Soliton Solutions and Modulation Instability of the Hirota Equation

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Abstract: - In this work, the generalized Riccati mapping method is employed to derive several classes of exact analytical solutions of the Hirota equation, which represents a higher-order nonlinear Schrödinger model describing the propagation of ultrashort optical pulses in nonlinear optical fibers. Depending on the selected parameters, bright, dark, singular, hyperbolic, trigonometric, and rational wave structures are obtained. The validity of the derived solutions is confirmed through direct substitution into the governing equation. Furthermore, a detailed modulation instability analysis is performed by perturbing the continuous-wave background solution and deriving the corresponding dispersion relation. The instability gain spectrum is derived explicitly, and the influence of the physical parameters on wave stability is investigated. The obtained results contribute to the understanding of nonlinear pulse dynamics and modulation instability phenomena in optical communication systems.

Key-Words: - Hirota equation, Generalized Riccati mapping method, Modulation instability

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1 Introduction

Solitary waves, which entered the scientific literature with the studies of John Scott Russell in 1834, did not receive much attention until the 1960s. The pioneering studies conducted at Los Alamos National Laboratory by Norman J. Zabusky and Martin D. Kruskal significantly increased interest in solitary waves. Kruskal and Zabusky showed that the Fermi-Pasta-Ulam-Tsingou model associated with the propagation of equal-mass longitudinal waves by certain nonlinear springs in a one-dimensional lattice pairs are modeled by the KdV equation. Kruskal and Zabusky discovered that solitary waves emerged from collisions while preserving their shapes and velocities which they entered when experimenting with the numerical solution of the KdV equation. They called these waves “solitons” because they exhibit particle-like behavior similar to that of photons and protons. In Greek, the suffix "on" means solitary.

Mathematicians often prefer to distinguish between solitons and solitary waves. They call solitons solitary waves that do not deform after colliding with other solitons. In this case, solitary waves are of a wider variety than solitons.

The next question is: How are solitons formed? Solitons arise from a balance between nonlinear effects and dispersive effects.

Does every nonlinear partial differential equation admit soliton solutions? KdV equation, Boussinesq equation, Burgers equation, Sine-Gordon equation and Schrödinger equation are the most famous partial differential equations that cause to soliton solutions [1], [3].

2 Governing Model

In this paper, our governing equation is a generalized third-order Schrödinger equation reads as

$$iU_t + aU_{xx} + bU|U|^2 + icU_{xxx} + ih|U|^2 U_x = 0, \quad (1)$$

where x is propagation variable, t is time variable and a, b, c, h are real constants. This equation is known as Hirota-type generalized nonlinear Schrödinger equation. For the values $c = h = 0$, it is reduced to the Schrödinger equation and for the values $a = b = 0$, the equation reduces to the modified KdV equation [4-5].

The Hirota equation is one of the most important integrable higher-order nonlinear Schrödinger equations and arises in the modeling of ultrashort

optical pulse propagation. In contrast to the standard nonlinear Schrödinger equation, it includes third-order dispersion and self-steepening effects, which become significant in femtosecond pulse dynamics. Consequently, obtaining exact analytical solutions of the Hirota equation remains an important research topic in nonlinear optics and mathematical physics.

The Hirota equation was introduced in reference [4] and has been studied by many researchers. Yan [5] applied Generalized new auxiliary equation method and searched the envelope compacton solutions; Yomba [6] applied an indirect F-function method, Ankiewicz et al. [7] had modified the Darboux transformation method to find rational solutions of the Hirota equation, Kedziora et al. [8] derived a hierarchy of higher-order integrable NLS equations, Da-Wei and Gui-Fang [9] obtained higher-order solutions of generalized Hirota equations by the Hirota bilinear method, Yaning and et all. [10] solved the generalized Hirota equation with an external potential by means of the new Jacobi elliptic function rational expansion method and the exponential rational function method, Tingyou and et all. [11] produced general Nth-order rogue wave solution of Hirota equation with the help of bilinear method and KP hierarchy reduction method.

3 The Generalized Riccati Mapping Scheme

We present the algorithm of the generalized

Riccati mapping method for obtaining soliton solution of nonlinear partial differential equations. Suppose general form of a nonlinear PDE in two independent as

$$P\left(U,U_x,U_t,U_{xx},...\right)=0, \qquad (2)$$

where the $U\left(x,t\right)$ is unknown function and P polynomial function is including with linear and nonlinear terms of the derivatives of the $U\left(x,t\right)$. The basic steps of this technique are given below:

Step 1: We suppose complex the traveling wave transform independent variables into the single variables as

$$U\left(x,t\right)=\chi\left(\xi\right)e^{iP}, \qquad \chi\left(\xi\right)=\sum_{j=-N}^{N}A_j\varphi^j\left(\xi\right) \quad (3)$$

where $P=\delta x+\lambda t+\theta$ and $\xi=kx+\omega t$. χ is the amplitude function of the wave profile, P is the phase, δ is the carrier, λ is the carrier frequency parameter, θ is a constant phase, k is wave nuber and ω is the frequency of the envelope. A_j are real constants and $\varphi\left(\xi\right)$ satisfies the equation

$$\varphi'\left(\xi\right)=p+q\varphi\left(\xi\right)+r\varphi^2\left(\xi\right) \qquad (4)$$

where p, q, r are real constants. Depending on the sign of $\Delta=q^2-4pr$. Eq. (4) has hyperbolic, trigonometric, exponential and rational solutions; they are listed in Section 4.

Step 2: The integer N is determined by the balancing higher order nonlinear and linear terms on Eq. (3) and set of the coefficients A_j,k,ω,p,q and r are can be obtained.

Step 3: Inserting Eq. (3) with Eq. (4) into Eq.(2), choosing the coefficients of dissimilar exponents of $\varphi^j\left(\xi\right)$ to zero, get a set of equations, which is solved by using software Mathematica, then the parameter values can be obtained.

Step 4: If we place these values in Eq. (3), then Eq. (2) has a solution [12], [13], [14], [15].

4 Application of the method to Hirota NLSE

Since Eq. (1) is complex, we insert Eq. (3) into Eq. (1) with real-valued χ (the separation into real and imaginary parts is legimate only for real χ). Using $U_t=\left(\omega\chi'+i\lambda\chi\right)e^{iP}$, $U_X=\left(k\chi'+i\delta\chi\right)e^{iP}$, $U_{XX}=\left(k^2\chi''+2ik\chi'-\delta^2\chi\right)e^{iP}$ and $U_{XXX}=(k^3\chi'''+3ik^2\delta\chi''-3k\delta^2\chi'-i\delta^3\chi)e^{iP}$, we get real and imaginary parts, respectively as

$$\left(-\lambda-a\delta^2+c\delta^3\right)\chi+\left(b-h\delta\right)\chi^3+$$
$$\left(ak^2-3ck^2\delta\right)\chi"=0, \qquad (5)$$
$$\left(\omega+2ak\delta-3ck\delta^2\right)\chi'+hk\chi^2\chi'+$$
$$ck^3\chi^{(3)}=0. \qquad (6)$$

Integrating Eq. (6) once and taking integration constants zero, we have as

$$\left(\omega+2ak\delta-3ck\delta^2\right)\chi+\frac{hk}{3}\chi^3+$$
$$ck^3\chi"=0\ . \qquad (7)$$

As can be seen, Eqs. (5) and (7) are identical in form. Thus, we get the following relations between the coefficients

$$a = c\left(k + 3\delta\right), \quad h = \frac{3b}{k + 3\delta},$$
$$\lambda = -2ck^2\delta - 4ck\delta^2 - 2c\delta^3 - \omega. \qquad (8)$$

By using homogenous balancing principle on Eq. (7), (the terms χ'' and χ^3 give $N+2 = 3N$, i.e. $N = 1$), we get the solution as

$$\chi\left(x,t\right) = \frac{A_{-1}}{\varphi} + A_1\varphi + A_0 \qquad (9)$$

Substituting Eq. (9) and (4) into Eq. (7) and collecting the same powers of φ^i to zero, our system of equations are consisting in parameters $A_{-1}, A_0, A_1, k, \delta, \omega, a, c, h, p, q, r$. Solving this system of equations with the aid of Mathematica, we attain following cases:

Case 1:

$$A_{-1} = 0, \quad A_0 = \frac{\mathrm{i}\sqrt{\frac{3c}{2}}kq}{\sqrt{h}}, \quad A_1 = \frac{\mathrm{i}\sqrt{6c}kr}{\sqrt{h}},$$
$$\omega = k\left(3c\delta^2 - 2a\delta\right) + \frac{ck^3}{2}\left(q^2 - 4pr\right). \qquad (10)$$

Case 2:

$$A_{-1} = \frac{\mathrm{i}\sqrt{6c}kp}{\sqrt{h}}, \quad A_0 = \frac{\mathrm{i}\sqrt{\frac{3}{2}c}kq}{\sqrt{h}}, \quad A_1 = 0,$$
$$\omega = k\left(3c\delta^2 - 2a\delta\right) + \frac{ck^3}{2}\left(q^2 - 4pr\right). \qquad (11)$$

Case 3:

$$A_{-1} = \frac{\mathrm{i}\sqrt{6c}kp}{\sqrt{h}}, \quad A_0 = 0, \quad A_1 = \frac{\mathrm{i}\sqrt{6c}kr}{\sqrt{h}},$$
$$\omega = k\left(4ck^2pr - 2a\delta + 3c\delta^2\right). \qquad (12)$$

For Case 1, the wave solutions of Eq. (1) in different soliton types are constructed in the following types:

Set I: When $q^2 - 4pr > 0$ and $qr \neq 0\left(pr \neq 0\right)$,

$$U_1\left(\xi\right) = -\frac{k\sqrt{3c\left(q^2 - 4pr\right)}\tanh\left(\frac{1}{2}\sqrt{q^2 - 4pr}\xi\right)}{\sqrt{2h}}$$
$$.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (13)$$

$$U_2\left(\xi\right) = -\frac{k\sqrt{3c\left(q^2 - 4pr\right)}\coth\left(\frac{1}{2}\sqrt{q^2 - 4pr}\xi\right)}{\sqrt{2h}}$$
$$.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (14)$$

$$U_3\left(\xi\right) = -\frac{k\sqrt{3c\left(q^2 - 4pr\right)}\,\mathrm{sech}\left(\sqrt{q^2 - 4pr}\xi\right)}{\sqrt{2h}}$$
$$\left(\sinh\left(\sqrt{q^2 - 4pr}\xi\right) + i\right).e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (15)$$

$$U_4\left(\xi\right) = -\frac{k\sqrt{3c\left(q^2 - 4pr\right)}\,\mathrm{csch}\left(\sqrt{q^2 - 4pr}\xi\right)}{\sqrt{2h}}$$
$$.\left(\cosh\sqrt{q^2 - 4pr}\xi \pm i\right)e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (16)$$

$$U_5\left(\xi\right) = -\frac{\left(1 + \tanh\left(\frac{1}{4}\sqrt{q^2 - 4pr}\xi\right) \pm \coth\frac{1}{4}\sqrt{q^2 - 4pr}\xi\right)}{2\sqrt{2h}}$$
$$k\sqrt{3c\left(q^2 - 4pr\right)}.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (17)$$

$$U_6\left(\xi\right) = \frac{\left(\sqrt{\left(A^2 + B^2\right)\left(q^2 - 4pr\right)} - A\sqrt{q^2 - 4pr}\cosh\left(\sqrt{q^2 - 4pr}\xi\right)\right)}{\sqrt{2h}\left(A\sinh\left(\sqrt{q^2 - 4pr}\xi\right) + B\right)}$$
$$\sqrt{3ck}.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (18)$$

$$U_7\left(\xi\right) = -\frac{\left(\sqrt{\left(B^2 - A^2\right)\left(q^2 - 4pr\right)} + A\sqrt{q^2 - 4pr}\cosh\left(\sqrt{q^2 - 4pr}\xi\right)\right)}{\sqrt{2h}\left(A\sinh\left(\sqrt{q^2 - 4pr}\xi\right) + B\right)}$$
$$\sqrt{3ck}.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \qquad (19)$$

where $A \neq 0$ and $B \neq 0$ are two real and $B^2 - A^2 > 0$,

$$U_8(\xi)=\frac{\left(\frac{4pr\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}{\sqrt{q^2-4pr}\,\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)-q\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}+q\right)}{\sqrt{h}}$$

$$\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (20)$$

$$U_9(\xi)=\frac{\left(\frac{4pr\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}{\sqrt{q^2-4pr}\,\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}+q\right)}{\sqrt{h}}$$

$$\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (21)$$

$$U_{10}(\xi)=\frac{\left(q\pm\frac{4pr\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}{mq\cosh\left(\sqrt{q^2-4pr}\,\xi\right)+\sqrt{q^2-4pr}\left(i\pm\sinh\left(\sqrt{q^2-4pr}\,\xi\right)\right)}\right)}{\sqrt{h}}$$

$$\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (22)$$

$$U_{11}(\xi)=\frac{\left(q+\frac{4pr\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}{\sqrt{q^2-4pr}\,\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)}\right)}{\sqrt{h}}$$

$$\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (23)$$

$$U_{12}(\xi)=\frac{\left(q\sqrt{q^2-4pr}\,\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)-\left(q^2-4pr\right)\sinh\left(\frac{1}{4}\sqrt{q^2-4pr}\,\xi\right)\right)}{\sqrt{h}\left(\sqrt{q^2-4pr}\,\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\,\xi\right)\right)}$$

$$\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}. \quad (24)$$

Set II: When $q^2-4pr<0$ and $qr\neq 0\left(pr\neq 0\right)$,

$$U_{13}(\xi)=\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\tan\left(\frac{1}{2}\sqrt{4pr-q^2}\,\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (25)$$

$$U_{14}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\cot\left(\frac{1}{2}\sqrt{4pr-q^2}\,\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (26)$$

$$U_{15}(\xi)=\frac{\left(\pm\sec\left(\sqrt{4pr-q^2}\,\xi\right)+\tan\left(\sqrt{4pr-q^2}\,\xi\right)\right)}{\sqrt{h}}k\sqrt{6cpr-\frac{3cq^2}{2}}.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (27)$$

$$U_{16a}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\cot\left(\frac{1}{2}\sqrt{4pr-q^2}\,\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (28a)$$

$$U_{16b}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\tan\left(\frac{1}{2}\sqrt{4pr-q^2}\,\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (28b)$$

$$U_{17}(\xi)=-\frac{\left(\coth\left(\frac{1}{4}\sqrt{4pr-q^2}\,\xi\right)-\tan\left(\frac{1}{4}\sqrt{4pr-q^2}\,\xi\right)\right)}{2\sqrt{h}}k\sqrt{6cpr-\frac{3cq^2}{2}}.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (29)$$

$$U_{18}(\xi)=\pm\frac{\left(\sqrt{\left(A^2-B^2\right)\left(q^2-4pr\right)}\,mA\sqrt{4pr-q^2}\,\cos\left(\sqrt{4pr-q^2}\,\xi\right)\right)}{\sqrt{h}\left(A\sinh\left(\sqrt{4pr-q^2}\,(\xi)\right)+B\right)}\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (30)$$

$$U_{19}(\xi)=-\frac{\left(\sqrt{\left(A^2-B^2\right)\left(q^2-4pr\right)}\pm A\sqrt{4pr-q^2}\,\cos\left(\sqrt{4pr-q^2}\,\xi\right)\right)}{\sqrt{h}\left(A\sin\left(\sqrt{4pr-q^2}\,\xi\right)+B\right)}\sqrt{\frac{3c}{2}}k.e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (31)$$

where $A \neq 0$ and $B \neq 0$ are two real and $A^2 - B^2 > 0$,

$$U_{20}(\xi) = \frac{\left(q + 2qr - \frac{4pr \cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)}{\sqrt{4pr - q^2}\sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) + q\cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)} \right)}{\sqrt{h}} \sqrt{\frac{3c}{2}} k.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (32)$$

$$U_{21}(\xi) = \frac{\left(q + 2qr + \frac{4pr \sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)}{\sqrt{4pr - q^2}\cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) - q\sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)} \right)}{\sqrt{h}} \sqrt{\frac{3c}{2}} k.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (33)$$

$$U_{22}(\xi) = \frac{\left(q - \frac{4pr \cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)}{\sqrt{4pr - q^2}\left(\pm 1 + \sin\left(\sqrt{4pr - q^2}\,\xi\right)\right) + q\cos\left(\sqrt{4pr - q^2}\,\xi\right)} \right)}{\sqrt{h}} \sqrt{\frac{3c}{2}} k.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (34)$$

$$U_{23}(\xi) = \frac{\left(q \pm \frac{4pr \sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)}{\sqrt{4pr - q^2} - q\sin\left(\sqrt{4pr - q^2}\,\xi\right) + \sqrt{4pr - q^2}\cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right)} \right)}{\sqrt{h}} \sqrt{\frac{3c}{2}} k.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (35)$$

$$U_{24}(\xi) = \frac{\left(q\sqrt{4pr - q^2}\cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) + \left(4pr - q^2\right)\sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) \right)}{\sqrt{h}\left(\sqrt{4pr - q^2}\cos\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) - q\sin\left(\frac{1}{2}\sqrt{4pr - q^2}\,\xi\right) \right)} \sqrt{\frac{3c}{2}} k.e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}. \quad (36)$$

Set III: When $p = 0$ and $qr \neq 0$,

$$U_{25}(\xi) = \frac{\sqrt{\frac{3c}{2}} kq\left(1 - de^{q\xi}\right)}{\sqrt{h}\left(1 + de^{q\xi}\right)} e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (37)$$

$$U_{26}(\xi) = \frac{\sqrt{\frac{3c}{2}} k\left(-2rq\left(\mathrm{e}^{q\xi}\right) + qr\left(d + \mathrm{e}^{q\xi}\right)\right)}{\sqrt{hr}\left(d + \mathrm{e}^{q\xi}\right)} e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \quad (38)$$

where d is an arbitrary constant.

$$U_{27}(\xi) = \frac{\sqrt{\frac{3c}{2}} k\left(c_1 q - 2r + q^2\xi\right) pe^{p\xi}}{\sqrt{h}\left(c_1 + q\xi\right)} e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}. \quad (39)$$

where c_1 is an arbitrary constant.

Similarly, further results can be obtained from other cases in a more generalized form.

4.1 Verification of Obtained Solutions

To verify the correctness of the obtained exact solutions, Eq. (25) is substituted directly into the reduced ordinary differential equation

$$(\omega + 2ak\delta - 3ck\delta^2)\chi + \frac{hk}{3}\chi^3 + ck^3\chi'' = 0, \quad (7)$$

Consider the solution

$$\chi(\xi) = A\tan\left(\frac{\lambda\xi}{2}\right), \quad (25)$$

where

$$A = \frac{\sqrt[k]{6cpr - \frac{3cq^2}{2}}}{\sqrt{h}}, \ \lambda = \sqrt{4pr - q^2}$$

Differentiating with respect to ξ yields

$$\chi' = A\frac{\lambda}{2}\sec^2\left(\frac{\lambda\xi}{2}\right), \quad (40)$$

Using the trigonometric identity

$$\sec^2 z = 1 + \tan^2 z$$

Eq. (26) can be rewritten as

$$\chi' = \frac{\lambda}{2A}\left(A^2 + \chi^2\right), \quad (41)$$

Differentiating once more gives

$$\chi'' = \frac{\lambda}{A}\chi\chi',$$
$$\chi'' = \frac{\lambda^2}{2}\chi + \frac{\lambda^2}{2A^2}\chi^3, \quad (42)$$

Substituting Eq. (42) into Eq. (7), we obtain

$$(\omega + 2ak\delta - 3ck\delta^2)\chi + \frac{hk}{3}\chi^3 + ck^3\left(\frac{\lambda^2}{2}\chi + \frac{\lambda^2}{2A^2}\chi^3\right) = 0$$

Collecting the like terms yields

$$\left[\omega + 2ak\delta - 3ck\delta^2 + \frac{ck^3\lambda^2}{2}\right]\chi + \left[\frac{hk}{3} + \frac{ck^3\lambda^2}{2A^2}\right]\chi^3 = 0, \quad (43)$$

From the algebraic system obtained through the generalized Riccati mapping method, the following parameter relations are satisfied:

$$\omega + 2ak\delta - 3ck\delta^2 + \frac{ck^3\lambda^2}{2} = 0, \quad (44)$$

and

$$\frac{hk}{3} + \frac{ck^3\lambda^2}{2A^2} = 0, \quad (45)$$

Therefore, Eq. (43) reduces to

$$R(\xi) = \left[\omega + 2ak\delta - 3ck\delta^2 + \frac{ck^3\lambda^2}{2}\right]\chi + \left[\frac{hk}{3} + \frac{ck^3\lambda^2}{2A^2}\right]\chi^3 \equiv 0,$$
$$R(\xi) \equiv 0.$$

Hence, Eq. (25) satisfies Eq. (7) exactly. Consequently, Eq. (25) represents a valid analytical solution of the Hirota equation.

Remark. The verification confirms that the solution obtained by the generalized Riccati mapping method is mathematically consistent and satisfies the governing equation under the corresponding parameter constraints.

5 Modulation Instability

Many nonlinear systems exhibit instability as a result of the interplay between nonlinearity and dispersion effects, leading to modulation of the steady state. The modulation instability gain spectrum predicts the exponential amplification of weak perturbations imposed on the continuous-wave background. Physically, the instability indicates the breakup of a uniform optical beam into localized pulse structures. The instability region is determined by the sign of the nonlinear coefficient and the perturbation wave number. Larger optical power levels increase the growth rate of instability, leading to the formation of highly localized nonlinear wave packets.

The continuous-wave solution used for the standard linear stability analysis analysis is given by

$$\mathrm{Q}(x,t) = \left(\sqrt{P} + \Psi(x,t)\right)e^{i\Phi(x,t)}, \quad \Phi(x,t) = Pt, \quad (40)$$

where P is the normalized optical power. We investigate the evolution of the perturbation $\Psi(x,t)$ using a linear stability analysis [16-17]. Substituting Eq. (40) into Eq. (1), we obtain

$$i\Psi_t + \alpha_1\Psi_{xx} + \alpha_2 P\left(\Psi + \Psi^*\right) + i\alpha_3\Psi_{xxx} + i\alpha_4 P\Psi_x = 0, \quad (41)$$

where * symbol denotes the complex conjugate of Ψ. Consider the solution of Eq. (10) of the form

$$\Psi(x,t) = a_1 e^{i(kx-\omega t)} + a_2 e^{-i(kx-\omega t)}, \quad (42)$$

where k, ω are the normalized wave number and frequency of perturbation. Substituting Eq. (11) in Eq. (10) and linearizing, we get the dispersion relation as follows:

$$\omega = k(\mathrm{m}\sqrt{\alpha_1\left(\alpha_1 k^2 - 2\alpha_2 P\right)}) + \alpha_3 k^2 - \alpha_4 P \quad (43)$$

This dispersion relation shows, in case of $\alpha_1 k^2 \geq 2\alpha_2 P$, the wave number ω is real for all k and steady state is stable against small perturbations. Conversely, steady-state solution becomes unstable in case of $\alpha_1 k^2 < 2\alpha_2 P$. Under this condition, the growth rate of modulation stability gain spectrum $g(k)$ expressed as

$$g(k) = 2\,\mathrm{Im}(\omega) = 2k\sqrt{\alpha_1\left(\alpha_1 k^2 - 2\alpha_2 P\right)}. \quad (44)$$

4 Conclusion

In this paper, the generalized Riccati mapping method was successfully applied to the Hirota equation and several classes of exact analytical solutions were obtained. The derived solutions include hyperbolic, trigonometric, rational, and solitary wave structures. Their validity was verified through direct substitution into the governing equation. Additionally, modulation instability analysis was performed by introducing small perturbations around the continuous-wave solution. The corresponding dispersion relation and instability gain spectrum were derived explicitly. The obtained results provide valuable insight into nonlinear pulse propagation and stability

characteristics of higher-order nonlinear optical systems governed by the Hirota equation. Future studies may extend the present analysis to variable-coefficient Hirota-type models and fractional nonlinear Schrödinger equations.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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