Catastrophe in the Two-regional Economics

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Abstract: On 4-dimensional canards having Brownian motion, we prove that there exists the rigidity for the motion from out side. Furthermore, it is confirmed that there exists "catastrophe" as a potential "hyperbolic umbilic" by R. Thom. Then, we have a question: in the two-region economic model, does it also exist "catastrophe"? Bifurcation for the pseudo singular point, which depends on some parameters has already been analyzed enough to realize it. In the two-regional economic model, it is, however, not yet analyzed still now though. The reason is complex nonlinearity, but the answer is yes. Even in this system, the model has a potential such as the hyperbolic umbilic. Why and how to construct corresponding potentials in order to catch up an orbit jumping by the bifurcation. In this paper, they will be described.

Key-Words: Canards flying, 4-dimensional slow-fast system, Hyper catastrophe

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1 Introduction

In our previous paper, "Hyper catastropheon 4-dimensional canards", Advanced in Pure Mathematics, vol.14, 196-203 (2024), it is confirmed that the neuron system induced from FitzHugh-Nagumo equations has Hyper Catastrophe. On the other hand, in the two-region economics, the nonlinearity is much complex than the neuron system. The investment functions $\phi_i,\ (i = 1, 2)$ cause complex nonlinearity and containing parameters. In this model, there are many parameters, for example, m_i and n_i, (i = 1, 2) regarding import and export factors. Especially, when the parameter $\alpha, (0 < \alpha < 1)$ giving the marginal consumption, the two-regional economy obtains the same potential as the neuron system in case is a linear-cubic polynomial. Under these conditions, the bifurcation structure is quite same as the neuron system. Although these systems look like different each other at a glance, they are composed of common factors on the bifurcation structure. Regarding the original catastrophe, see [1],[2],[3] and [4].

2 Preliminary

Now, let us consider the following slow-fast system:

$$\begin{cases} \varepsilon \frac{dx}{dt} = h\left(x, y\right) \\ \dfrac{dy}{dt} = g\left(x, y\right) \end{cases}, \tag{1}$$

where ϵ is infinitesimal and

$$x = (x_1, x_2) \in R^2, \quad y = (y_1, y_2) \in R^2,$$

$$h = \left(h_1\left(x, y\right), h_2\left(x, y\right)\right) : R^4 \to R^2,$$

$$g = \left(g_1\left(x, y\right), g_2\left(x, y\right)\right) : R^4 \to R^2.$$

Then, assume that the origin is a singular point.

Furthermore, we assume that the system (1) satisfies the following conditions (A1) $\sim$ (A6), they are the same as describing our previous paper.

(A1) h is of class C^1 and g is of class C^2.

(A2) The slow manifold $S = \left\{(x, y) \in \mathrm{R}^4 \left| \mathrm{h}\left(x, y\right) = 0\right.\right\}$ is a two-dimensional differential manifold and intersects the set

$$T = \left\{ (x, y) \in \mathrm{R}^4 \middle| \det\left[\frac{\partial h}{\partial x}\left(x, y\right)\right] = 0 \right\} \tag{2}$$

transversely, where

$$\frac{\partial h}{\partial x}=\left[\begin{array}{cc}\dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial x_2}\\ \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial x_2}\end{array}\right] \tag{3}$$

Then, the pli set

$$PL=\{(x,y)\in S\cap T\} \tag{4}$$

is a one-dimensional differentiable manifold.

(A3) Either the value of g_1 or that of g_2 is nonzero at any point of PL.

Note that the pli set PL divides the slow manifolds $S\backslash\mathrm{PL}$ into three parts depending on the signs of the two eigenvalues of $\frac{\partial h}{\partial x}(x,y)$.

First, consider the following reduced system which is obtained from (1) with $\varepsilon=0$:

$$\begin{cases} 0=h(x,y)\\ \dfrac{dy}{dt}=g(x,y)\end{cases} \tag{5}$$

By differentiating $h(x,y)$ with respect to t, we have

$$\frac{\partial h}{\partial x}(x,y)\frac{dx}{dt}+\frac{\partial h}{\partial y}(x,y)\,g(x,y)=0 \tag{6}$$

Then (4) becomes the following:

$$\begin{cases} \dfrac{dx}{dt}=-\left[\dfrac{\partial h}{\partial x}(x,y)\right]^{-1}\dfrac{\partial h}{\partial y}(x,y)\,g(x,y)\\ \dfrac{dy}{dt}=g(x,y)\end{cases}, \tag{7}$$

where $(x,y)\in S\backslash\mathrm{PL}$. To avoid degeneracy in (6), we consider the time-scaled-reduced system:

$$\begin{cases} \dfrac{dx}{dt}=\left\{-\det\left[\dfrac{\partial h}{\partial x}(x,y)\right]^{-1}\right\}\left[\dfrac{\partial h}{\partial x}(x,y)\right]^{-1}\\ \qquad\qquad\times\dfrac{\partial h}{\partial y}(x,y)\,g(x,y)\\ \dfrac{dy}{dt}=\left\{\det\left[\dfrac{\partial h}{\partial x}(x,y)\right]^{-1}\right\}g(x,y)\end{cases}. \tag{8}$$

The phase portrait of the system (8) is the same as that of (7) except the region where $\det\left[\frac{\partial h}{\partial x}(x,y)\right]=0$, but only the orientation of the orbit is different.

Definition 1. A singular point of (8), which is on PL, is called a pseudo singular point of (1). The set of pseudo singular points is denoted by PS.

(A4) $\quad rank\left[\dfrac{\partial h}{\partial x}(x,y)\right]=2$, and $rank\left[\dfrac{\partial h}{\partial y}(x,y)\right]=2$ for any $(x,y)\in S\backslash\mathrm{PL}$.

From (A4), the implicit function theorem guarantees the existence of a unique function $y=\varphi(x)$ such that $h(x,\varphi(x))=0$. By using $y=\varphi(x)$, we obtain the following system:

$$\frac{dx}{dt}=\left\{-\det\left[\frac{\partial h}{\partial x}(x,\varphi(x))\right]^{-1}\right\} \tag{9}$$

$$\times\left[\frac{\partial h}{\partial x}(x,\varphi(x))\right]^{-1}\frac{\partial h}{\partial y}(x,\varphi(x))\,g(x,\varphi(x)).$$

(A5) All singular points of (8) are non-degenerate, that is, the linearization of (8) at a singular point has two nonzero eigenvalues.

Now, let us introduce a definition of ”symmetry”. It is a key word through this paper.

Definition 2. If $h_1(x_1,x_2,y_1,y_2)=h_2(x_2,x_1,y_2,y_1)$, and $g_1(x_1,x_2,y_1,y_2)=g_2(x_2,x_1,y_2,y_1)$, then the system is ”symmetric” for the subspace $I=\{(x_1,x_2,y_1,y_2)|x_1=x_2,y_1=y_2\}$.

(A6) I intersects PL transversely.

Definition 3. Let $\lambda_1,\ \lambda_2$ be two eigenvalues of the linearization of (8) at a pseudo singular point. The pseudo singular point with real eigenvalues is called a pseudo singular saddle point if $\lambda_1<0<\lambda_2$ and a pseudo singular node point if $\lambda_1<\lambda_2<0$ or $\lambda_1>\lambda_2>0$.

The following Theorems 1 is already established in our previous paper (Trends in Mathematics, 11, 193-199 (2018)).

Theorem 1. Let (x_0,y_0) be a pseudo singular saddle or node point. If trace $\left[\dfrac{\partial h}{\partial x}(x_0,y_0)\right]<0$, then there exists a solution which first follows the attractive part and the repulsive part after crossing PL near the pseudo singular point.

Remark 1. The solution in Theorem 1 is called “canard”.

Remark 2. The condition trace $\left[\frac{\partial h}{\partial x}(x_0, y_0)\right] < 0$ implies that one of eigenvalues of $\left[\frac{\partial h}{\partial x}(x_0, y_0)\right]$ is equal to zero and the other one is negative. Notice that the system has two kinds of vector fields: one is 2-dimensional slow and the other is 2-dimensional fast one. The condition provides the state of the fast vector field.

Remark 3. The singular solution in Theorem 1 is called a canard in R^4 with 2-dimensional slow manifold. As a result, it causes a delayed jumping. The study of canards requires still more precise topological analysis on the slow vector field.

Remark 4. On the subspace I, the following system is established for some b. I is an invariant manifold.

$$\begin{cases} \varepsilon \dfrac{dx_1}{dt} = h_1(x_1, y_1) \\ \dfrac{dy_1}{dt} = g_1(x_1, y_1) \end{cases}. \tag{10}$$

Remark 5. On the set PL, $det\left[\frac{\partial h}{\partial x}\right] = 0$ is satisfied and at $(x_0, y_0) \in PS$ the following equation is established:

$$\begin{aligned} &\frac{\partial h_1}{\partial x_1}(x_0, \varphi(x_0)) \frac{\partial h_2}{\partial x_2}(x_0, \varphi(x_0)) \\ &\times g_1(x_0, \varphi(x_0)) \\ &-\frac{\partial h_1}{\partial x_2}(x_0, \varphi(x_0)) \frac{\partial h_2}{\partial x_1}(x_0, \varphi(x_0)) \\ &\times g_1(x_0, \varphi(x_0)) = 0. \end{aligned} \tag{11}$$

Note that there exists $y = \phi(x)$ because of assuming $rank\left[\frac{\partial h}{\partial y}\right] = 2$.

3 Bifurcation on slow/fast vectors

From now on, let us consider the following system extended having parameters for slow/fast vectors:

$$\begin{cases} \varepsilon \frac{dx}{dt} = h(\alpha x, \beta y) \\ \dfrac{dy}{dt} = g(\alpha x, \beta y) \end{cases}, \tag{12}$$

where $\alpha = (b_1, b_2)$, $\beta = (b_3, b_4)$ and b_1, b_2, b_3, b_4 are bifurcation parameters.

On the other hand,

$$det\left[\frac{\partial h}{\partial x}\right] = b_1 b_2 \left\{\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right\}. \tag{13}$$

In fact,

$$\begin{aligned} &\frac{\partial h_1(b_1x_1, b_2x_2, b_3y_1, b_4y_2)}{\partial x_1} \\ &= b_1 \frac{\partial h_1(x_1, x_2, b_3y_1, b_4y_2)}{\partial x_1}, \\ &\frac{\partial h_1(b_1x_1, b_2x_2, b_3y_1, b_4y_2)}{\partial x_2} \\ &= b_2 \frac{\partial h_1(x_1, x_2, b_3y_1, b_4y_2)}{\partial x_2}, \\ &\frac{\partial h_2(b_1x_1, b_2x_2, b_3y_1, b_4y_2)}{\partial x_2} \\ &= b_1 \frac{\partial h_2(x_1, x_2, b_3y_1, b_4y_2)}{\partial x_2}, \\ &\frac{\partial h_2(b_1x_1, b_2x_2, b_3y_1, b_4y_2)}{\partial x_2} \\ &= b_2 \frac{\partial h_2(x_1, x_2, b_3y_1, b_4y_2,)}{\partial x_2}. \end{aligned} \tag{14}$$

Lemma 1.

$$det\left[\frac{\partial h}{\partial x}\right] = \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} = 0. \tag{15}$$

It does not depend on bifurcation parameter b_1, b_2.

Theorem 2. If the system having parameters for slow/fast vectors is "symmetric", then it has a potential classified by R. Thom.

Proof. Under the rank condition $(A4)$, proceeding a projection (changing the co-ordinates):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{16}$$

like as $y_1 + y_2 = Y, x_1 + x_2 = X$, then, the potentials are obtained as "elementary catastrophe" under the following conditions.
At around $(x_0, y_0) \in PS$,

(i) $\frac{\partial^3 h_1}{\partial x_1^3} \neq 0, \quad \frac{\partial h_1}{\partial x_1} \neq 0.$

(ii) $\frac{\partial^4 h_1}{\partial x_1^4} \neq 0, \quad \frac{\partial^2 h_1}{\partial x_1^2} \neq 0, \quad \frac{\partial h_1}{\partial x_1} \neq 0.$

(iii) $\frac{\partial^5 h_1}{\partial x_1^5} \neq 0, \frac{\partial^2 h_1}{\partial x_1^2} \neq 0, \frac{\partial h_1}{\partial x_1} \neq 0, \frac{\partial^4 h_1}{\partial x_1^4} = 0$.

(iv) $\frac{\partial^6 h_1}{\partial x_1^6} \neq 0, \frac{\partial^4 h_1}{\partial x_1^4} \neq 0, \frac{\partial^3 h_1}{\partial x_1^3} \neq 0, \frac{\partial^2 h_1}{\partial x_1^2} \neq 0, \frac{\partial h_1}{\partial x_1} \neq 0.$

(v) $\frac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \frac{\partial^3 h_2}{\partial x_2^3} \neq 0, \frac{\partial^2 h_1}{\partial x_1^2} \neq 0, \frac{\partial h_2}{\partial x_2} \neq 0,$
$\frac{\partial^3 h_1}{\partial x_2^2 \partial x_1} = 0, \frac{\partial^2 h_1}{\partial x_1 \partial x_2} = 0.$
(vi) $\frac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \frac{\partial^3 h_2}{\partial x_2^3} \neq 0, \frac{\partial^2 h_1}{\partial x_1^2} \neq 0,$
$\frac{\partial h_1}{\partial x_2} \neq 0, \frac{\partial h_1}{\partial x_1} \neq 0.$
(vii) $\frac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \frac{\partial^4 h_2}{\partial x_2^4} \neq 0, \frac{\partial^2 h_2}{\partial x_2^2} \neq 0, \frac{\partial^2 h_1}{\partial x_2^1} \neq 0,$
$\frac{\partial h_2}{\partial x_2} \neq 0, \frac{\partial h_1}{\partial x_1} \neq 0.$

□

Remark 6. As the system is "symmetric", the conditions are described exclusively, for example, the condition (1) is as the following,

$$\frac{\partial^3 h_2}{\partial x_2^3} \neq 0, \quad \frac{\partial h_2}{\partial x_2} \neq 0.$$

Remark 7. It is called "Hyper Catastrophe", which is composed of the potential reduced from the slow manifold ($\epsilon = 0$). Although it is using non-standard analysis, for example ϵ is infinitesimal, "Transfer Principle" ensures that it is established in standard analysis. Then, the slow manifold is obtained as the singular limit (ϵ tends to zero). They are "dynamical catastrophe" but not "statical one".

Theorem 3. There exists another bifurcation such that the X-coordinate is changing by the parameters.

Proof. The capital $Y = y_1 + y_2$ in the equation (5) includes bifurcation parameters α and β implicitly. Because the implicit function theorem ensures that there exists a function $y = \phi(x, \alpha, \beta)$ where $y = (y_i = \phi_i(x, b_1, b_2, b_3, b_4))(i = 1, 2)$ by taking $\epsilon = 0$. Then, the corresponding capital X axis, which satisfies $Y = 0$, sometimes changes the sign, i.e., switching the direction. In fact, let $y = (y_1, y_2) = (b_2 x_2, b_1 x_1)$ be satisfied for the simplicity. There are two possibility, i.e., $b_1 + b_2 > 0$ or $b_1 + b_2 < 0$. It causes another bifurcation as the jumping direction changes. □

4 Concrete example of the neuron system

Consider the equation,

$$\begin{cases} \varepsilon \frac{dx_1}{dt} = b_2 x_2 + b_3 y_1 + \frac{b_1^3 x_1^3}{3} \\ \varepsilon \frac{dx_2}{dt} = b_1 x_1 + b_4 y_2 + \frac{b_2^3 x_2^3}{3} \\ \frac{dy_1}{dt} = -(b_1 x_1 + b_3 y_1) \\ \frac{dy_2}{dt} = -(b_2 x_2 + b_4 y_2) \end{cases}, \tag{17}$$

where $b_i (i = 1, ..., 4)$ are bifurcation parameters.

Changing the coordinates like as

$$\begin{cases} x_1 + x_2 = X \\ \frac{3b_3}{b_1^3} y_1 + \frac{3b_4}{b_2^3} y_2 = Y \end{cases}, \tag{18}$$

then on the axis X, getting the function

$$Y = x_1^3 + x_2^3 - \frac{3b_2}{b_1^3} x_2 - \frac{3b_1}{b_2^3} x_1. \tag{19}$$

This potential function is "hyperbolic umbilic" classified by R. Thom.

Using variables X, Y, when satisfying $b_1 = b_2$,

$$Y = X^3 - kX + O(X^3) \quad \text{and} \quad k = \frac{3}{b_1^2}. \tag{20}$$

Corollary 1. If $\frac{\partial^2 h_1}{\partial x_1 \partial x_2}$ or $\frac{\partial^2 h_2}{\partial x_2 \partial x_1}$ takes nearly equal 0, there exists "hyperbolic umbilic". Changing the coordinates, it is "fold" catastrophe.

Remark 8. In the concrete example, the values corresponding partial derivatives are at around the origin. For the simplicity, it is not at the pseudo singular point.

In our previous paper "Hyper Catastrophe on 4-dimensional canards" , Advances in Pure Mathematics, 2024, 14, 196-203, the following is established.

Theorem 4. The system induced from FitzHugh-Nagumo equation has "Hyper catastrophe" at around the pseudo singular point. The corresponding potential is "hyperbolic umbilic".

5 The two-regional economic model

As a concrete model, we consider a two-region business cycle model between two nations A and B as followings. See Nonlinear Studies, 19, 39-55 (2012) for more details of the two-region business cycle model.

$$
\begin{cases}
\varepsilon \dfrac{dx_1}{dt} = -\dfrac{1-\alpha+m_1}{\theta} y_1 + \dfrac{m_2}{\theta} y_2 \\
-\left(\dfrac{\varepsilon}{\theta} + 1 - \alpha\right)(x_1 - a) \\
+\dfrac{1-n_1}{\theta} \varphi_1 (x_1 - a) + \dfrac{n_2}{\theta} \varphi_2 (x_2 - a), \\
\varepsilon \dfrac{dx_2}{dt} = \dfrac{m_1}{\theta} y_1 - \dfrac{1-\alpha+m_2}{\theta} y_2 \\
+\dfrac{1-n_1}{\theta} \varphi_1 (x_1 - a) \\
-\left(\dfrac{\varepsilon}{\theta} + 1 - \alpha\right)(x_2 - a) + \dfrac{1-n_2}{\theta} \varphi_2 (x_2 - a), \\
dy_1 = (x_1 - a)\, dt \\
dy_2 = (x_2 - a)\, dt
\end{cases} \tag{21}
$$

for $0 \leq t \leq T$, where $x_1(t)$ and $x_2(t)$ are exports of A and B, $m_1(t)$ and $m_2(t)$ are imports of A and B, $y_1(t) - \frac{q}{1-\alpha}$ and $y_2(t) - \frac{q}{1-\alpha}$ are national income identities of A and B for some constants q and α, respectively.

Now, let us introduce a difference equations for the system (21). Then, the relations $\Delta t = \frac{T}{N}$ and $t_k = k\frac{T}{N},\ k = 0, 1, \cdots, N$ are satisfied, where N is a hyper finite. Put

$$
\begin{aligned}
&\frac{\varepsilon}{\Delta t}\{x_1(t_k) - x_1(t_{k-1})\} \qquad (22)\\
&= -\frac{1-\alpha+m_1}{\theta} y_1(t_{k-1}) + \frac{m_2}{\theta} y_2(t_{k-1}) \\
&\quad -\left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_1(t_{k-1}) - a) \\
&\quad +\frac{1-n_1}{\theta} \varphi_1 (x_1(t_{k-1}) - a) \\
&\quad +\frac{n_2}{\theta} \varphi_2 (x_2(t_{k-1}) - a),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\varepsilon}{\Delta t}\{x_2(t_k) - x_2(t_{k-1})\} \qquad (23)\\
&= \frac{m_1}{\theta} y_1(t_{k-1}) - \frac{1-\alpha+m_2}{\theta} y_2(t_{k-1}) \\
&\quad +\frac{1-n_1}{\theta} \varphi_1 (x_1(t_{k-1}) - a) \\
&\quad -\left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_2(t_{k-1}) - a) \\
&\quad +\frac{1-n_2}{\theta} \varphi_2 (x_2(t_{k-1}) - a),
\end{aligned}
$$

$$y_1(t_k) - y_1(t_{k-1}) = (x_1(t_{k-1}) - a)\, \Delta t, \tag{24}$$

$$y_2(t_k) - y_2(t_{k-1}) = (x_2(t_{k-1}) - a)\, \Delta t. \tag{25}$$

Furthermore put

$$\varphi_1(x-\alpha) = \varphi_2(x-\alpha) = (1-\alpha)\left(\theta x + x^2 - \frac{x^3}{3}\right). \tag{26}$$

$$Y = y_1(t_k) - \sigma_1 \Delta B_{1k} + y_2(t_k) - \sigma_2 \Delta B_{2k} \tag{27}$$

$$\Delta B_{1k} - \Delta B_{1k-1} = \mu_{k-1}, \tag{28}$$

where

$$\Delta B_{11} - \Delta B_{10} = \mu_0, \quad \Delta B_{10} = 0. \tag{29}$$

Take a note that μ_k for each k is a step function, that is, a constant.

Applying the conditions in the poof of Theorem 2, we provide the following main theorem.

Theorem 5. The two-regional economics and the neuron system induced from FitzHugh-Nagumo equations are consist of the common catastrophe.

Proof. When $\epsilon = 0$,

$$
\begin{gathered}
(\alpha-1)y_1 + (\alpha-1)y_2 + n_1\phi_1(x_1-\alpha) \qquad (30)\\
-\theta(1-\alpha)x_1 + (1-n_1)\phi_1(x_1-\alpha) \\
+n_2\phi_2(x_2-\alpha) + (1-n_2)\phi_2(x_2-\alpha) \\
-\theta(1-\alpha)x_2 = 0.
\end{gathered}
$$

Furthermore, we have

$$\frac{y_1+y_2}{\theta} = -\phi_1(x_1) - \phi_2(x_2) + \frac{\epsilon(x_1+x_2)}{\theta}. \tag{31}$$

□

6 Simulation results

In this section, let us provide computer simulations for the equations (22), (23), (24) and (25). When doing simulations $\Delta t = 1/N$ and N takes a standard number.

In Appendix in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6, the line $x_1 = x_2$ is an invariant manifold. Furthermore, $\varepsilon = 0.01$ and $\Delta \mathrm{t} = 0.0005$ in (22), (23), (24) and (25). The Pli set satisfying $det[h_x] = 0$ is the fixed curves.

Remark 9. Take a note that the curves of $det[h_x] = 0$ are changed when the parameters $(m_i, n_i)(i = 1, 2)$ and α are changed. Because the functions ϕ_1 and ϕ_2 are composed of these parameters. If only the parameter α takes nearly equal 1, the parameters $b_i(i = 1, 2, 3, 4)$ are nearly equal to $(m_i, n_i)(i = 1, 2)$. If not, it is not satisfied with the condition in Lemma 1.

Figure 1 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 50\}$ satisfying the equation (22) with $\alpha = 0.01, m_1 = 0.4, m_2 = 0.1, n_1 = 0.1, n_2 = 0.4$ and starting from $(x_1(t_0), x_2(t_0)) = (1.0, -1.0)$ near the the pseudo singular point $\left(\sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)}, -\sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)}\right)$. The orbit converges to the invariant manifold $x_1 = x_2$. Then, corresponding potential is $Y = x_1^3 + x_2^3 - 3x_1 - 3x_2$.

Figure 2 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 50\}$ satisfying the equations (22), (23), (24) and (25) with $\alpha = 0.01, m_1 = 0.1, m_2 = 0.8, n_1 = 0.8, n_2 = 0.1$ and starting from $(1.0, -1.0)$ near the the pseudo singular point.. From Fig.2 we observe that the orbit does not converge to $x_1 = -x_2$, which is not the invariant manifold.

Figure 3 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 50\}$ satisfying the equations (22), (23), (24) and (25) with $\alpha = 0.01, m_1 = 0.8, m_2 = 0.1, n_1 = 0.1, n_2 = 0.8$ and starting from $(1.0, -1.0)$ near the the pseudo singular point.. From Fig.2 we observe that the orbit does not converge to $x_1 = -x_2$, which is not the invariant manifold. Take a note that the Pli set, which is consist of $det[h_x] = 0$, is changed like as pointing out in "remark 9".

Figure 4 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 50\}$ satisfying the equations (22), (23), (24) and (25) with $\alpha = 0.99, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$ and starting from $(1.0, -1.0)$ near the the pseudo singular point. From Fig.4 we observe that the orbit does not converge to $x_1 = -x_2$, which is not the invariant manifold.

Figure 5 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 2\}$ satisfying the equations (22), (23), (24) and (25) with $\alpha = 0.4, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$ and starting from $(1.0, -1.0)$ near the the pseudo singular point. The orbit converges to the invariant manifold $x_1 = x_2$.

Figure 6 shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \le t \le T = 2\}$ satisfying the equations (22), (23), (24) and (25) with $\alpha = 0.7, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$ and starting from $(1.0, -1.0)$ near the the pseudo singular point.

7 Conclusion

In this paper, we confirm that the neuron system and the 2-regional economics has common catastrophe. Our brain-storming should be reflected in the social effects. It might be a very natural conclusion in the sense of standard world. When the parameter α takes nearly equal to 1, the marginal cost is very high level. It may look like very strange, however, it sometimes happens. When α takes nearly equal to 0, the system has the same potential. The investment are controlled using computer with high frequency. It is confirmed on Fourier analysis, see [5]. Regarding the mathematical economics, it has been already described in [6], [7], [8], [9], [10] and [11].

Furthermore, in some countries, the national debt becomes very big. It causes the same thing as the parameter α takes nearly equal 1. Therefore, those situations are always realistic in our current economy. The parameters m_i and n_i, $(i = 1, 2)$ regarding import and export factors make an important role to characterize the bifurcation structure. Take a note that when the parameter α takes nearly equal to 1, (m_i, n_i), $(i = 1, 2)$ nearly equal (α, β) for the bifurcation on all slow/fast vectors.

In general, it is hard for us to regulate them in our mind. When we can choose suitable values of these parameters, it might be possible to moderate the current economy before crashing. For example, if we choose an economy downsizing or doing decentralization, it makes us much easier to operate the parameters.

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Appendix

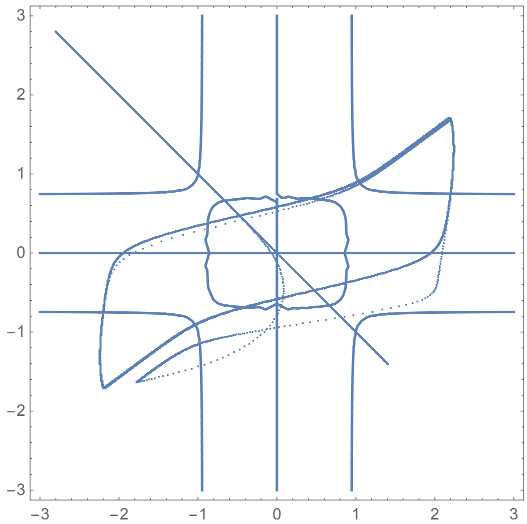

Figure 1. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.01, m_1 = 0.4, m_2 = 0.1, n_1 = 0.1, n_2 = 0.4$

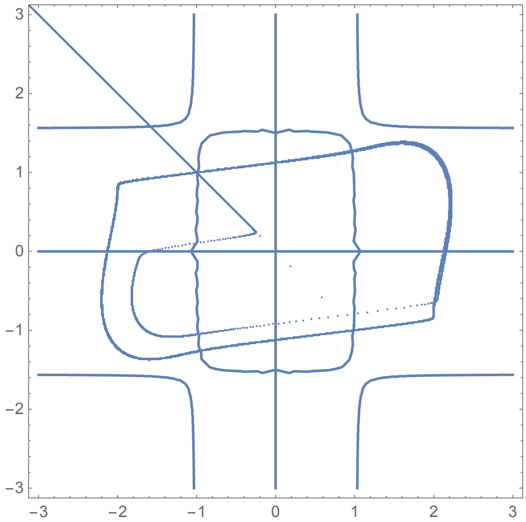

Figure 2. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.01, m_1 = 0.8, m_2 = 0.1, n_1 = 0.1, n_2 = 0.8$

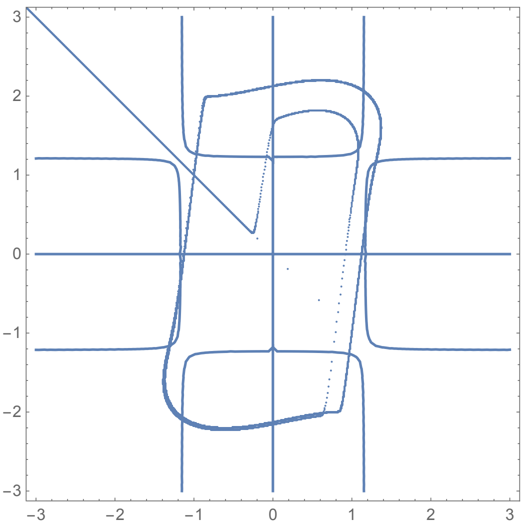

Figure 3. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.01, m_1 = 0.1, m_2 = 0.8, n_1 = 0.8, n_2 = 0.1$

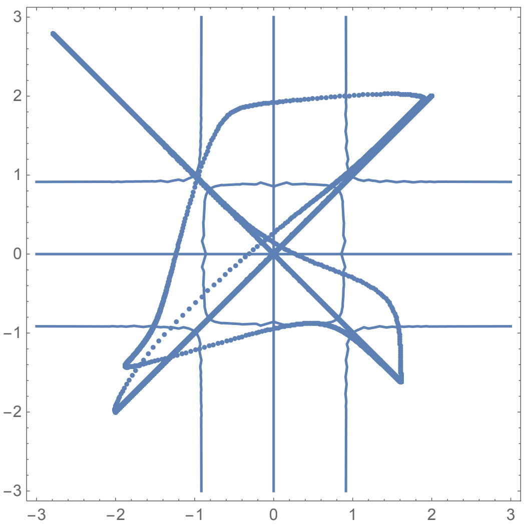

Figure 5. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.4, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$

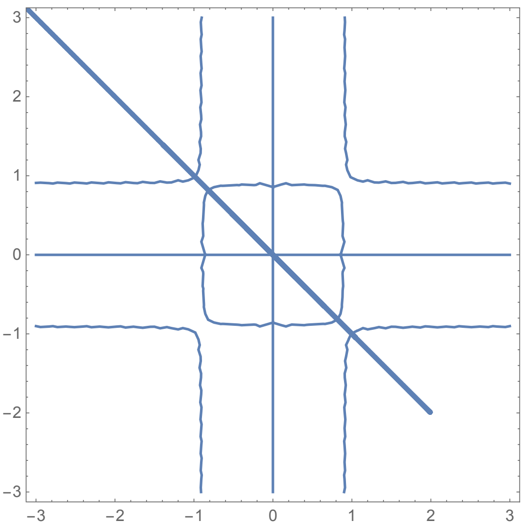

Figure 4. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.99, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$

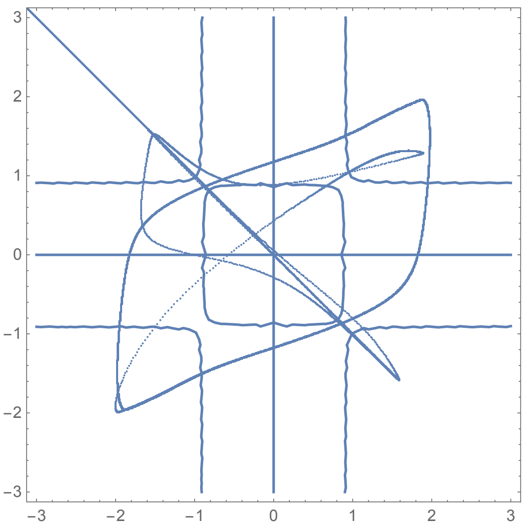

Figure 6. $(x_1(0), x_2(0)) = (1.0, -1.0)$, $\alpha = 0.7, m_1 = 0.3, m_2 = 0.2, n_1 = 0.1, n_2 = 0.8$