}_2(\gamma) + s_1\ln\hat{q}_1(\alpha)$$
$$+s_2\ln\hat{q}_2(\gamma) + (n+a-1)\ln\alpha + (n+c-1)ln\gamma$$
$$-\sum_{i=1}^{n}\ln(1+\boldsymbol{x}^{-\boldsymbol{\alpha}}) - \alpha b - (1+\alpha)\sum_{i=1}^{n} lnx_i)]$$
$$-\boldsymbol{\gamma}(\boldsymbol{d} + \sum_{i=1}^{n}(1+R_i)\ln(\boldsymbol{1}+\boldsymbol{x}^{-\boldsymbol{\alpha}}) + \delta R_T^*\ln(1+\boldsymbol{T}^{-\boldsymbol{\alpha}}))]. \quad (16)$$

The derivatives of $H(\alpha,\gamma)$ and $H^*(\alpha,\gamma)$ have been derived in Appendix A.

3 Simulation Study

The simulation studies are carried out for studying the performance of the Picard, Runge Kutta. and Bayes methods through the absolute average bais (AVB) and the mean squared error (MSE), which are given respictively as follows:

$$AVB = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta| \text{ and } MSE(\theta^*) = \sum_{i=1}^{L}(\theta_i - \theta^*)^2/L$$

where $\theta^*$is the point estimate of the unknown parameter θ and L is the number of replications.
In our simulation study, we choose the hyper parameters of α and β as: $a = c = 0.5$, and $b = d = 0.3$, for which the prior distribution will be decreasing function for each parameter respectively. We choose two values for the parameter $\alpha = (1, 2)$ and two values for the parameter $\gamma = (2, 3)$. Using the above values of the parameters for generating different samples from the Burr-III distribution with sizes $n = 20, 40,$ and 60 to represent small, moderate and large sizes. We choose two values for the termination time T = (1.5, 3). To assess the performance of these estimates, the AVB and the MSE for each sample were calculated using 1000 replications.

From the simulation results in Tables 3-6, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

1. In general, the estimated values of the MSE for the parameters α and γ based on Picard and R-K methods are often less than the estimated values based on the Bayes method estimates using the different priors.
2. The estimated MSE values increase as the values of α and γ increase.
3. The estimated MSE values decrease as the hyperparameters of the informative priors decrease and as the termination time of the experiment T increases as expected.
4. It is immediately apparent that, as the sample sizes increases, the estimated MSE values decreases.
5. The estimated MSE values based on the characteristic and kernel priors are less than the ones based on the informative gamma prior.

As a conclusion, it appears that the point estimates based on the Picard and R-K methods compete and outperform the Bayes method based on the different priors.

4 Real Data Analysis

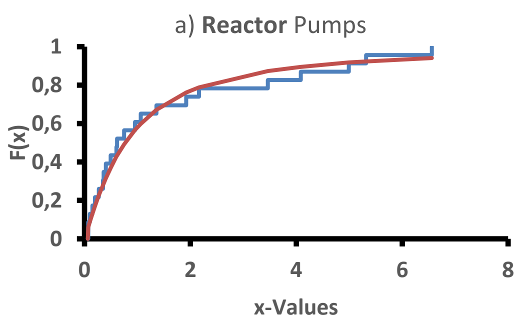
In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the Burr–III distribution in practice and to illustrate that this distribution can be considered as a good lifetime model for some new area of applications, comparing to many known distributions such as the Weibull distribution. We have fitted these datasets using some goodness-of- fit tests, such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH2) tests for significance level test equal to 0.05.

4.1 The Reactor Pumps Data

In this section, a real dataset for secondary nuclear pumps has been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the recirculating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as those in the single-phase flow case. Thus, the secondary reactor pump is a feedwater pump that takes feedwater from the desecrator storage tank, pressurized by the booster pump, and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high-temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following dataset represents the time between the failures of the secondary reactor pumps. [25] have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this dataset. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320

For studying the reliability of these reactor pumps, based on this dataset, we found the parameter estimates that represent the shape of the failures between pumps, using our model to determine the behavior of the failed pumps. We noticed that Runge-Kutta and Bayes estimates for α lie in the interval [1, 1.3] and for γ lie in the interval [0.5, 0.9], indicating the above dataset is heavily right-skewed, which means the failure rate decreases with increasing time, see Fig. (2b). Thus, the reliability of safety mechanisms will decrease with increasing time.



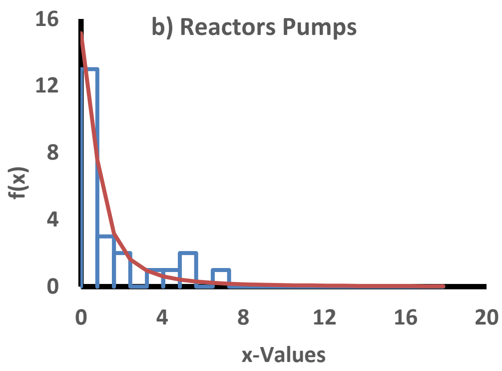


Fig. 2: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

4.2 The Vinyl Chloride Data

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. It is known that a concentration of vinyl chloride in drinking water of 0.5 mg/liter was calculated to be associated with an excess risk in liver and brain tumors for exposure beginning in adulthood and would double the cancer risk for continuous exposure from birth. Therefore, we consider the dataset used by [6], which represents 34 data points in mg/L from the vinyl chloride that was obtained from clean upgrade monitoring wells as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

For studying the concentration of vinyl chloride in the water of these wells based on this dataset, the Burr-III distribution parameters have been estimated, which represent the shape of the concentration of vinyl chloride in the water. We noticed that Runge-Kutta and Bayes estimates for α are close to 1.4, which indicates that the above dataset is moderately highly right-skewed, which means the concentration decreases with increasing time, see Fig. (3 b). Also, the Picard, Runge-Kutta, and Bayes estimates for γ are close to 1.1, which ensures the dataset is heavily right- skewed and the vinyl chloride concentration will decrease with increasing time, and therefore monitoring these wells is very significant.

The results in Table 1 indicate that the Burr-III model is a good fit for these datasets, as shown in Fig. (2a, 3a), where the calculated values for the goodness- of- fit test statistics are less than the critical values and the power of the tests is greater than the significance level of the tests, which is equal to 0.05. The results in Table 2 indicate that the estimated MSE values based on the Picard and Runge-Kutta methods are less than those based on the Bayes' method for large values of T when considering the MLEs as the true values of the parameters. Thus, the results of these datasets ensure the simulation results.

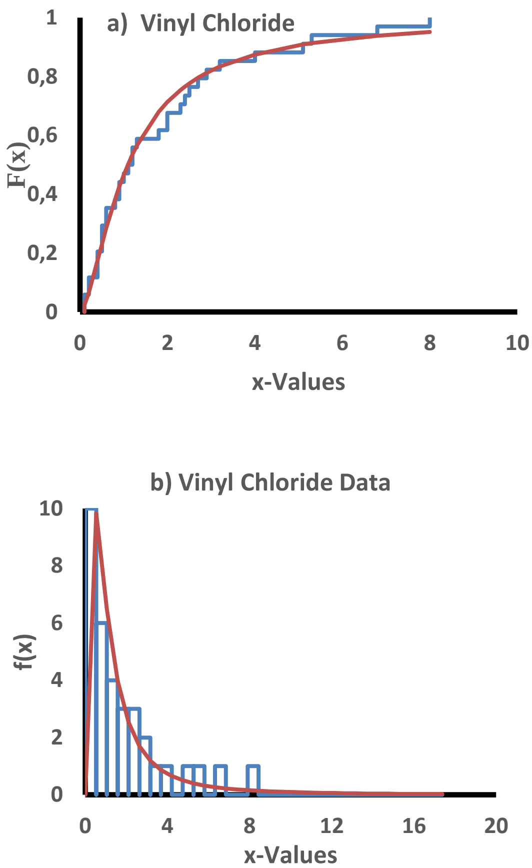


Fig. 3: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

5 Conclusion

In statistical inference, it is well known that the parameter estimations using Bayes' method based on the informative priors are more efficient than most of the estimation methods in reliability analysis. Thus, in this work, we introduced optimal numerical iteration techniques such as the Picard and Runge-Kutta methods for estimating the Burr-III distribution parameters based on the dual generalized progressive hybrid censoring scheme. As a result of this work, we noticed that the Picard and Runge -Kutta methods are more efficient and outperform the Bayes' method based on the informative gamma, characteristic, and kernel priors.

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Appendix A:

The log of the posterior density function (16) can be derived as follows:

$$H(\alpha,\gamma|\underline{x}) = [p_1 \ln\hat{g}_1(\alpha) + p_2 \ln\hat{g}_2(\gamma) + s_1 \ln\hat{q}_1(\alpha)$$

$$+s_2 \ln\hat{q}_2(\gamma) + (n+a-1)\ln\alpha + (n+c-1)ln\gamma$$

$$-\sum_{i=1}^{n} \ln(1+x^{-\alpha}) - \alpha b - (1+\alpha)\sum_{i=1}^{n} lnx_i)]$$

$$-\gamma(d + \textstyle\sum_{i=1}^{n}(1+R_i)\ln(1+x^{-\alpha}) + \delta R_T^* \ln(1+T^{-\alpha}))].$$

$$\frac{\partial H}{\partial \alpha} = p_1 \frac{\hat{g}_1^{/}(\alpha)}{\hat{g}_1(\alpha)} + s_1 \frac{\hat{q}_1^{/}(\alpha)}{\hat{q}_1(\alpha)} + (n+a-1)/\alpha$$

$$-b - \sum_{i=1}^{n} lnx + \sum_{i=1}^{n} \frac{x^{-\alpha}\ln x}{1+x^{-\alpha}}$$

$$+\gamma[\sum_{i=1}^{n}(1+R_i)\frac{x^{-\alpha}\ln x}{1+x^{-\alpha}} + \delta R_T^* \frac{T^{-\alpha}\ln T}{1+T^{-\alpha}}]$$

$$\frac{\partial^2 H}{\partial \alpha^2} = p_1 \frac{\hat{g}_1(\alpha)\hat{g}_1^{//}(\alpha) - \hat{g}^{/2}_1(\alpha)}{\hat{g}_1^2(\alpha)}$$

$$+s_1 \frac{\hat{q}_1(\alpha)\hat{q}_1^{//}(\alpha) - \hat{q}^{/2}_1(\alpha)}{q_1^2(\alpha)} - (n+a-1)/\alpha^2$$

$$-\sum_{i=1}^{n} \frac{x^{-\alpha}(\ln x)^2}{(1+x^{-\alpha})^2} - \gamma[\sum_{i=1}^{n}(1+R_i)\frac{x^{-\alpha}(\ln x)^2}{(1+x^{-\alpha})^2} + \frac{T^{-\alpha}(\ln T)^2}{(1+T^{-\alpha})^2}$$

$$\frac{\partial H}{\partial \gamma} = p_2 \frac{\hat{g}_2'(\gamma)}{\hat{g}_2(\gamma)} + s_2 \frac{\hat{q}_2'(\gamma)}{q_2(\gamma)} + (n+c-1)/\gamma - d$$

$-[\sum_{i=1}^{n}(1+R_i)\ln(1+x^{-\alpha})+\delta R_T^* ln(1+T^{-\alpha})],$

$$\frac{\partial^2 H}{\partial \gamma^2}=\mathrm{p}_2\frac{\hat{\mathrm{g}}_2(\gamma)\mathrm{g}_2^{//}(\gamma)-\hat{\mathrm{g}}_2^{/^2}(\gamma)}{\hat{\mathrm{g}}_2{}^2(\gamma)}$$

$$+\mathrm{s}_2\frac{\hat{\mathrm{q}}_2(\gamma)\mathrm{g}_2^{//}(\gamma)-\mathrm{q}_2^{/^2}(\gamma)}{\hat{\mathrm{q}}_2{}^2(\gamma)}-\frac{(n+c-1)}{\gamma^2}$$

$$\frac{\partial^2 H}{\partial \gamma \partial \alpha}=[\sum_{i=1}^{n}(1+R_i)\frac{\mathrm{x}^{-\alpha} lnx}{1+\mathrm{x}^{-\alpha}}+\delta R_T^*\frac{\mathrm{T}^{-\alpha} lnT}{1+\mathrm{T}^{-\alpha}}]$$

$$\frac{\partial^2 H}{\partial \alpha \partial x}=-\sum_{i=1}^{n}\frac{1}{x}-\sum_{i=1}^{n}\frac{\mathrm{x}^{-\alpha-1}(\alpha lnx-\mathrm{x}^{-\alpha}-1)}{(1+\mathrm{x}^{-\alpha})^2}$$

$$-\gamma[\sum_{i=1}^{n}(1+R_i)\frac{\mathrm{x}^{-\alpha-1}(\alpha lnx-\mathrm{x}^{-\alpha}-1)}{(1+\mathrm{x}^{-\alpha})^2}$$

$$\frac{\partial^2 H}{\partial \gamma \partial x}=\alpha\sum_{i=1}^{n}(1+R_i)\frac{\mathrm{x}^{-\alpha-1}}{1+\mathrm{x}^{-\alpha}}$$

where the r^{th} derivative of the kernel density estimation can be defined as follows:

$$\frac{\mathrm{d}^{\mathrm{r}}\hat{\mathrm{g}}_1(\alpha)}{\mathrm{d}\alpha^{\mathrm{r}}}=\hat{\mathrm{g}}_1{}^{\mathrm{r}}(\alpha)=\frac{1}{\mathrm{nh}_1^{\mathrm{r}+1}}\sum_{\mathrm{i}=1}^{n}\mathrm{K}^{\mathrm{r}}\left(\frac{\alpha-\hat{\alpha}_{\mathrm{i}}}{\mathrm{h}_1}\right), \qquad (17)$$

where r =0,1,2,3, -----.

Using the Gaussian kernel and (17), we have

$$\hat{\mathrm{g}}_1(\alpha)=\frac{1}{\mathrm{nh}_1\sqrt{2\pi}}\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{e}^{-0.5(\frac{\alpha-\hat{\alpha}_{\mathrm{i}}}{\mathrm{h}_1})^2},$$

$$\hat{\mathrm{g}}_1'(\alpha)=-\frac{1}{\mathrm{nh}_1^2\sqrt{2\pi}}\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\frac{\alpha-\hat{\alpha}_{\mathrm{i}}}{\mathrm{h}_1}\right)\mathrm{e}^{-0.5(\frac{\alpha-\hat{\alpha}_{\mathrm{i}}}{\mathrm{h}_1})^2},$$

$$\hat{\mathrm{g}}_1''(\alpha)=\frac{1}{\mathrm{nh}_1^3\sqrt{2\pi}}\sum_{\mathrm{i}=1}^{\mathrm{n}}[(\frac{\alpha-\alpha_{\mathrm{i}}}{\mathrm{h}_1})^2-1]\,\mathrm{e}^{-0.5(\frac{\alpha-\alpha_{\mathrm{i}}}{\mathrm{h}_1})^2}.$$

Similarly for the kernel priors $\hat{\mathrm{g}}_2(\gamma)$.

For the parameter α the Characteristic prior and their derivatives can be derived as follows:

$$\hat{\mathrm{q}}_1(\alpha)=\frac{1}{2\pi\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{|(\alpha-\hat{\alpha}_{\mathrm{i}})|},$$

$$\hat{q}_1'(\alpha)=-\frac{1}{2\pi\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{(\alpha-\hat{\alpha}_{\mathrm{i}})^2}$$

and $\hat{q}_1^{"}(\alpha)=\frac{1}{\pi\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{(\alpha-\hat{\alpha}_{\mathrm{i}})^3}$.

Similarly for the characteristic priors $\hat{\mathrm{q}}_2(\gamma)$.

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Table 1: The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p- values) The MLE's for the parameters α, γ for these datasets have been calculated.

Data	The Tests	Critical value	Calculated value	P-values	$\hat{\alpha}$	$\hat{\gamma}$
Reactor Pumps Data N=23	K-S	0.8998	0.4290	0.9177	1.3326	0.7836
	A-D	0.7686	0.2615	0.7229		
	CH2	24.3376	1.0609	0.7773		
Vinyl Chloride Data N=34	K-S	0.9089	0.5534	0.6216	1.4746	1.1026
	A-D	0.7731	0.2719	0.6904		
	CH2	25.0930	1.0731	0.7585		

Table 2: The estimate (Est.) and the MSEs for the parameters α and γ based on the Picard, R-K and Bayes methods using the Charactric, gamma and Kernel priors based on the DGHPCS scheme for m=n/2 , and k=m/2, with a=0.3, b=0.2, c=0.3, d=0.2.

Sam.	T	Par.	Picard Estimate		R-K Estamite		Chara. Prior		Gamma Prior		Kernel Prior	
			Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
Data 4.1 N=23	0.15	α	1.3008	0.0010	1.2856	0.0022	1.1519	0.0327	1.1031	0.0527	1.1083	0.0503
		γ	0.7940	0.0001	0.7739	0.0009	0.6797	0.0108	0.6411	0.0203	0.6443	0.0194
	4.5	α	1.2897	0.0018	1.2587	0.0055	1.1488	0.0338	1.0686	0.0697	1.0874	0.0602
		γ	0.5608	0.0496	0.7542	0.0086	0.6770	0.0114	0.5948	0.0356	0.6150	0.0284
Data 4.2 N=34	0.5	α	1.4364	0.0015	1.4169	0.0033	1.2836	0.0365	1.2323	0.0587	1.2356	0.0571
		γ	1.1171	0.0002	1.0924	0.0001	0.9553	0.0217	0.9134	0.0358	0.9172	0.0344
	5.5	α	1.4254	0.0024	1.3914	0.0069	1.2779	0.0387	1.2055	0.0724	1.2204	0.0646
		γ	1.0029	0.0099	1.0632	0.0016	0.9518	0.0227	0.8562	0.0607	0.8852	0.0473

Table 3: The ABS and the MSEs in parentheses for the parameter α using Picard and Bayes methods at T=1.5 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) .

n	m	k	α	γ	Picard Estimators	R-K Estimations	Bayes Estimations		
							Chara-Prior	Kernel Prior	Gamma Prior
20	10	5	1	2	0.1157(0.0135)	0.1637(0.0269)	0.1453(0.0215)	0.1938(0.0382)	0.2606(0.2843)
				3	0.1200(0.0146)	0.1676(0.0281)	0.1448(0.0210)	0.1808(0.0330)	0.1468(0.0485)
			2	2	0.2102(0.0442)	0.3548(0.1259)	0.2943(0.0866)	0.3886(0.1527)	0.5845(0.2461)
				3	0.2155(0.0465)	0.3555(0.1264)	0.2937(0.0863)	0.3573(0.1283)	0.2756(0.1578)
		8	1	2	0.1139(0.0130)	0.1604(0.0258)	0.1453(0.0211)	0.1747(0.0305)	0.1846(0.0341)
				3	0.1190(0.0143)	0.1633(0.0267)	0.1448(0.0214)	0.1684(0.0284)	0.1981(0.0405)
			2	2	0.2093(0.0438)	0.3542(0.1255)	0.2943(0.0866)	0.3401(0.1157)	0.3707(0.1376)
				3	0.2167(0.0470)	0.3551(0.1261)	0.2939(0.0864)	0.3319(0.1102)	0.3985(0.1638)
	15	8	1	2	0.1138(0.0130)	0.1607(0.0259)	0.1453(0.0213)	0.1785(0.0319)	0.2058(0.0427)
				3	0.1190(0.0143)	0.1633(0.0267)	0.1447(0.0209)	0.1717(0.0295)	0.2733(0.3163)
			2	2	0.2092(0.0438)	0.3542(0.1255)	0.2943(0.0866)	0.3408(0.1162)	0.3853(0.1524)
				3	0.2175(0.0474)	0.3552(0.1262)	0.2939(0.0864)	0.3316(0.1100)	0.3948(0.1631)
		11	1	2	0.1100(0.0122)	0.1571(0.0247)	0.1453(0.0215)	0.1700(0.0289)	0.1737(0.0302)
				3	0.1175(0.0139)	0.1595(0.0255)	0.1447(0.0209)	0.1663(0.0276)	0.1741(0.0303)
			2	2	0.2071(0.0429)	0.3551(0.1261)	0.2943(0.0866)	0.3333(0.1111)	0.3493(0.1220)
				3	0.2155(0.0465)	0.3550(0.1261)	0.2938(0.0863)	0.3281(0.1076)	0.3492(0.1220)
40	20	10	1	2	0.1142(0.0131)	0.1611(0.0260)	0.1433(0.0205)	0.1698(0.0288)	0.2187(0.0764)
				3	0.1176(0.0139)	0.1663(0.0277)	0.1393(0.0194)	0.1656(0.0274)	0.1717(0.1246)
			2	2	0.2093(0.0438)	0.3556(0.1264)	0.2929(0.0858)	0.3383(0.1146)	0.5607(0.1352)
				3	0.2148(0.0462)	0.3558(0.1266)	0.2919(0.0852)	0.3221(0.1038)	0.5856(0.2463)
		15	1	2	0.1124(0.0127)	0.1587(0.0252)	0.1434(0.0206)	0.1651(0.0273)	0.1723(0.0297)
				3	0.1173(0.0138)	0.1625(0.0265)	0.1390(0.0193)	0.1619(0.0262)	0.2232(0.0973)
			2	2	0.2082(0.0434)	0.3553(0.1263)	0.2931(0.0859)	0.3248(0.1055)	0.3397(0.1155)
				3	0.2154(0.0464)	0.3558(0.1266)	0.2916(0.0850)	0.3195(0.1021)	0.3486(0.1221)
		15	1	2	0.1133(0.0129)	0.1596(0.0255)	0.1434(0.0206)	0.1659(0.0275)	0.1752(0.0307)
				3	0.1173(0.0138)	0.1633(0.0267)	0.1393(0.0194)	0.1623(0.0263)	0.2367(0.1012)
			2	2	0.2081(0.0433)	0.3553(0.1262)	0.2931(0.0859)	0.3248(0.1055)	0.3400(0.1157)

				3	0.2152(0.0464)	0.3557(0.1265)	0.2916(0.0850)	0.3196(0.1021)	0.3489(0.1226)
	30	23	1	2	0.1102(0.0122)	0.1552(0.0241)	0.1432(0.0205)	0.1600(0.0256)	0.1611(0.0260)
				3	0.1158(0.0134)	0.1568(0.0246)	0.1398(0.0195)	0.1583(0.0251)	0.1606(0.0258)
			2	2	0.2087(0.0436)	0.3564(0.1270)	0.2926(0.0856)	0.3188(0.1016)	0.3245(0.1053)
				3	0.2144(0.0460)	0.3559(0.1267)	0.2912(0.0848)	0.3158(0.0997)	0.3228(0.1042)
60	30	15	1	2	0.1142(0.0131)	0.1601(0.0257)	0.1344(0.0181)	0.1637(0.0268)	0.1839(0.0526)
				3	0.1167(0.0136)	0.1652(0.0273)	0.1808(0.0335)	0.1612(0.0260)	0.1752(0.1106)
			2	2	0.2084(0.0435)	0.3557(0.1266)	0.2886(0.0833)	0.3529(0.1290)	0.3781(0.9052)
				3	0.2135(0.0456)	0.3559(0.1267)	0.2599(0.0677)	0.3668(0.1485)	0.2882(0.0831)
		23	1	2	0.1128(0.0128)	0.1576(0.0249)	0.1341(0.0180)	0.1607(0.0258)	0.1648(0.0272)
				3	0.1168(0.0137)	0.1616(0.0262)	0.1779(0.0327)	0.1587(0.0252)	0.2096(0.1127)
			2	2	0.2074(0.0430)	0.3557(0.1265)	0.2907(0.0845)	0.3180(0.1011)	0.3263(0.1065)
				3	0.2151(0.0463)	0.3561(0.1268)	0.2845(0.0809)	0.3144(0.0989)	0.3298(0.1088)
	45	23	1	2	0.1130(0.0128)	0.1583(0.0251)	0.1354(0.0183)	0.1609(0.0259)	0.1653(0.0273)
				3	0.1167(0.0136)	0.1626(0.0265)	0.1893(0.0392)	0.1587(0.0252)	0.1965(0.0452)
			2	2	0.2077(0.0431)	0.3558(0.1266)	0.2905(0.0844)	0.3184(0.1014)	0.3281(0.1077)
				3	0.2149(0.0462)	0.3560(0.1268)	0.2845(0.0810)	0.3145(0.0989)	0.3301(0.1090)
		34	1	2	0.1103(0.0122)	0.1551(0.0241)	0.1318(0.0174)	0.1569(0.0246)	0.1576(0.0248)
				3	0.1161(0.0135)	0.1562(0.0244)	0.1861(0.0377)	0.1558(0.0243)	0.1573(0.0247)
			2	2	0.2105(0.0443)	0.3566(0.1272)	0.2866(0.0822)	0.3140(0.0986)	0.3174(0.1008)
				3	0.2147(0.0461)	0.3561(0.1268)	0.2700(0.0823)	0.3117(0.0971)	0.3160(0.0999)

Table 4: The ABS and the MSEs in parentheses for the parameter γ using Picard and Bayes methods at T=1.5 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4).

n	m	k	α	γ	Picard Estimators	R-K Estimation	Bayes Estimation		
							Chara-Prior	Kernel Prior	Gamma Prior
20	10	5	1	2	0.1189(0.0324)	0.1983(0.0393)	0.2914(0.0849)	0.1938(0.0382)	0.9189(0.5436)
				3	0.0636(0.0041)	0.2985(0.0891)	0.4406(0.1941)	0.1808(0.0330)	0.9073(0.3423)
			2	2	0.1630(0.0460)	0.1936(0.0375)	0.2901(0.0841)	0.3888(0.1528)	0.6947(0.5436)
				3	0.1051(0.0119)	0.2917(0.0851)	0.4392(0.1929)	0.3575(0.1285)	0.3808(0.3253)
		8	1	2	0.0961(0.0403)	0.1978(0.0391)	0.2909(0.0846)	0.1747(0.0305)	0.4527(0.2053)
				3	0.0665(0.0045)	0.2976(0.0886)	0.4400(0.1936)	0.1684(0.0284)	0.8372(0.7283)
			2	2	0.2428(0.1362)	0.1923(0.0370)	0.2896(0.0839)	0.3401(0.1157)	0.4489(0.2017)
				3	0.1038(0.0119)	0.2897(0.0839)	0.4389(0.1926)	0.3320(0.1102)	0.8368(0.7297)
	15	8	1	2	0.1017(0.1269)	0.1978(0.0391)	0.2909(0.0846)	0.1785(0.0319)	0.5387(0.2931)
				3	0.0689(0.0049)	0.2976(0.0886)	0.4398(0.1935)	0.1717(0.0295)	0.6573(0.2432)
			2	2	0.2050(0.1994)	0.1924(0.0370)	0.2896(0.0839)	0.3408(0.1162)	0.4775(0.2372)
				3	0.1014(0.0111)	0.2902(0.0842)	0.4389(0.1926)	0.3316(0.1100)	0.8243(0.7100)
		11	1	2	0.2868(0.3263)	0.1968(0.0387)	0.2901(0.0841)	0.1700(0.0289)	0.3862(0.1492)
				3	0.0878(0.0196)	0.2965(0.0879)	0.4393(0.1929)	0.1663(0.0276)	0.6224(0.3878)
			2	2	0.3077(0.4123)	0.1921(0.0369)	0.2893(0.0837)	0.3333(0.1111)	0.3841(0.1476)
				3	0.1220(0.0259)	0.2886(0.0833)	0.4385(0.1923)	0.3281(0.1076)	0.6165(0.3804)
40	20	10	1	2	0.0979(0.4497)	0.1981(0.0393)	0.2874(0.0826)	0.1698(0.0288)	0.6389(0.9895)
				3	0.0643(0.0042)	0.2983(0.0890)	0.4820(0.2882)	0.1656(0.0274)	0.9623(0.6743)
			2	2	0.1593(0.0498)	0.1933(0.0374)	0.2886(0.0833)	0.3384(0.1146)	0.9135(0.5632)
				3	0.0957(0.0094)	0.2914(0.0850)	0.4381(0.1920)	0.3222(0.1038)	0.5632(0.2362)
		15	1	2	0.1195(0.2711)	0.1976(0.0391)	0.2856(0.0816)	0.1651(0.0273)	0.4019(0.1617)
				3	0.0688(0.0048)	0.2976(0.0886)	0.4841(0.4025)	0.1619(0.0262)	0.3252(0.2142)
			2	2	0.1966(0.3990)	0.1925(0.0370)	0.2886(0.0833)	0.3248(0.1055)	0.3875(0.1504)
				3	0.1028(0.0111)	0.2900(0.0841)	0.4373(0.1912)	0.3195(0.1021)	0.6689(0.4538)
	30	15	1	2	0.0900(0.0173)	0.1978(0.0391)	0.2868(0.0823)	0.1659(0.0275)	0.4201(0.1769)
				3	0.0672(0.0046)	0.2978(0.0887)	0.5373(0.2412)	0.1623(0.0263)	0.3452(0.3241)
			2	2	0.1847(0.0959)	0.1924(0.0370)	0.2886(0.0833)	0.3248(0.1055)	0.3882(0.1511)
				3	0.1024(0.0108)	0.2900(0.0841)	0.4373(0.1913)	0.3196(0.1021)	0.6674(0.4530)
		23	1	2	0.2717(0.7635)	0.1968(0.0387)	0.3142(0.3922)	0.1600(0.0256)	0.3397(0.1154)
				3	0.1667(0.2421)	0.2960(0.0876)	0.4421(0.1955)	0.1583(0.0251)	0.5245(0.2751)
			2	2	0.1466(0.0387)	0.1931(0.0373)	0.2872(0.0825)	0.3188(0.1016)	0.3380(0.1142)
				3	0.1242(0.0279)	0.2888(0.0834)	0.4360(0.1901)	0.3158(0.0997)	0.5200(0.2705)
			1	2	0.0777(0.0070)	0.1981(0.0392)	0.2981(0.0892)	0.1637(0.0268)	0.4486(0.2065)
				3	0.0654(0.0043)	0.2981(0.0888)	0.4435(0.1967)	0.1612(0.0260)	0.5324(0.3521)

60	30	15	2	2	0.1547(0.0261)	0.1932(0.0373)	0.2786(0.0776)	0.3541(0.1337)	0.6219(0.3523)
				3	0.1065(0.0116)	0.2915(0.0850)	0.5729(0.3523)	0.3665(0.1490)	0.3296(0.1106)
		23	1	2	0.0936(0.0143)	0.1977(0.0391)	0.2963(0.0878)	0.1607(0.0258)	0.3681(0.1355)
				3	0.0688(0.0048)	0.2976(0.0885)	0.4430(0.1962)	0.1587(0.0252)	0.8329(0,3242)
			2	2	0.1699(0.0394)	0.1923(0.0370)	0.2846(0.0810)	0.3180(0.1011)	0.3543(0.1256)
				3	0.1014(0.0105)	0.2901(0.0842)	0.4218(0.1779)	0.3144(0.0989)	0.5834(0.3414)
	45	23	1	2	0.0873(0.0121)	0.1978(0.0391)	0.2984(0.0917)	0.1609(0.0259)	0.3727(0.1390)
				3	0.0672(0.0045)	0.2977(0.0886)	0.4429(0.1962)	0.1587(0.0252)	0.8090(0.7680)
			2	2	0.1707(0.0826)	0.1924(0.0370)	0.2841(0.0807)	0.3184(0.1014)	0.3594(0.1294)
				3	0.1019(0.0106)	0.2900(0.0841)	0.4217(0.1779)	0.3145(0.0989)	0.5845(0.3426)
		34	1	2	0.4024(0.2351)	0.1969(0.0388)	0.2903(0.0843)	0.1569(0.0246)	0.3266(0.1066)
				3	0.1109(0.0563)	0.2961(0.0877)	0.4403(0.1939)	0.1558(0.0243)	0.5009(0.2509)
			2	2	0.1450(0.3908)	0.1935(0.0375)	0.2795(0.0781)	0.3140(0.0986)	0.3251(0.1057)
				3	0.1137(0.0158)	0.2891(0.0836)	0.4132(0.1750)	0.3117(0.0971)	0.4964(0.2464)

Table 5: The ABS and the MSEs in parentheses for the parameter α using Picard and Bayes methods at T=3 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) .

n	m	k	α	γ	Picard Estimators	R-K Estimation	Bayes Estimation		
							CH-Prior	Kernel Prior	Gamma Prior
20	10	5	1	2	0.0905(0.0083)	0.0981(0.0096)	0.1452(0.0211)	0.2093(0.0449)	0.1434(0.0628)
				3	0.1019(0.0106)	0.1482(0.0220)	0.1445(0.0209)	0.1872(0.0363)	0.1330(0.0182)
			2	2	0.1521(0.0232)	0.0940(0.0089)	0.2943(0.0866)	0.3475(0.1208)	0.4951(0.4694)
				3	0.1551(0.0241)	0.1414(0.0200)	0.2940(0.0864)	0.3353(0.1125)	0.4407(0.4023)
		8	1	2	0.0907(0.0084)	0.0975(0.0095)	0.1454(0.0211)	0.1747(0.0305)	0.1847(0.0342)
				3	0.1051(0.0114)	0.1474(0.0217)	0.1449(0.0210)	0.1684(0.0284)	0.1973(0.0400)
			2	2	0.1570(0.0247)	0.0952(0.0091)	0.2942(0.0866)	0.3365(0.1132)	0.3600(0.1297)
				3	0.1575(0.0248)	0.1420(0.0202)	0.2938(0.0863)	0.3298(0.1088)	0.3717(0.1394)
	15	8	1	2	0.0914(0.0085)	0.0977(0.0095)	0.1454(0.0211)	0.1746(0.0305)	0.1845(0.0341)
				3	0.1049(0.0113)	0.1474(0.0217)	0.1449(0.0210)	0.1684(0.0284)	0.1978(0.0415)
			2	2	0.1567(0.0246)	0.0954(0.0091)	0.2942(0.0866)	0.3362(0.1131)	0.3590(0.1290)
				3	0.1574(0.0248)	0.1422(0.0202)	0.2938(0.0863)	0.3296(0.1086)	0.3693(0.1373)
		11	1	2	0.0926(0.0087)	0.0974(0.0095)	0.1453(0.0211)	0.1701(0.0289)	0.1743(0.0304)
				3	0.1020(0.0106)	0.1463(0.0214)	0.1448(0.0210)	0.1662(0.0276)	0.1742(0.0304)
			2	2	0.1565(0.0245)	0.0956(0.0091)	0.2940(0.0865)	0.3313(0.1097)	0.3453(0.1192)
				3	0.1576(0.0248)	0.1429(0.0204)	0.2937(0.0862)	0.3260(0.1063)	0.3433(0.1179)
40	20	10	1	2	0.0882(0.0079)	0.0985(0.0097)	0.1434(0.0206)	0.3052(0.4147)	0.1310(0.0204)
				3	0.0971(0.0095)	0.1484(0.0220)	0.1399(0.0196)	0.2216(0.1171)	0.1431(0.0205)
			2	2	0.1523(0.0232)	0.0946(0.0089)	0.2933(0.0860)	0.3287(0.1080)	0.3683(0.1396)
				3	0.1550(0.0240)	0.1415(0.0200)	0.2923(0.0855)	0.3217(0.1035)	0.4532(0.8358)
		15	1	2	0.0882(0.0079)	0.0977(0.0096)	0.1440(0.0207)	0.1640(0.0269)	0.1686(0.0285)
				3	0.1005(0.0102)	0.1475(0.0218)	0.1422(0.0202)	0.1606(0.0258)	0.1742(0.0306)
			2	2	0.1578(0.0249)	0.0961(0.0092)	0.2928(0.0857)	0.3215(0.1034)	0.3308(0.1095)
				3	0.1574(0.0248)	0.1431(0.0205)	0.2918(0.0851)	0.3177(0.1009)	0.3344(0.1119)
	30	15	1	2	0.0877(0.0077)	0.0977(0.0095)	0.1440(0.0208)	0.1641(0.0269)	0.1690(0.0286)
				3	0.1007(0.0103)	0.1475(0.0218)	0.1422(0.0202)	0.1605(0.0258)	0.1732(0.0301)
			2	2	0.1577(0.0249)	0.0955(0.0091)	0.2929(0.0858)	0.3223(0.1039)	0.3328(0.1108)
				3	0.1579(0.0249)	0.1427(0.0204)	0.2918(0.0852)	0.3179(0.1011)	0.3357(0.1128)
		23	1	2	0.0976(0.0096)	0.0977(0.0096)	0.1434(0.0206)	0.1603(0.0257)	0.1617(0.0261)
				3	0.1067(0.0115)	0.1468(0.0215)	0.1414(0.0200)	0.1584(0.0251)	0.1610(0.0259)
			2	2	0.1632(0.0266)	0.0983(0.0097)	0.2916(0.0850)	0.3156(0.0996)	0.3191(0.1018)
				3	0.1636(0.0268)	0.1462(0.0214)	0.2905(0.0844)	0.3137(0.0984)	0.3183(0.1013)
60	30	15	1	2	0.0873(0.0077)	0.0985(0.0097)	0.1391(0.0194)	0.2316(0.0874)	0.1370(0.0565)
				3	0.0964(0.0093)	0.1485(0.0220)	0.0806(0.0073)	0.2154(0.2107)	0.1439(0.0207)
			2	2	0.1527(0.0233)	0.0950(0.0090)	0.2917(0.0851)	0.3210(0.1031)	0.3397(0.1156)
				3	0.1552(0.0241)	0.1415(0.0200)	0.2895(0.0838)	0.3165(0.1001)	0.5060(0.3523)
		23	1	2	0.0871(0.0076)	0.0977(0.0095)	0.1414(0.0200)	0.1597(0.0255)	0.1622(0.0263)
				3	0.0994(0.0100)	0.1475(0.0217)	0.1316(0.0173)	0.1575(0.0248)	0.1645(0.0271)
			2	2	0.1559(0.0243)	0.0960(0.0092)	0.2907(0.0845)	0.3158(0.0997)	0.3212(0.1032)
				3	0.1570(0.0247)	0.1433(0.0206)	0.2882(0.0830)	0.3130(0.0979)	0.3222(0.1038)
			1	2	0.0871(0.0076)	0.0977(0.0095)	0.1414(0.0200)	0.1597(0.0255)	0.1623(0.0264)

	45	23		3	0.0991(0.0099)	0.1475(0.0218)	0.1315(0.0173)	0.1575(0.0248)	0.1644(0.0270)
			2	2	0.1593(0.0254)	0.0960(0.0092)	0.2908(0.0845)	0.3160(0.0999)	0.3215(0.1034)
				3	0.1568(0.0246)	0.1436(0.0206)	0.2881(0.0830)	0.3129(0.0979)	0.3218(0.1036)
		34	1	2	0.0976(0.0096)	0.0978(0.0096)	0.1393(0.0194)	0.1572(0.0247)	0.1580(0.0250)
				3	0.1084(0.0118)	0.1471(0.0216)	0.1242(0.0155)	0.1559(0.0243)	0.1575(0.0248)
			2	2	0.1641(0.0269)	0.0985(0.0097)	0.2886(0.0833)	0.3113(0.0969)	0.3132(0.0981)
				3	0.1644(0.0270)	0.1466(0.0215)	0.2850(0.0812)	0.3100(0.0961)	0.3125(0.0977)

Table 6: The ABS and the MSEs in parentheses for the parameter γ using Picard and Bayes methods at T=3 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) .

n	m	k	α	γ	Picard Estimators	R-K Estimation	Bayes Estimation		
							CH-Prior	Kernel Prior	Gamma Prior
20	10	5	1	2	0.1320(0.7698)	0.1838(0.0339)	0.2814(0.0792)	0.2093(0.0449)	0.8084(0.2353)
				3	0.0961(0.0093)	0.2834(0.0805)	0.4236(0.1795)	0.1872(0.0363)	0.2686(0.3900)
			2	2	0.1615(0.0370)	0.1460(0.0222)	0.2821(0.0796)	0.3475(0.1208)	0.7122(0.5589)
				3	0.1271(0.0168)	0.2197(0.0506)	0.4255(0.1811)	0.3353(0.1125)	0.5634(0.2452)
		8	1	2	0.1083(0.0208)	0.1786(0.0321)	0.2811(0.0790)	0.1747(0.0305)	0.4534(0.2059)
				3	0.0975(0.0097)	0.2770(0.0770)	0.4241(0.1798)	0.1684(0.0284)	0.8350(0.7245)
			2	2	0.0846(0.0235)	0.1576(0.0251)	0.2810(0.0790)	0.3365(0.1132)	0.4454(0.1986)
				3	0.1222(0.0153)	0.2275(0.0525)	0.4242(0.1799)	0.3298(0.1088)	0.7705(0.6032)
	15	8	1	2	0.1059(0.0232)	0.1801(0.0326)	0.2811(0.0790)	0.1746(0.0305)	0.4539(0.2062)
				3	0.0976(0.0098)	0.2769(0.0770)	0.4241(0.1798)	0.1684(0.0284)	0.8314(0.7225)
			2	2	0.1299(0.0318)	0.1587(0.0255)	0.2810(0.0790)	0.3362(0.1131)	0.4440(0.1973)
				3	0.1206(0.0148)	0.2296(0.0534)	0.4241(0.1799)	0.3296(0.1086)	0.7663(0.5953)
		11	1	2	0.1794(0.2435)	0.1780(0.0318)	0.2798(0.0783)	0.1701(0.0289)	0.3909(0.1529)
				3	0.1161(0.0355)	0.2687(0.0725)	0.4224(0.1785)	0.1662(0.0276)	0.6251(0.3913)
			2	2	0.1151(0.0273)	0.1616(0.0263)	0.2797(0.0782)	0.3313(0.1097)	0.3907(0.1526)
				3	0.1188(0.0174)	0.2372(0.0568)	0.4225(0.1785)	0.3260(0.1063)	0.6159(0.3796)
40	20	10	1	2	0.0941(0.0094)	0.1870(0.0350)	0.2752(0.0757)	0.3052(0.4147)	0.2521(0.8506)
				3	0.0946(0.0090)	0.2856(0.0816)	0.3917(0.1535)	0.2216(0.1171)	0.3086(0.0983)
			2	2	0.1474(0.0240)	0.1506(0.0230)	0.2797(0.0782)	0.3287(0.1080)	0.5104(0.2721)
				3	0.1275(0.0165)	0.2205(0.0497)	0.4216(0.1778)	0.3217(0.1035)	0.4635(0.1325)
		15	1	2	0.1063(0.0266)	0.1806(0.0327)	0.2761(0.0762)	0.1640(0.0269)	0.3895(0.1519)
				3	0.0966(0.0094)	0.2778(0.0773)	0.4134(0.1709)	0.1606(0.0258)	0.6768(0.4663)
			2	2	0.1108(0.0126)	0.1652(0.0274)	0.2768(0.0766)	0.3215(0.1034)	0.3801(0.1445)
				3	0.1168(0.0137)	0.2369(0.0564)	0.4175(0.1743)	0.3177(0.1009)	0.6231(0.3894)
	30	15	1	2	0.1030(0.0146)	0.1801(0.0325)	0.2761(0.0762)	0.1641(0.0269)	0.3907(0.1531)
				3	0.0966(0.0094)	0.2780(0.0774)	0.4135(0.1710)	0.1605(0.0258)	0.6702(0.4551)
			2	2	0.1171(0.0139)	0.1599(0.0256)	0.2768(0.0766)	0.3223(0.1039)	0.3830(0.1467)
				3	0.1184(0.0141)	0.2339(0.0550)	0.4175(0.1743)	0.3179(0.1011)	0.6277(0.3955)
		23	1	2	0.1397(0.6687)	0.1810(0.0328)	0.2686(0.0721)	0.1603(0.0257)	0.3431(0.1177)
				3	0.1142(0.0402)	0.2726(0.0744)	0.4012(0.1610)	0.1584(0.0251)	0.5293(0.2802)
			2	2	0.0808(0.0065)	0.1841(0.0339)	0.2701(0.0729)	0.3156(0.0996)	0.3411(0.1164)
				3	0.1018(0.0104)	0.2652(0.0704)	0.4086(0.1669)	0.3137(0.0984)	0.5212(0.2716)
60	30	15	1	2	0.0922(0.0088)	0.1869(0.0350)	0.2238(0.0504)	0.2316(0.0874)	0.3567(0.4326)
				3	0.0938(0.0088)	0.2860(0.0818)	0.4692(0.2201)	0.2154(0.2107)	0.3293(0.1109)
			2	2	0.1409(0.0203)	0.1540(0.0239)	0.2749(0.0756)	0.3210(0.1031)	0.4323(0.1884)
				3	0.1273(0.0164)	0.2210(0.0495)	0.4103(0.1683)	0.3165(0.1001)	0.5635(0.2534)
		23	1	2	0.1085(0.0982)	0.1803(0.0326)	0.2576(0.0664)	0.1597(0.0255)	0.3579(0.1281)
				3	0.0967(0.0094)	0.2777(0.0772)	0.3526(0.0536)	0.1575(0.0248)	0.5909(0.3504)
			2	2	0.1109(0.0126)	0.1644(0.0271)	0.2655(0.0705)	0.3158(0.0997)	0.3522(0.1240)
				3	0.1154(0.0134)	0.2393(0.0574)	0.3909(0.1529)	0.3130(0.0979)	0.5588(0.3125)
	45	23	1	2	0.1028(0.0153)	0.1803(0.0326)	0.2591(0.0694)	0.1597(0.0255)	0.3585(0.1286)
				3	0.0966(0.0094)	0.2779(0.0773)	0.8813(0.1536)	0.1575(0.0248)	0.5901(0.3495)
			2	2	0.1111(0.0125)	0.1638(0.0269)	0.2653(0.0704)	0.3160(0.0999)	0.3527(0.1244)
				3	0.1142(0.0131)	0.2411(0.0582)	0.3908(0.1527)	0.3129(0.0979)	0.5577(0.3113)
		34	1	2	0.1052(0.0195)	0.1816(0.0330)	0.1693(0.0295)	0.1572(0.0247)	0.3293(0.1084)
				3	0.1014(0.0104)	0.2749(0.0756)	0.5239(0.2789)	0.1559(0.0243)	0.5050(0.2550)
			2	2	0.0796(0.0063)	0.1862(0.0347)	0.2383(0.0568)	0.3113(0.0969)	0.3274(0.1072)
				3	0.1005(0.0101)	0.2688(0.0723)	0.3100(0.0969)	0.3100(0.0961)	0.4971(0.2471)