Global Correctness Theorem to the Mixed Problem for a Homogeneous Two-Velocity Model Telegraph Equation with Non-Characteristic Time-Dependent Oblique Derivatives at the Ends of a String

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Abstract: We have proved the global correctness theorem to new mixed problem for two-velocity homogeneous model telegraph equation with non-characteristic time-dependent oblique derivatives at the ends of string. An explicit recurrent formula for a unique and stable classic solutions with a correctness criterion for mixed problem in the set of twice continuously differentiable functions are derived. Its correctness criterion consists of necessary and sufficient four matching conditions and three smoothness requirements on the initial and boundary data of non-characteristic mixed problem. These results are obtained by author's "method of auxiliary mixed problems for wave equations on a half-line" and "implicit characteristic method".

Key–Words: Non-characteristic oblique derivative, method of auxiliary mixed problems, implicit characteristics method, global correctness theorem, Hadamard's correctness criterion.

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1 Introduction

In this paper, we prove a global correctness theorem to new mixed (initial-boundary) problem for two-velocity homogeneous model telegraph equation with non-characteristic time-dependent oblique derivatives in a half-band of the plane. This global correctness theorem is proved by "the method of auxiliary mixed problems for a semi-bounded string (wave equation on a half-line)" from the previously established global correctness theorem of an auxiliary non-characteristic mixed problem for two-velocity homogeneous model telegraph equation in the first quarter of the plane [9]. In a special case, global correctness theorem with explicit formulas for the unique and stable classical solutions of the auxiliary mixed problem for our two-speed homogeneous model telegraph equation in a system of twice continuously differentiable functions by implicit characteristic method are also available in [9]. The correctness criterion to non-characteristic mixed problem of this article includes the necessary and sufficient smoothness requirements for the initial and boundary data and matching conditions the boundary regimes with the initial conditions and the equation for its unique and stable everywhere solvability. By global correctness theorems to mixed problems for partial differential equations we mean theorems with necessary and sufficient conditions for Hadamard correctness, i.e. its existence, uniqueness and stability on mixed problem data. Thus, in this paper, complete, final and unimprovable smoothness requirements and matching conditions for the mixed problem data are obtained. The implicit characteristics method was first proposed by F.E. Lomovtsev in [10] for the first mixed problem for the general single-speed telegraph equation with variable coefficients. For the case of a non-homogeneous two-speed telegraph equation in [9], the application of the method of correcting trial solutions into classical solutions from article [4] is inevitable (see remark 1 after Theorem 3 of the second part of this article). The implicit characteristics method was used to derive the Riemann formulas of classical solutions and Hadamard's correctness criteria in articles [5], [6], [7], respectively, for the generalized Cauchy problem with Cauchy conditions on curved lines of the plane and two second mixed problems for the model telegraph equation and the general telegraph equation with variable coefficients.

The Fourier method explicitly solves linear initial-boundary (mixed) problems for some wave equations in articles [3], [11]. The uniqueness, local and global solvability of a mixed problem for a semilinear wave equation with a nonlinear boundary condition are investigated. Cases of not only global but also local absence of a solution are considered, as well as the case when this problem has a collapsing solution [2]. The existence of solutions to nonlinear pulse wave equations is the subject of the article [1]. The existence of local and global solutions of the boundary value problem for the relativistic wave equation, the continuous dependence of solutions on the initial data and the blow-up of solutions are investigated [12]. Only in the works of the author of this article and his students were complete and final correctness criteria according to Hadamard found with explicit formulas for classical solutions of many linear mixed problems for telegraph equations with real variable coefficients. In them, for classical solutions, the minimum possible necessary (obligatory) and sufficient (admissible) conditions on the right-hand sides of the wave equations, the initial and boundary data of these mixed problems are strictly proven. In sets of classical solutions, such results for linear mixed problems were obtained by us by recurrent continuations of the right-hand sides of the wave equations and initial data along the time axis, and not by periodic or other continuations along the spatial axis in works of other authors.

2 Auxiliary Mixed Problem for the Model Telegraph Equation with Non-Characteristic Time-Dependent Oblique Derivative on the Half-Line

In the next section 3, we derive the classic solutions and correctness criterion for the main mixed problem from the classic solutions and correctness criterion for the auxiliary mixed problem. In the first quarter of the plane $\dot{G}_\infty =]0, +\infty[\times]0, +\infty[$, the following mixed problem was investigated by a new "implicit characteristic method" [9]:

$$u_{tt}(x,t) + (a_1(x,t) - a_2(x,t))u_{xt}(x,t) -$$

$$-a_1(x,t)a_2(x,t)u_{xx}(x,t) - \frac{(a_2)_t(x,t)}{a_2(x,t)}u_t(x,t) -$$

$$-a_1(x,t)(a_2)_x(x,t)u_x(x,t) = 0,\ \{x,t\} \in \dot{G}_\infty, \tag{1}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x > 0, \tag{2}$$

$$[\alpha(t)u_t(x,t) + \beta(t)u_x(x,t) + \gamma(t)u(x,t)]\,|_{x=0} =$$
$$= \mu(t),\ t > 0, \tag{3}$$

where the coefficients $a_{3-i}(x,t) \geq a_{3-i}^{(0)} > 0$, $\{x,t\} \in G_\infty = [0,+\infty[\times[0,+\infty[$, $i = 1, 2$, of the equation and the mixed problem data φ, ψ, μ are given real functions of their variables x and t. We denote the orders of their partial derivatives by the number of subscripts of the functions.

Let $C^k(\Omega)$ be a set of k times continuously differentiable functions on a subset $\Omega \subset R^2$, $R =]-\infty, +\infty[$, and $C^0(\Omega) = C(\Omega)$. Equation (1) has the characteristic equations [9]:

$$dx = (-1)^i a_{3-i}(x,t)dt,\ i = 1, 2, \tag{4}$$

which correspond to implicit general integrals $g_i(x,t) = C_i$, $C_i \in R$, $i = 1, 2$. If the coefficients a_{3-i} is strictly positive $a_{3-i}(x,t) \geq a_{3-i}^{(0)} > 0$, $\{x,t\} \in G_\infty$, then the variable t on the characteristics $g_1(x,t) = C_1$, $C_1 \in R$, strictly decreases, on the characteristics $g_2(x,t) = C_2$, $C_2 \in R$, it strictly increases in the plane Oxt along with the growth of variable x. Therefore, implicit functions $y_i = g_i(x,t) = C_i$, $x, t \geq 0$, have strictly monotone implicit inverse functions $x = h_i\{y_i, t\}$, $t \geq 0$, and $t = h^{(i)}[x, y_i]$, $x \geq 0$, $i = 1, 2$. By the definition of inverse mappings, these implicit functions satisfy the following inversion identities [9]:

$$g_i(h_i\{y_i,\ t\},\ t) = y_i,\ t \geq 0,$$
$$h_i\{g_i(x,\ t),\ t\} = x,\ x \geq 0,\ i = 1,2, \tag{5}$$

$$g_i(x,\ h^{(i)}[x, y_i]) = y_i,\ x \geq 0,$$
$$h^{(i)}[x,\ g_i(x,\ t)] = t,\ t \geq 0,\ i = 1,2, \tag{6}$$

$$h_i\{y_i,\ h^{(i)}[x, y_i]\} = x,\ x \geq 0,$$
$$h^{(i)}[h_i\{y_i,\ t\}, y_i] = t,\ t \geq 0,\ i = 1,\ 2. \tag{7}$$

In the right-hand parts of identities (5)–(7), together with mutually inverse implicit functions, variables that repeat twice in the left-hand parts are excluded, even if only one of the possible values of these variables is repeated twice in the left-hand parts. If the coefficients be $a_{3-i}(x,t) \geq a_{3-i}^{(0)} > 0$, $\{x,t\} \in G_\infty$, $a_{3-i} \in C^2(G_\infty)$, $G_\infty = [0,+\infty[\times[0,+\infty[$, $i = 1, 2$, then the implicit functions g_i, h_i, $h^{(i)}$ are twice continuously differentiable by x, t, y_i, $i = 1, 2$, [9].

Definition 1 *The classic solution of mixed problem (1)–(3) is a function $u \in C^2(G_\infty)$, that satisfies equation (1) in the usual sense on $\dot{G}_\infty$, and the initial conditions (2) and the boundary regime (3) in the sense of the values of the corresponding limits of the function $u(\dot{x}, \dot{t})$ or its derivatives $u_{\dot{t}}(\dot{x}, \dot{t})$, $u_{\dot{x}}(\dot{x}, \dot{t})$ for internal points $\{\dot{x}, \dot{t}\} \in \dot{G}_\infty$, tending to the boundary points $\{x, t\}$, which indicated in (2) and (3).*

Definition 2 *The characteristic $g_2(x,t)=g_2(0,0)$ is called critical for equation (1) in the first quarter of the plane G_∞.*

Classic solutions and correctness criterion (necessary and sufficient conditions) according to Hadamard (existence, uniqueness of the solution and its stability with respect to the mixed problem data $\varphi,\ \psi,\ \mu$ to the mixed problem (1)–(3) on the half-line are found in explicit form.

From Definition 1 of classic solutions to the mixed problem (1)–(3) in G_∞ and its formulation, obvious necessary (mandatory) smoothness requirements directly follow

$$\varphi\in C^2[0,+\infty[,\ \psi\in C^1[0,+\infty[,\ \mu\in C^1[0,+\infty[. \tag{8}$$

Putting $t=0$ in the boundary mode (3), due to the initial conditions (2) at $x=0$ we find the necessary first matching condition:

$$\alpha(0)\psi(0)+\beta(0)\varphi'(0)+\gamma(0)\varphi(0)=\mu(0). \tag{9}$$

In the first derivative with respect to t the boundary regime (3) we assume $t=0$ and, due to the initial conditions (2) and the second derivative $u_{tt}(x,t)$ from equation (1) at $x=0,\ t=0$, we find the necessary second matching condition

$$\begin{gathered}\alpha'(0)\psi(0)+\beta'(0)\varphi'(0)+\gamma'(0)\varphi(0)+\beta(0)\psi'(0)+\\+\gamma(0)\psi(0)+\alpha(0)[(a_2(0,0)-a_1(0,0))\,\psi'(0)+\\+a_1(0,0)a_2(0,0)\varphi''(0)+\frac{(a_2)_t(0,0)}{a_2(0,0)}\psi(0)+\\+a_1(0,0)(a_2)_x(0,0)\varphi'(0)]=\mu'(0).\end{gathered} \tag{10}$$

We denote by the number of strokes above functions of one variable the corresponding orders of their ordinary derivatives with respect to these variables.

Then the sufficiency of the smoothness and matching conditions (8)–(10) is proved.

The critical characteristic $g_2(x,t)=g_2(0,0)$ divides the first quarter G_∞ into two sets

$$\begin{gathered}G_-=\{\{x,t\}\in G_\infty: g_2(x,t)>g_2(0,0)\},\\ G_+=\{\{x,t\}\in G_\infty: g_2(x,t)\le g_2(0,0)\}.\end{gathered} \tag{11}$$

Theorem 3 *[9]. Let the coefficients be $a_{3-i}(x,t)\ge a_{3-i}^{(0)}>0$, $\{x,t\}\in G_\infty$, $a_{3-i}\in C^2(G_\infty)$, $i=1,\,2$, $\alpha,\ \beta,\ \gamma\in C^1[0,+\infty[$, $a_1(0,t)\alpha(t)\ne\beta(t)$, $t\ge 0$, and also $\alpha(0)=0$ in (3) or*

$$\begin{gathered}a_1(0,0)=a_2(0,0),\ (a_1)_t(0,0)=\\=(a_2)_t(0,0),\ (a_1)_x(0,0)=(a_2)_x(0,0).\end{gathered} \tag{12}$$

The mixed problem (1)–(3) in G_∞ has a unique and stable on $\varphi,\ \psi,\ \mu$ classic solutions $u\in C^2(G_\infty)$ if and only if the smoothness requirements (8) and the matching conditions (9), (10) are satisfied.

These classic solutions $u\in C^2(G_\infty)$ to the mixed problem (1)–(3) in G_∞ are the functions

$$\begin{gathered}u_-(x,t)=\varphi(h_2\{g_2(x,t),0\})+\\+\int_{h_2\{g_2(x,t),0\}}^{h_1\{g_1(x,t),0\}}\frac{a_1(\nu,0)\varphi'(\nu)+\psi(\nu)}{a_1(\nu,0)+a_2(\nu,0)}d\nu,\\ \{x,t\}\in G_-,\end{gathered} \tag{13}$$

$$\begin{gathered}u_+(x,t)=\varphi(0)+\\+\int_0^{h_1\{g_1(x,t),0\}}\frac{a_1(\nu,0)\varphi'(\nu)+\psi(\nu)}{a_1(\nu,0)+a_2(\nu,0)}d\nu+\\+\frac{1}{\eta(h^{(2)}[0,g_2(x,t)])}\int_0^{h^{(2)}[0,g_2(x,t)]}\eta(\rho)\times\\ \times\Big\{\frac{a_1(0,\rho)\mu(\rho)}{a_1(0,\rho)\alpha(\rho)-\beta(\rho)}-\frac{[a_1(0,\rho)\alpha(\rho)+\beta(\rho)]}{[a_1(0,\rho)\alpha(\rho)-\beta(\rho)]}\times\\ \times\frac{a_1(0,\rho)}{a_2(0,\rho)}\frac{a_1(\nu,0)\varphi'(\nu)+\psi(\nu)}{a_1(\nu,0)+a_2(\nu,0)}\Big|_{\nu=h_1\{g_1(0,\rho),0\}}\times\\ \times\frac{\partial h_1\{g_1(0,\rho),0\}}{\partial\rho}-\frac{a_1(0,\rho)\gamma(\rho)}{a_1(0,\rho)\alpha(\rho)-\beta(\rho)}\Big[\varphi(0)+\\+\int_0^{h_1\{g_1(0,\rho),0\}}\frac{a_1(\nu,0)\varphi'(\nu)+\psi(\nu)}{a_1(\nu,0)+a_2(\nu,0)}d\nu\Big]\Big\}d\rho,\\ \{x,t\}\in G_+.\end{gathered} \tag{14}$$

In Theorem 3, the uniqueness of the classic solution $u\in C^2(G_\infty)$ to the mixed problem (1)–(3) is ensured by the algorithm for searching it in the method of implicit characteristics. For any $T>0$, the continuous dependence of the solution $u\in C^2(G_\infty)$ on the mixed problem data $\varphi,\ \psi,\ \mu$ of this mixed problem in the corresponding Banach spaces follows from its existence and uniqueness, thanks to the well-known Banach closed graph theorem or Banach open mapping theorem. On the other hand, formulas (13) and (14) also imply its continuous dependence.

Remark 1. This Theorem 3 with zero right-hand side $f=0$ of the homogeneous two-velocity model telegraph equation is a special case of the theorem with non-zero right-hand side $f\ne 0$ from [9] in the theorem on the global correctness to the non-characteristic mixed problem for the inhomogeneous two-velocity model telegraph equation in the first quadrant of the plane. In it, by the author's methods of implicit characteristics and correction of trial solutions into classic solutions, formulas for its unique and stable classic (twice continuously differentiable) solutions are derived and a criterion for Hadamard correctness of unique and stable everywhere solvability is established. The necessity of the corresponding integral smoothness requirements on the continuous right-hand side $f \in C(G_\infty)$ of the two-velocity model telegraph equation is justified using the correcting Goursat problem in [4]. In the paper [9] also given are the particular classical solutions $F_1(x,t)$ and $F_2(x,t)$ of inhomogeneous two-velocity model telegraph equation on the sets G_- and G_+ from (11), calculated by the correction method in [4], with the corresponding necessary and sufficient smoothness on G_∞ with critical characteristic $g_2(x,t) = g_2(0,0)$.

3 The main mixed problem for the model telegraph equation on a segment

In the half-band of the plane $\dot{G} =]0,d[\times]0,+\infty[$, we study and solve the mixed problem:

$$u_{tt}(x,t) + (a_1(x,t) - a_2(x,t))u_{xt}(x,t) -$$
$$-a_1(x,t)a_2(x,t)u_{xx}(x,t) - \frac{(a_2)_t(x,t)}{a_2(x,t)}u_t(x,t) -$$
$$-a_1(x,t)(a_2)_x(x,t)u_x(x,t) = 0,\ \{x,t\} \in \dot{G}, \quad (15)$$
$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ 0 < x < d, \quad (16)$$
$$[\alpha_i(t)u_t(x,t) + \beta_i(t)u_x(x,t) + \gamma_i(t)u(x,t)]\,|_{x=\hat{d}_i} =$$
$$= \mu_i(t),\ i = 1,\,2,\ t > 0, \quad (17)$$

here the coefficients $a_{3-i}(x,t) \geq a^{(0)}_{3-i} > 0$, $\{x,t\} \in G = [0,d] \times [0,+\infty[$, $\hat{d}_i = (i-1)d$, $i = 1,\,2$, of the equation and the mixed problem data φ, ψ, μ_1, μ_2 are given real functions of their variables x and t.

Definition 4 *A classic solution to problem (15)–(17) is a function $u \in C^2(G)$, that satisfies equation (15) in the usual sense on $\dot{G}$, and initial conditions (16) and boundary conditions (17) in the sense of the values of the corresponding limits of the function $u(\dot{x},\dot{t})$ or its derivatives $u_{\dot{t}}(\dot{x},\dot{t})$ and $u_{\dot{x}}(\dot{x},\dot{t})$ for internal points $\{\dot{x},\dot{t}\} \in \dot{G}$, tending to the boundary points $\{x,t\}$, indicated in (16) and (17).*

The formulation of the mixed problem (15)–(17) and the definition 4 of its classic solutions $u \in C^2(G)$ imply obvious necessary smoothness requirements

$$\varphi \in C^2[0,d],\ \psi \in C^1[0,d],\ \mu_1,\ \mu_2 \in C^1[0,d_{n+1}]. \quad (18)$$

Putting $t = 0$ in the boundary mode (17) at $t = 0$, $i = 2$, due to the initial conditions (16) at $x = d$ we trove the necessary first matching condition for $x = d$:

$$\alpha_2(0)\psi(d) + \beta_2(0)\varphi'(d) + \gamma_2(0)\varphi(d) = \mu_2(0). \quad (19)$$

In the first derivative with respect to t the boundary regime (17) at $i = 2$ we assume $t = 0$ and, due to the initial conditions (16) and the second derivative $u_{tt}(x,t)$ from equation (15) at $x = d$, $t = 0$, we find the necessary second matching condition for $x = d$:

$$\alpha'_2(0)\psi(d) + \beta'_2(0)\varphi'(d) + \gamma'_2(0)\varphi(d) +$$
$$+\beta_2(0)\psi'(d) + \gamma_2(0)\psi(d) + \alpha_2(0)[(a_2(d,0) -$$
$$-a_1(d,0))\psi'(d) + a_1(d,0)a_2(d,0)\varphi''(d) +$$
$$+\frac{(a_2)_t(d,0)}{a_2(d,0)}\psi(d) + a_1(d,0)(a_2)_x(d,0)\varphi'(d)] = \mu'_2(0). \quad (20)$$

The global correctness theorem 5 of the non-characteristic mixed problem (15)–(17) is derived from Theorem 3 by "the method of auxiliary mixed problems for a semi-bounded string (wave equation on a semi-line)" from [8].

The system of two equations $g_2(x,t) = g_2(0,0)$, $g_1(x,t) = g_1(d,0)$ for $0 \leq x \leq d$, $t \geq 0$ has a unique solution, because according to the characteristic equations (4), the characteristic $g_2(x,t) = g_2(0,0)$ strictly increases, and the characteristic $g_1(x,t) = g_1(d,0)$ strictly decreases in t as x increases in the plane Oxt. Let $\{x_0,t_0\}$ be the unique solution to this system of equations. The half-band $G = [0,d] \times [0,+\infty[$ is divided into rectangles $G_k = [0,d] \times [d_k,d_{k+1}]$, where $d_k = (k-1)t_0$, $k = 1,\,2,\ldots$ Each rectangle G_k is divided by critical characteristics $g_2(x,t) = g_2(0,d_k)$, $g_1(x,t) = g_1(d,d_k)$, $k = 1,\,2,\ldots$, of the equation (15) into triangles

$$\Delta_{3k-2} = \{\{x,t\} : g_2(x,t) \geq g_2(0,d_k),$$
$$g_1(x,t) \leq g_1(d,d_k), x \in [0,d]\},$$
$$\Delta_{3k-1} = \{\{x,t\} :\ g_2(x,t) \leq g_2(0,d_k),$$
$$x \in [0,x_0],\ t \in [d_k,d_{k+1}]\},$$
$$\Delta_{3k} = \{\{x,t\} :\ g_1(x,t) \geq g_1(d,d_k),$$
$$x \in [x_0,d],\ t \in [d_k,d_{k+1}]\},\ k = 1,\,2,\ \ldots$$

Theorem 5 *Let the coefficients be* $a_{3-i}(x,t) \geq a^{(0)}_{3-i} > 0$, $\{x,t\} \in Q_n = [0,d] \times [0,d_{n+1}]$, $a_{3-i} \in C^2(Q_n)$, α_i, β_i, $\gamma_i \in C^1[0,d_{n+1}]$, $\beta_i(t) \neq (-1)^{i+1}a_i(\hat{d}_i,t)\alpha_i(t), t \geq 0$, $i = 1,\,2$. *Let the coefficients be $\alpha_i(0) = 0$ in (17) or*

$$a_1(\hat{d}_i,0) = a_2(\hat{d}_i,0),\ (a_1)_t(\hat{d}_i,0) =$$

$$= (a_2)_t(\hat{d}_i, 0),\ (a_1)_x(\hat{d}_i, 0) = (a_2)_x(\hat{d}_i, 0) \quad (21)$$

for $i = 1$ or $i = 2$. Then, for the existence of unique and stable classic solutions $u \in C^2(Q_n)$ to the mixed problem (15)–(17) on $\dot{Q}_n$ it is necessary and sufficient to have the smoothness requirements (18) and matching conditions:

$$\alpha_i(0)\psi(\hat{d}_i) + \beta_i(0)\varphi'(\hat{d}_i) + \gamma_i(0)\varphi(\hat{d}_i) = \mu_i(0), \quad (22)$$

$$\alpha_i'(0)\psi(\hat{d}_i) + \beta_i'(0)\varphi'(\hat{d}_i) + \gamma_i{}'(0)\varphi(\hat{d}_i)+$$
$$+\beta_i(0)\psi'(\hat{d}_i) + \gamma_i(0)\psi(\hat{d}_i)+$$
$$+\alpha_i(0)[(a_2(\hat{d}_i, 0) - a_1(\hat{d}_i, 0))\psi'(\hat{d}_i)+$$
$$+a_1(\hat{d}_i, 0)a_2(\hat{d}_i, 0)\varphi''(\hat{d}_i)+$$
$$+((a_2)_t(\hat{d}_i, 0)/a_2(\hat{d}_i, 0))\psi(\hat{d}_i)+$$
$$+a_1(\hat{d}_i, 0)(a_2)_x(\hat{d}_i, 0)\varphi'(\hat{d}_i)] = \mu_i'(0),\ i = 1,\ 2. \quad (23)$$

This classic solutions to mixed problem (15)–(17) in $\dot{Q}_n$ are the functions:

$$u_{3k-2}(x,t) = \varphi_k(h_2\{g_2(x,t), d_k\})+$$
$$+\int_{h_2\{g_2(x,t),d_k\}}^{h_1\{g_1(x,t),d_k\}} \frac{a_1(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)} d\nu,$$
$$\{x,t\} \in \Delta_{3k-2}, \quad (24)$$

$$u_{3k-1}(x,t) = \varphi_k(0)+$$
$$+\int_0^{h_1\{g_1(x,t),d_k\}} \frac{a_1(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)} d\nu+$$
$$+\frac{1}{\eta_1(h^{(2)}[0, g_2(x,t)])}\int_{d_k}^{h^{(2)}[0,g_2(x,t)]} \eta_1(\rho)\times$$
$$\times\Big\{\frac{a_1(0,\rho)\mu_1(\rho)}{a_1(0,\rho)\alpha_1(\rho) - \beta_1(\rho)} - \frac{[a_1(0,\rho)\alpha_1(\rho) + \beta_1(\rho)]}{[a_1(0,\rho)\alpha_1(\rho) - \beta_1(\rho)]}\times$$
$$\times\frac{a_1(0,\rho)}{a_2(0,\rho)}\frac{a_1(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)}\Big|_{\nu = h_1\{g_1(0,\rho), d_k\}}\times$$
$$\times\frac{\partial h_1\{g_1(0,\rho), d_k\}}{\partial\rho} - \frac{a_1(0,\rho)\gamma_1(\rho)}{a_1(0,\rho)\alpha_1(\rho) - \beta_1(\rho)}\Big[\varphi_k(0)+$$
$$+\int_0^{h_1\{g_1(0,\rho),d_k\}} \frac{a_1(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)} d\nu\Big]\Big\}d\rho,$$
$$\{x,t\} \in \Delta_{3k-1}, \quad (25)$$

$$u_{3k}(x,t) = \varphi_k(d)+$$
$$+\int_{h_2\{g_2(x,t),d_k\}}^{d} \frac{a_2(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)} d\nu+$$
$$+\frac{1}{\eta_2(h^{(1)}[d, g_1(x,t)])}\int_{d_k}^{h^{(1)}[d,g_1(x,t)]} \eta_2(\rho)\times$$
$$\times\Big\{\frac{a_2(d,\rho)\mu_2(\rho)}{a_2(d,\rho)\alpha_2(\rho) + \beta_2(\rho)} + \frac{[a_2(d,\rho)\alpha_2(\rho) - \beta_2(\rho)]}{[a_2(d,\rho)\alpha_2(\rho) + \beta_2(\rho)]}\times$$
$$\frac{a_2(d-\nu, d_k)\varphi_k'(d-\nu) + \psi_k(d-\nu)}{a_1(d-\nu, d_k) + a_2(d-\nu, d_k)}\Big|_{\nu = d - h_2\{g_2(d,\rho), d_k\}}$$
$$\times\frac{a_2(d,\rho)}{a_1(d,\rho)}\frac{\partial h_2\{g_2(d,\rho), d_k\}}{\partial\rho} -$$
$$-\frac{a_2(d,\rho)\gamma_2(\rho)}{a_2(d,\rho)\alpha_2(\rho) + \beta_2(\rho)}\Big[\varphi_k(d)+$$
$$+\int_{h_2\{g_2(d,\rho),d_k\}}^{d} \frac{a_2(\nu, d_k)\varphi_k'(\nu) + \psi_k(\nu)}{a_1(\nu, d_k) + a_2(\nu, d_k)} d\nu\Big]\Big\}d\rho,$$
$$\{x,t\} \in \Delta_{3k}, \quad (26)$$

where $k = 1, 2, ..., n$, $n = 1, 2, \ldots$, u_{3k-l} are the restrictions of the classical solution $u \in C^2(Q_n)$ of the non-characteristic mixed problem (15)–(17) to triangles Δ_{3k-l}, $l = 0, 1, 2$, and in rectangles G_k the recurrent initial data along the axis Ot have the values

$$\varphi_1(x) = \varphi(x),\ \psi_1(x) = \psi(x),\ x \in [0, d],$$
$$\varphi_k(x) = u_{3k+j-4}(x, d_k),$$
$$\psi_k(x) = \frac{\partial u_{3k+j-4}}{\partial t}\Big|_{t=d_k},\ x \in [jx_0, max\{x_0, jd\}],$$
$$j = 0,\ 1,\ k = 2,\ 3, ..., n,\ n = 2,\ 3, ... \quad (27)$$

Proof. The proof of Theorem 5 can be carried out by the method of mathematical induction on rectangles $Q_n = [0, d] \times [0, d_{n+1}]$, $n = 1, 2, \ldots$ At the first step of mathematical induction to the mixed problem (15)–(17) on a rectangle $Q_1 = G_1$, we verify the existence of a unique and stable classic solutions u of the form (24)–(26) with correctness criterion (18), (22), (23) for $n = k = 1$.

Restrictions of formulas (13), (14) for $\mu = \mu_1$ and necessary and sufficient conditions (8), (9), (10) for $\mu = \mu_1$ to the trapezoid $\Delta_1 \cup \Delta_2$ from Theorem 3, respectively, coincide with formulas (22), (23) and correctness criterion (18), (22), (23) for $n = k = 1$, $i = 1$ from Theorem 5. In a trapezoid $\Delta_1 \cup \Delta_2$, the correctness criterion consists of the smoothness requirements

$$\varphi \in C^2[0, x_0],\ \psi \in C^1[0, x_0],\ \mu_1 \in C^1[0, d_2], \quad (28)$$

and the matching conditions (9), (10) at $\mu = \mu_1$ from Theorem 3.

To find a classic solutions and correctness criterion for a mixed problem consisting of equation (15), initial conditions (16), and the boundary mode from

(17) for $x = d$, $i = 2$ in a trapezoid $\Delta_1 \cup \Delta_3$, we reduce it by replacing $x = d - \tilde{x}$, $t = \tilde{t}$ to an equivalent mixed problem

$$\tilde{u}_{tt}(\tilde{x},t) + (\tilde{a}_2(\tilde{x},t) - \tilde{a}_1(\tilde{x},t))\tilde{u}_{\tilde{x}t}(\tilde{x},t) -$$
$$-\tilde{a}_1(\tilde{x},t)\tilde{a}_2(\tilde{x},t)\tilde{u}_{\tilde{x}\tilde{x}}(\tilde{x},t) -$$
$$-((\tilde{a}_2)_t(\tilde{x},t)/\tilde{a}_2(\tilde{x},t))\tilde{u}_t(\tilde{x},t) +$$
$$+\tilde{a}_1(\tilde{x},t)(\tilde{a}_2)_{\tilde{x}}(\tilde{x},t)\tilde{u}_{\tilde{x}}(\tilde{x},t) = 0,\ \{\tilde{x},t\} \in \tilde{\Delta}_1 \cup \tilde{\Delta}_2, \tag{29}$$

$$\tilde{u}|_{t=0} = \tilde{\varphi}(\tilde{x}),\ (\partial\tilde{u}/\partial t)|_{t=0} = \tilde{\psi}(\tilde{x}),\ \tilde{x} \in [0,d], \tag{30}$$

$$[\alpha_2(t)\tilde{u}_t(\tilde{x},t) - \beta_2(t)\tilde{u}_{\tilde{x}}(\tilde{x},t) +$$
$$\gamma_2(t)\tilde{u}(\tilde{x},t)]|_{\tilde{x}=0} = \mu_2(t),\ t \in [0,d_2], \tag{31}$$

with respect to a new function $\tilde{u}(\tilde{x},t) = u(d-\tilde{x},t) = u(x,t)$ with new coefficients $\tilde{a}_i(\tilde{x},t) = a_i(d-\tilde{x},t) = a_i(x,t)$, $i = 1,\ 2$, and initial data $\tilde{\varphi}(\tilde{x}) = \varphi(d-\tilde{x}) = \varphi(x)$, $\tilde{\psi}(\tilde{x}) = \psi(d-\tilde{x}) = \psi(x)$. In the equation (29) and the boundary condition (31) there are a minus, because the derivative is $u_x(x,t) = -\tilde{u}_{\tilde{x}}(\tilde{x},t)$. Here the trapezium $\tilde{\Delta}_1 \cup \tilde{\Delta}_2$ is formed by two triangles

$$\tilde{\Delta}_1 = \{\{\tilde{x},t\}:\ g_1(\tilde{x},t) \geq g_1(0,0),$$
$$g_2(\tilde{x},t) \leq g_2(d,0),\ \tilde{x} \in [0,d],\ t \in [0,d_2]\},$$
$$\tilde{\Delta}_2 = \{\{\tilde{x},t\}:\ g_1(\tilde{x},t) \leq g_1(0,0),$$
$$\tilde{x} \in [0,\tilde{x}_0],\ t \in [0,d_2]\},$$

where the point $\{\tilde{x}_0, t_0\}$ is the unique solution of a similar system of equations $g_1(\tilde{x},t) = g_1(0,0)$, $g_2(\tilde{x},t) = g_2(d,0)$ for $0 \leq \tilde{x} \leq d$, $t \geq 0$.

After this non-degenerate change $x = d - \tilde{x}$, $t = \tilde{t}$, the characteristic functions $y_i = g_i(x,t) = C_i$, $x,\ t \geq 0$, and their inverse functions $x = h_i\{y_i,t\}$, $t \geq 0$, $t = h^{(i)}[x,y_i]$, $x \geq 0$, $i = 1,\ 2$, become implicit functions:

$$\tilde{y}_i = \tilde{g}_i(\tilde{x},t) = g_i(d-\tilde{x},t) = g_i(x,t),\ x,\ t \geq 0, \tag{32}$$

$$\tilde{x} = \tilde{h}_i\{\tilde{y}_i,t\} = d - h_i\{\tilde{y}_i,t\},\ t \geq 0, \tag{33}$$

$$t = \tilde{h}^{(i)}[\tilde{x},\tilde{y}_i] = h^{(i)}[d-\tilde{x},\tilde{y}_i] = h^{(i)}[x,\tilde{y}_i],\ x \geq 0. \tag{34}$$

For them, from the inversion identities (5)–(7), respectively, we derive similar inversion identities:

$$\tilde{g}_i(\tilde{h}_i\{\tilde{y}_i,t\},t) = \tilde{y}_i,\ t \geq 0,$$
$$\tilde{h}_i\{\tilde{g}_i(\tilde{x},\ t),\ t\} = \tilde{x},\ \tilde{x} \geq 0,\ i = 1,\ 2,$$
$$\tilde{g}_i(\tilde{x},\tilde{h}^{(i)}[\tilde{x},\tilde{y}_i]) = \tilde{y}_i,$$
$$\tilde{x} \geq 0,\ \tilde{h}^{(i)}[\tilde{x},\ \tilde{g}_i(\tilde{x},\ t)] = t,\ t \geq 0,\ i = 1,\ 2,$$
$$\tilde{h}_i\{\tilde{y}_i,\ \tilde{h}^{(i)}[\tilde{x},\tilde{y}_i]\} = \tilde{x},\ \tilde{x} \geq 0,$$
$$\tilde{h}^{(i)}[\tilde{h}_i\{\tilde{y}_i,\ t\},\tilde{y}_i] = t,\ t \geq 0,\ i = 1,\ 2.$$

In view of Theorem 3, the unique and stable classical solution of the mixed problem (29)–(31) in a triangle $\tilde{\Delta}_1$ is obtained by restricting the unique and stable classic solutions $\tilde{u}_-(\tilde{x},t)$ to $\tilde{\Delta}_1$ the form (13), in which the characteristic functions $\tilde{g}_1,\ \tilde{h}_1,\ \tilde{h}^{(1)}$ are replaced by functions $\tilde{g}_2,\ \tilde{h}_2,\ \tilde{h}^{(2)}$ and vice versa. As a result, from formula (13) for $\varphi(x) = \tilde{\varphi}_1(\tilde{x})$, $\psi(x) = \tilde{\psi}_1(\tilde{x})$, $\tilde{x} \in [0,d]$, at $(\tilde{x},t) \in \tilde{\Delta}_1$ we have the classic solutions

$$\tilde{u}_1(\tilde{x},t) = \tilde{\varphi}_1(\tilde{h}_1\{\tilde{g}_1(\tilde{x},t),0\}) +$$
$$+\int_{\tilde{h}_1\{\tilde{g}_1(\tilde{x},t),0\}}^{\tilde{h}_2\{\tilde{g}_2(\tilde{x},t),0\}} \frac{\tilde{a}_2(\nu,0)\tilde{\varphi}_1'(\nu) + \tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0) + \tilde{a}_2(\nu,0)}d\nu,$$
$$\{\tilde{x},t\} \in \tilde{\Delta}_1. \tag{35}$$

In these solutions (35), we make an inverse substitution $\tilde{x} = d - x$. To do this, using the functions (33) and (34), we derive equalities

$$\tilde{\varphi}_1(\tilde{h}_i\{\tilde{g}_i(\tilde{x},t),0\}) = \varphi_1(d - \tilde{h}_i\{\tilde{g}_i(\tilde{x},t),0\}) =$$
$$= \varphi_1(h_i\{\tilde{g}_i(\tilde{x},t),0\}) = \varphi_1(h_i\{g_i(d-\tilde{x},t),0\}) =$$
$$= \varphi_1(h_i\{g_i(x,t),0\}),\ \tilde{a}_i(\nu,0) = a_i(d-\nu,0),$$
$$\tilde{\varphi}_1'(\nu) = -\varphi_1'(d-\nu),\ \tilde{\psi}_1(\nu) = \psi_1(d-\nu),$$
$$\tilde{h}_i\{\tilde{g}_i(\tilde{x},t),0\} = d - h_i\{\tilde{g}_i(\tilde{x},t),0\} =$$
$$= d - h_i\{g_i(d-\tilde{x},t),0\} =$$
$$= d - h_i\{g_i(x,t),0\},\ i = 1,\ 2,$$

$$\int_{\tilde{h}_1\{\tilde{g}_1(\tilde{x},t),0\}}^{\tilde{h}_2\{\tilde{g}_2(\tilde{x},t),0\}} \frac{\tilde{a}_1(\nu,0)\tilde{\varphi}_1'(\nu) + \tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0) + \tilde{a}_2(\nu,0)}d\nu =$$
$$= \int_{d-h_1\{g_1(x,t),0\}}^{d-h_2\{g_2(x,t),0\}} \left[\frac{-a_1(d-\nu,0)(\varphi_1)_\nu'(d-\nu)}{a_1(d-\nu,0) + a_2(d-\nu,0)} + \right.$$
$$\left. + \frac{\psi_1(d-\nu)}{a_1(d-\nu,0) + a_2(d-\nu,0)}\right]d\nu =$$
$$= \int_{d-h_1\{g_1(x,t),0\}}^{d-h_2\{g_2(x,t),0\}} \left[\frac{a_1(d-\nu,0)(\varphi_1)'(d-\nu)}{a_1(d-\nu,0) + a_2(d-\nu,0)} + \right.$$
$$\left. + \frac{\psi_1(d-\nu)}{a_1(d-\nu,0) + a_2(d-\nu,0)}\right]d\nu =$$
$$= \int_{h_2\{g_2(x,t),0\}}^{h_1\{g_1(x,t),0\}} \frac{a_1(\rho,0)(\varphi_1)'(\rho) + \psi_1(\rho)}{a_1(\rho,0) + a_2(\rho,0)}d\rho,$$

where we made changes of integration variables $\rho = d - \nu$.

These equalities indicate that the classic solutions (35) coincide with the solutions

$$\tilde{u}_1(\tilde{x},t) = \varphi_1(h_2\{g_2(x,t),0\}) +$$

$$+\int_{h_2\{g_2(x,t),0\}}^{h_1\{g_1(x,t),0\}} \frac{a_1(\rho,0)\varphi_1'(\rho)+\psi_1(\rho)}{a_1(\rho,0)+a_2(\rho,0)}d\rho = u_1(x,t),$$

$$\{x,t\}\in\Delta_1, \tag{36}$$

which is equal to solution $u_1(x,t)$ on Δ_1 the form (24) for $n=k=1$ from Theorem 5.

From the formula (14) at the functions $\varphi(x)=\varphi_1(x)$, $\psi(x)=\psi_1(x)$, $\mu=\mu_2$, $\alpha=\alpha_1$, $\beta=\beta_1$, $\gamma=\gamma_1$ for the characteristic functions $g_1,\ h_1,\ h^{(1)}$ replaced by the functions $\tilde{g}_2,\ h_2,\ h^{(2)}$, coefficients $a_1,\alpha_1,\ \beta_1,\ \gamma_1$ replaced by coefficients a_2, $\alpha_2,\ -\beta_2,\ \gamma_2$ and vice versa for $\{\tilde{x},t\}\in\tilde{\Delta}_2$, similarly we obtain classical solutions

$$\tilde{u}_3(\tilde{x},t)=\tilde{\varphi}_1(0)+$$

$$+\int_0^{\tilde{h}_2\{\tilde{g}_2(\tilde{x},t),0\}} \frac{\tilde{a}_2(\nu,0)\tilde{\varphi}_1'(\nu)+\tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0)+\tilde{a}_2(\nu,0)}d\nu+$$

$$+\frac{1}{\tilde{\eta}_2(\tilde{h}^{(1)}[0,\tilde{g}_1(\tilde{x},t)])}\int_0^{\tilde{h}^{(1)}[0,\tilde{g}_1(\tilde{x},t)]}\tilde{\eta}_2(\rho)\times$$

$$\left\{\frac{\tilde{a}_2(0,\rho)\mu_2(\rho)}{\tilde{a}_2(0,\rho)\alpha_2(\rho)+\beta_2(\rho)}-\frac{\tilde{a}_2(0,\rho)\alpha_2(\rho)-\beta_2(\rho)}{\tilde{a}_2(0,\rho)\alpha_2(\rho)+\beta_2(\rho)}\times\right.$$

$$\times\frac{\tilde{a}_2(0,\rho)}{\tilde{a}_1(0,\rho)}\frac{\tilde{a}_2(\nu,0)\tilde{\varphi}_1'(\nu)+\tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0)+\tilde{a}_2(\nu,0)}\Big|_{\nu=\tilde{h}_2\{\tilde{g}_2(0,\rho),0\}}\times$$

$$\frac{\partial\tilde{h}_2\{\tilde{g}_2(0,\rho),0\}}{\partial\rho}-\frac{\tilde{a}_2(0,\rho)\gamma_2(\rho)}{\tilde{a}_2(0,\rho)\alpha_2(\rho)+\beta_2(\rho)}\Big[\tilde{\varphi}_1(0)+$$

$$\left.+\int_0^{\tilde{h}_2\{\tilde{g}_2(0,\rho),0\}}\frac{\tilde{a}_2(\nu,0)\tilde{\varphi}_1'(\nu)+\tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0)+\tilde{a}_2(\nu,0)}d\nu\Big]\right\}d\rho, \tag{37}$$

in which, due to the equality $\tilde{a}_2(0,\rho)=a_2(d,\rho)$, the integrating factor is equal to the factor

$$\tilde{\eta}_2(t)=\exp\left\{\int_0^t\frac{\tilde{a}_2(0,\rho)\gamma_2(\rho)}{\tilde{a}_2(0,\rho)\alpha_2(\rho)+\beta_2(\rho)}d\rho\right\}=$$

$$=\exp\left\{\int_0^t\frac{a_2(d,\rho)\gamma_2(\rho)}{a_2(d,\rho)\alpha_2(\rho)+\beta_2(\rho)}d\rho\right\}=\eta_2(t).$$

The integrating factor $\tilde{\eta}_2(\rho)$ is taken from above with the wave, since in the second integral of formula (37) the upper limit of integration is equal to the values $\tilde{h}^{(1)}[0,\tilde{g}_1(\tilde{x},t)]$ with waves and before this integral in the denominator there is this factor $\tilde{\eta}_2$ from the values $\tilde{h}^{(1)}[0,\tilde{g}_1(\tilde{x},t)]$ with waves.

In these solutions (37), we make an inverse substitution $\tilde{x}=d-x$ and, using functions (32)–(34), we derive the auxiliary relations

$$\tilde{\varphi}_1(0)=\varphi_1(d),\ \tilde{\psi}_1(0)=\psi_1(d),$$

$$\tilde{a}_i(\nu,\rho)=a_i(d-\nu,\rho),\ \tilde{g}_i(0,\rho)=g_i(d,\rho),$$

$$\tilde{h}^{(i)}[0,\tilde{g}_i(\tilde{x},t)]=h^{(i)}[d,\tilde{g}_i(\tilde{x},t)]=$$

$$=h^{(i)}[d,g_i(d-\tilde{x},t)]=h^{(i)}[d,g_i(x,t)],\ i=1,\ 2.$$

$$\int_0^{\tilde{h}_2\{\tilde{g}_2(\tilde{x},t),0\}}\frac{\tilde{a}_2(\nu,0)\tilde{\varphi}_1'(\nu)+\tilde{\psi}_1(\nu)}{\tilde{a}_1(\nu,0)+\tilde{a}_2(\nu,0)}d\nu=$$

$$=\int_0^{d-h_2\{g_2(x,t),0\}}\left[\frac{-a_2(d-\nu,0)(\varphi_1)'_\nu(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}+\right.$$

$$\left.+\frac{\psi_1(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}\right]d\nu=$$

$$=\int_0^{d-h_2\{g_2(x,t),0\}}\left[\frac{a_2(d-\nu,0)\varphi_1'(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}+\right.$$

$$\left.+\frac{\psi_1(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}\right]d\nu=$$

$$=\int_{h_2\{g_2(x,t),0\}}^d\frac{a_2(\delta,0)\varphi_1'(\delta)+\psi_1(\delta)}{a_1(\delta,0)+a_2(\delta,0)}d\delta,$$

where we made changes of integration variables $\delta=d-\nu$.

Based on them, from the solutions (37) by inverse replacing $\tilde{x}=d-x$ with similar transformations, we find the functions

$$\tilde{u}_3(\tilde{x},t)=\varphi_1(d)+$$

$$+\int_{h_2\{g_2(x,t),0\}}^d\frac{a_2(\rho,0)\varphi_1'(\rho)+\psi_1(\rho)}{a_1(\rho,0)+a_2(\rho,0)}d\rho+$$

$$+\frac{1}{\eta_2(h^{(1)}[d,g_1(x,t)])}\int_0^{h^{(1)}[d,g_1(x,t)]}\eta_2(\rho)\times$$

$$\left\{\frac{a_2(d,\rho)\mu_2(\rho)}{a_2(d,\rho)\alpha_2(\rho)+\beta_2(\rho)}+\frac{a_2(d,\rho)\alpha_2(\rho)-\beta_2(\rho)}{a_2(d,\rho)\alpha_2(\rho)+\beta_2(\rho)}\times\right.$$

$$\times\left[\frac{a_2(d-\nu,d_k)\varphi_1'(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}+\right.$$

$$\left.+\frac{\psi_1(d-\nu)}{a_1(d-\nu,0)+a_2(d-\nu,0)}\right]\Big|_{\nu=d-h_2\{g_2(d,\rho),0\}}\times$$

$$\times\frac{a_2(d,\rho)}{a_1(d,\rho)}\frac{\partial h_2\{g_2(d,\rho),0\}}{\partial\rho}-$$

$$-\frac{a_2(d,\rho)\gamma_2(\rho)}{a_2(d,\rho)\alpha_2(\rho)+\beta_2(\rho)}\Big[\varphi_1(d)+$$

$$+\int_{h_2\{g_2(d,\delta),0\}}^{d}\frac{a_2(\delta,0)\varphi_1'(\delta)+\psi_1(\delta)}{a_1(\delta,0)+a_2(\delta,0)}d\delta\Big]\Big\}d\rho=$$

$$=u_3(x,t),\ \{x,t\}\in\Delta_3, \tag{38}$$

which is equal to the classic solutions $u_3(x,t)$ for $\{x,t\}\in\Delta_3$ the form (26) for $n=k=1$ from Theorem 5.

By virtue of correctness criterion from Theorem 3, the correctness criterion of auxiliary mixed problem (29)–(31) at $\mu=\mu_2$ for $\{\tilde{x},t\}$ on trapezoid $\tilde{\Delta}_1\cup\tilde{\Delta}_2$ is the smoothness requirements

$$\tilde{\varphi}\in C^2[0,\tilde{x}_0],\ \tilde{\psi}\in C^1[0,\tilde{x}_0],\ \mu_2\in C^1[0,d_2]. \tag{39}$$

and matching conditions (9), (10) at $\mu=\mu_2$ for variables $\{\tilde{x},t\}$.

Here we apply the relations obtained above, and from the above correctness criterion with the smoothness requirements (39) after the reverse replacement of the variable $\tilde{x}=d-x$ we arrive at the smoothness requirements

$$\varphi\in C^2[x_0,d],\ \psi\in C^1[x_0,d],\ \mu_2\in C^1[0,d_2], \tag{40}$$

with matching conditions 22), (23) for $i=2$ on trapezoid $\Delta_1\cup\Delta_3$ from Theorem 5.

From the condition $\alpha_1(0)=0$ and the equalities (12) of Theorem 3 for the solvability of the mixed problem (29)–(31) in $\tilde{\Delta}_1\cap\tilde{\Delta}_2$, after replacing the coefficients $a_1,\ \alpha_1,\ \beta_1,\ \gamma_1$ with the coefficients $a_2,\ \alpha_2,\ -\beta_2,\ \gamma_2$ and vice versa and the variable $\tilde{x}=d-x$, follow the condition $\alpha_2(0)=0$ and the equalities (21) for $i=2$ for the solvability of the mixed problem (15)–(17) in the trapezoid $\triangle_1\cap\triangle_3$ of Theorem 5.

The smoothness requirements (28) on trapezoid $\Delta_1\cup\Delta_2$ and (40) on trapezoid $\Delta_1\cup\Delta_3$ coincide with the smoothness requirements (18) for $n=k=1$ in rectangle Q_1. The matching conditions (9), (10) for $\mu=\mu_1$ on trapezoid $\Delta_1\cup\Delta_2$ and the matching conditions (22), (23) for $\mu=\mu_2$ on trapezoid $\Delta_1\cup\Delta_3$ coincide with matching conditions (22), (23) from Theorem 5. Thus, the validity of Theorem 5 for mixed problem (15)–(17) on rectangle Q_1 is justified, because the solutions (36), (14), (38) are the same as the solutions from (24)–(26) for $n=k=1$ on rectangle Q_1 in the Theorem 5. In a rectangle Q_1, twice continuous differentiability of functions u_{3-l} in triangles $\triangle_{3-l},\ l=2,1,0$, and on characteristics $g_2(x,t)=g_2(0,0)$, $g_1(x,t)=g_1(d,0)$ follows from article [9], since in (36) classic solutions $\tilde{u}_1(\tilde{x},t)=u_1(x,t)$ on Δ_1.

At the second step of mathematical induction, we assume that formulas (24)–(26) for $k=\overline{1,n}$ of unique and stable classic solutions $u\in C^2(Q_n)$ and the correctness criterion for problem (15)–(17) indicated in Theorem 5 are true on a rectangle Q_n, and show that they are true on the rectangle Q_{n+1}. The mixed problem (15)–(17) in a rectangle G_{n+1} for a function $u=u(x,t)$ by a non-degenerate change of variables $x=\breve{x},\ t=\breve{t}+d_2$ reduces for a function $\breve{u}(x,\breve{t})=u(x,\breve{t}+d_2)=u(x,t)$ in $\breve{G}_n=[0,d]\times[d_n,d_{n+1}]$ to an equivalent mixed problem:

$$\breve{u}_{\breve{t}\breve{t}}(x,\breve{t})+(\breve{a}_1(x,\breve{t})-\breve{a}_2(x,\breve{t}))\breve{u}_{x\breve{t}}(x,\breve{t})-$$

$$-\breve{a}_1(x,\breve{t})\breve{a}_2(x,\breve{t})\breve{u}_{xx}(x,\breve{t})-\frac{(\breve{a}_2)_{\breve{t}}(x,\breve{t})}{\breve{a}_2(x,\breve{t})}u_{\breve{t}}(x,\breve{t})-$$

$$-\breve{a}_1(x,\breve{t})(a_2)_x(x,\breve{t})\breve{u}_x(x,\breve{t})=0,\ \{x,\breve{t}\}\in\breve{G}_n, \tag{41}$$

$$\breve{u}|_{\breve{t}=d_n}=\varphi_{n+1}(x),\ \frac{\partial\breve{u}}{\partial\breve{t}}|_{\breve{t}=d_n}=\psi_{n+1}(x),\ x\in[0,d], \tag{42}$$

$$[\breve{\alpha}_i(\breve{t})\breve{u}_{\breve{t}}(x,\breve{t})+\breve{\beta}_i(\breve{t})\breve{u}_x(x,\breve{t})+\breve{\gamma}_i(\breve{t})\breve{u}(x,\breve{t})]_{x=\hat{d}_i}=$$

$$=\breve{\mu}_i(\breve{t}),\ i=1,2,\ \breve{t}\in[d_n,d_{n+1}], \tag{43}$$

with the coefficients $\breve{a}_i(x,\breve{t})=a_i(x,\breve{t}+d_2)=a_i(x,t)$, and the boundary data $\breve{\mu}_i(\breve{t})=\mu_i(\breve{t}+d_2)=\mu_i(t),\ i=1,2$.

In the telegraph equation (15) after the implemented non-degenerate replacement $x=\breve{x},t=\breve{t}+d_2$ the implicit characteristic functions $y_i=g_i(x,t)=C_i,\ x,t\geq 0$, and their implicit inverse functions $x=h_i\{y_i,t\},\ t\geq 0,\ t=h^{(i)}[x,y_i],\ x\geq 0,\ i=1,2$, turn into functions:

$$\breve{y}_i=\breve{g}_i(x,\breve{t})=g_i(x,\breve{t}+d_2)=g_i(x,t),\ x,\ t\geq 0, \tag{44}$$

$$x=\breve{h}_i\{\breve{y}_i,\breve{t}\}=h_i\{\breve{y}_i,\breve{t}+d_2\}=h_i\{\tilde{y}_i,t\},\ t\geq 0, \tag{45}$$

$$\breve{t}=\breve{h}^{(i)}[x,\breve{y}_i]=h^{(i)}[x,\breve{y}_i]-d_2,\ x\geq 0. \tag{46}$$

From the inversion identities (5)–(7), the equivalent inversion identities are derived, respectively:

$$\breve{g}_i(\breve{h}_i\{\breve{y}_i,\ \breve{t}\},\ \breve{t})=\breve{y}_i,\ \breve{t}\geq 0,$$

$$\breve{h}_i\{\breve{g}_i(x,\breve{t}),\ \breve{t}\}=x,\ x\geq 0,\ i=1,2,$$

$$\breve{g}_i(x,\ \breve{h}^{(i)}[x,\breve{y}_i])=\breve{y}_i,\ x\geq 0,$$

$$\breve{h}^{(i)}[x,\breve{g}_i(x,\breve{t})]=\breve{t},\ \breve{t}\geq 0,\ i=1,2,$$

$$\breve{h}_i\{\breve{y}_i,\breve{h}^{(i)}[x,\breve{y}_i]\}=x,\ x\geq 0,$$

$$\breve{h}^{(i)}[\breve{h}_i\{\breve{y}_i,\ \breve{t}\},\breve{y}_i]=\breve{t},\ \breve{t}\geq 0,\ i=1,2.$$

By the assumption of mathematical induction, the mixed problem (41)–(43) in $\breve{G}_n$ has a unique and stable classical solution $\breve{u}\in C^2(\breve{G}_n)$, which is expressed by formulas (24)–(26) from Theorem 5 for $k=n$ with a variable $\breve{t}$ instead of a variable t. In particular, from formula (24) for $k=n$ and initial data $\varphi_{n+1},\ \psi_{n+1}$

from (27) in the triangle $\breve{\Delta}_{3n-2}$ we find the classic solutions

$$\breve{u}_{3n-2}(x,\breve{t})=\varphi_{n+1}(\breve{h}_2\{\breve{g}_2(x,\breve{t}),d_n\})+$$
$$+\int_{\breve{h}_2\{\breve{g}_2(x,\breve{t}),d_n\}}^{\breve{h}_1\{\breve{g}_1(x,\breve{t}),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu,$$
$$\{x,\breve{t}\}\in\breve{\Delta}_{3n-2}, \tag{47}$$

in which we return to the old variable $\breve{t}=t-d_2$. Using the functions (45) and (46), we calculate

$$\breve{a}_i(\nu,d_n)=a_i(\nu,d_n+d_2)=a_i(\nu,d_{n+1}),$$
$$\breve{h}_i\{\breve{g}_i(x,\breve{t}),\,d_n\}=\breve{h}_i\{g_i(x,\breve{t}+d_2),\,d_n\}=$$
$$=h_i\{g_i(x,\breve{t}+d_2),d_n+d_2\}=h_i\{g_i(x,t),d_{n+1}\},$$
$$\int_{\breve{h}_2\{\breve{g}_2(x,\breve{t}),d_n\}}^{\breve{h}_1\{\breve{g}_1(x,\breve{t}),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_{n+1})+\breve{a}_2(\nu,d_{n+1})}d\nu=$$
$$=\int_{h_2\{g_2(x,t),d_{n+1}\}}^{h_1\{g_1(x,t),d_{n+1}\}}\left[\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+\right.$$
$$\left.+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\right]d\nu,\ i=1,\,2.$$

As a result of the reverse change $\breve{t}=t-d_2$ from the solution (47), we arrive at the solutions $u_{3n+1}(x,t)$ for $(x,t)\in\Delta_{3n+1}$ the form (24) for $k=n+1$ from Theorem 5.

The assumption of mathematical induction for the mixed problem (41)–(43) in $\breve{G}_n$ formula (24) with $k=n$ and initial data φ_{n+1}, ψ_{n+1} from the formula (25) in a triangle $\breve{\Delta}_{3n-1}$ gives us the classic solutions

$$\breve{u}_{3n-1}(x,\breve{t})=\varphi_{n+1}(0)+$$
$$+\int_0^{\breve{h}_1\{\breve{g}_1(x,\breve{t}),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu+$$
$$+\frac{1}{\breve{\eta}_1(\breve{h}^{(2)}[0,\breve{g}_2(x,\breve{t})])}\int_{d_n}^{\breve{h}^{(2)}[0,\breve{g}_2(x,\breve{t})]}\breve{\eta}_1(\rho)\times$$
$$\left\{\frac{\breve{a}_1(0,\rho)\breve{\mu}_1(\rho)}{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)}-\frac{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)+\breve{\beta}_1(\rho)}{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)}\times\right.$$
$$\times\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}\Big|_{\nu=\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}\times$$
$$\times\frac{\breve{a}_1(0,\rho)}{\breve{a}_2(0,\rho)}\frac{\partial\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}{\partial\rho}-$$
$$-\frac{\breve{a}_1(0,\rho)\breve{\gamma}_1(\rho)}{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)}\left[\varphi_{n+1}(0)+\right.$$
$$\left.\left.\int_0^{\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu\right]\right\}d\rho. \tag{48}$$

Here we make a change of variable $\breve{t}=t-d_2$ and use functions (44)–(46) with transformations:

$$\breve{h}_i\{\breve{g}_i(x,\breve{t}),\tau\}=\breve{h}_i\{g_i(x,\breve{t}+d_2),\tau\}=$$
$$=h_i\{g_i(x,t),\tau+d_2\},\ \breve{h}_i\{\breve{g}_i(0,\rho),\tau\}=$$
$$=\breve{h}_i\{g_i(0,\rho+d_2),\tau\}=h_i\{g_i(0,\rho+d_2),\tau+d_2\},$$
$$\breve{h}^{(i)}[0,\breve{g}_i(x,\breve{t})]=\breve{h}^{(i)}[0,g_i(x,\breve{t}+d_2)]=$$
$$=h^{(i)}[0,g_i(x,t)]-d_2,\ \breve{\eta}_i(\breve{t})=\eta_i(\breve{t}+d_2),\ i=1,\,2,$$
$$\breve{\eta}_1(\breve{h}^{(2)}[0,\breve{g}_2(x,\breve{t})])=\eta_1(h^{(2)}[0,g_2(x,t)]),$$

$$\int_0^{\breve{h}_1\{\breve{g}_1(x,\breve{t}),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu=$$
$$=\int_0^{h_1\{g_1(x,t),d_{n+1}\}}\left[\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+\right.$$
$$\left.+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\right]d\nu,$$

$$\int_0^{\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu=$$
$$=\int_0^{h_1\{g_1(0,\rho+d_2),d_{n+1}\}}\left[\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+\right.$$
$$\left.+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\right]d\nu,$$

$$\int_{d_n}^{\breve{h}^{(2)}[0,\breve{g}_2(x,\breve{t})]}\breve{\eta}_1(\rho)\left\{\frac{\breve{a}_1(0,\rho)\breve{\mu}_1(\rho)}{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)}-\right.$$
$$-\frac{\breve{a}_1(0,\rho)[\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)+\breve{\beta}_1(\rho)]}{\breve{a}_2(0,\rho)[\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)]}\times$$
$$\times\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}\Big|_{\nu=\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}\times$$
$$\times\frac{\partial\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}{\partial\rho}-\frac{\breve{a}_1(0,\rho)\breve{\gamma}_1(\rho)}{\breve{a}_1(0,\rho)\breve{\alpha}_1(\rho)-\breve{\beta}_1(\rho)}\times$$
$$\times\int_0^{\breve{h}_1\{\breve{g}_1(0,\rho),d_n\}}\left[\frac{\breve{a}_1(\nu,d_n)\varphi'_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}+\right.$$
$$\left.\left.+\frac{\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}\right]d\nu\right\}d\rho=$$

$$= \int_{d_n}^{h^{(2)}[0,g_2(x,t)]-d_2} \eta_1(\rho+d_2)\times$$

$$\times\Big\{\frac{a_1(0,\rho+d_2)\mu_1(\rho+d_2)}{a_1(0,\rho+d_2)\alpha_1(\rho+d_2)-\beta_1(\rho+d_2)}-$$

$$\frac{a_1(0,\rho+d_2)}{a_2(0,\rho+d_2)}\frac{a_1(0,\rho+d_2)\alpha_1(\rho+d_2)+\beta_1(\rho+d_2)}{a_1(0,\rho+d_2)\alpha_1(\rho+d_2)-\beta_1(\rho+d_2)}\times$$

$$\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big|_{\nu=h_1\{g_1(0,\rho+d_2),d_{n+1}\}}$$

$$\times\frac{\partial h_1\{g_1(0,\rho+d_2),d_{n+1}\}}{\partial\rho}-$$

$$-\frac{a_1(0,\rho+d_2)\gamma_1(\rho+d_2)}{a_1(0,\rho+d_2)\alpha_1(\rho+d_2)-\beta_1(\rho+d_2)}\times$$

$$\times\int_0^{h_1\{g_1(0,\rho+d_2),d_{n+1}\}}\Big[\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+$$

$$+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu\Big\}d\rho=$$

$$= \int_{d_{n+1}}^{h^{(2)}[0,g_2(x,t)]} \eta_1(\delta)\Big\{\frac{a_1(0,\delta)\mu_1(\delta)}{a_1(0,\delta)\alpha_1(\delta)-\beta_1(\delta)}-$$

$$-\frac{a_1(0,\delta)}{a_2(0,\delta)}\frac{a_1(0,\delta)\alpha_1(\delta)+\beta_1(\delta)}{a_1(0,\delta)\alpha_1(\delta)-\beta_1(\delta)}\times$$

$$\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big|_{\nu=h_1\{g_1(0,\delta),d_{n+1}\}}$$

$$\times\frac{\partial h_1\{g_1(0,\delta),d_{n+1}\}}{\partial\rho}-\frac{a_1(0,\delta)\gamma_1(\delta)}{a_1(0,\delta)\alpha_1(\delta)-\beta_1(\delta)}$$

$$\times\int_0^{h_1\{g_1(0,\delta),d_{n+1}\}}\Big[\frac{a_1(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+$$

$$+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu\Big\}d\delta,$$

where we made a replacement of the integration variable $\delta=\rho+d_2$.

After the change $\breve{t}=t-d_2$, from solutions (48) we obtain a solution u_{3n+2} in Δ_{3n+2} the form (25) under $k=n+1$ of the Theorem 5.

In view of the assumption of mathematical induction for the mixed problem (41)–(43) in a triangle $\breve{\Delta}_{3n}$, according to formula (26) with $k=n$, initial data φ_{n+1}, ψ_{n+1} from (27) and variable $\breve{t}=t-d_2$, we have the classic solutions

$$\breve{u}_{3n}(x,\breve{t})=\varphi_{n+1}(d)+$$

$$+\int_{\breve{h}_2\{\breve{g}_2(x,\breve{t}),d_n\}}^{d}\frac{\breve{a}_2(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu+$$

$$+\frac{1}{\breve{\eta}_2(\breve{h}^{(1)}[d,\breve{g}_1(x,\breve{t})])}\times$$

$$\times\int_{d_n}^{\breve{h}^{(1)}[d,\breve{g}_1(x,\breve{t})]}\breve{\eta}_2(\rho)\Big\{\frac{\breve{a}_2(d,\rho)\breve{\mu}_2(\rho)}{\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)+\breve{\beta}_2(\rho)}+$$

$$+\frac{\breve{a}_2(d,\rho)[\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)-\breve{\beta}_2(\rho)]}{\breve{a}_1(d,\rho)[\breve{a}_2(d,\rho)\breve{\alpha}_{(}\rho)+\breve{\beta}_2(\rho)]}\times$$

$$\times\Big[\frac{\breve{a}_2(d-\nu,d_n)\varphi'_{n+1}(d-\nu)}{\breve{a}_1(d-\nu,d_n)+\breve{a}_2(d-\nu,d_n)}+$$

$$+\frac{\psi_{n+1}(d-\nu)}{\breve{a}_1(d-\nu,d_n)+\breve{a}_2(d-\nu,d_n)}\Big]\Big|_{\nu=d-\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}$$

$$\times\frac{\partial\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}{\partial\rho}-\frac{\breve{a}_2(d,\rho)\breve{\gamma}_2(\rho)}{\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)+\breve{\beta}_2(\rho)}\times$$

$$\times\Big[\varphi_{n+1}(d)+\int_{\breve{h}_1\{\breve{g}_1(d,\rho),d_n\}}^{d}\Big(\frac{\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}+$$

$$+\frac{\breve{a}_2(\nu,d_n)\varphi'_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}\Big)d\nu\Big]\Big\}d\rho. \tag{49}$$

Again applying functions (44)–(46) and more transformations:

$$\breve{h}^{(i)}[d,\breve{g}_i(x,\breve{t})]=\breve{h}^{(i)}[d,g_i(x,\breve{t}+d_2)]=$$

$$=h^{(i)}[d,g_i(x,t)]-d_2,\ \breve{\eta}_i(\breve{t})=\eta_i(\breve{t}+d_2),$$

$$\breve{h}_i\{\breve{g}_i(d,\rho),\tau\}=\breve{h}_i\{g_i(d,\rho+d_2),\tau\}=$$

$$=h_i\{g_i(d,\rho+d_2),\tau+d_2\},$$

$$\breve{\eta}_2(\breve{h}^{(1)}[d,\breve{g}_1(x,\breve{t})])=\eta_2(h^{(1)}[d,g_2(x,t)]),$$

$$\int_{\breve{h}_2\{\breve{g}_2(x,\breve{t}),d_n\}}^{d}\frac{\breve{a}_2(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu=$$

$$=\int_{h_2\{g_2(x,t),d_{n+1}\}}^{d}\Big[\frac{a_2(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+$$

$$+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu,$$

$$\int_{\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}^{d}\Big[\frac{\breve{a}_2(\nu,d_n)\varphi'_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}+$$

$$+\frac{\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}\Big]d\nu=$$

$$=\int_{h_2\{g_2(d,\rho+d_2),d_{n+1}\}}^{d}\Big[\frac{a_2(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}d\nu+$$

$$+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu,$$

$$\int_{d_n}^{\breve{h}^{(1)}[d,\breve{g}_1(x,\breve{t})]}\breve{\eta}_2(\rho)\Big\{\frac{\breve{a}_2(d,\rho)\breve{\mu}_2(\rho)}{\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)+\breve{\beta}_2(\rho)}+$$
$$+\frac{\breve{a}_2(d,\rho)[\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)-\breve{\beta}_2(\rho)]}{\breve{a}_1(d,\rho)[\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)+\breve{\beta}_2(\rho)]}\times$$
$$\times\Big[\frac{\breve{a}_2(d-\nu,d_n)\varphi'_{n+1}(d-\nu)}{\breve{a}_1(d-\nu,d_n)+\breve{a}_2(d-\nu,d_n)}+$$
$$+\frac{\psi_{n+1}(d-\nu)}{\breve{a}_1(d-\nu,d_n)+\breve{a}_2(d-\nu,d_n)}\Big]\Big|_{\nu=d-\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}\times$$
$$\times\frac{\partial\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}{\partial\rho}-\frac{\breve{a}_2(d,\rho)\breve{\gamma}_2(\rho)}{\breve{a}_2(d,\rho)\breve{\alpha}_2(\rho)+\breve{\beta}_2(\rho)}\times$$
$$\int_{\breve{h}_2\{\breve{g}_2(d,\rho),d_n\}}^{d}\frac{\breve{a}_2(\nu,d_n)\varphi'_{n+1}(\nu)+\psi_{n+1}(\nu)}{\breve{a}_1(\nu,d_n)+\breve{a}_2(\nu,d_n)}d\nu\Big\}d\rho=$$

$$=\int_{d_n}^{h^{(1)}[d,g_1(x,t)]-d_2}\eta_2(\rho+d_2)\times$$
$$\times\Big\{\frac{a_2(d,\rho+d_2)\mu_2(\rho+d_2)}{a_2(d,\rho+d_2)\alpha_2(\rho+d_2)+\beta_2(\rho+d_2)}+$$
$$+\frac{a_2(d,\rho+d_2)}{a_1(d,\rho+d_2)}\times$$
$$\times\frac{a_2(d,\rho+d_2)\alpha_2(\rho+d_2)-\beta_2(\rho+d_2)}{a_2(d,\rho+d_2)\alpha_2(\rho+d_2)+\beta_2(\rho+d_2)}\times$$
$$\times\Big[\frac{a_2(d-\nu,d_{n+1})\varphi'_{n+1}(d-\nu)}{a_1(d-\nu,d_{n+1})+a_2(d-\nu,d_{n+1})}+$$
$$+\frac{\psi_{n+1}(d-\nu)}{a_1(d-\nu,d_{n+1})+a_2(d-\nu,d_{n+1})}\Big]$$
$$\Big|_{\nu=d-h_2\{g_2(d,\rho+d_2),d_{n+1}\}}\frac{\partial h_2\{g_2(d,\rho+d_2),d_{n+1}\}}{\partial\rho}-$$
$$-\frac{a_2(d,\rho+d_2)\gamma_2(\rho+d_2)}{a_2(d,\rho+d_2)\alpha_2(\rho+d_2)+\beta_2(\rho+d_2)}\times$$
$$\times\int_{h_2\{g_2(d,\rho+d_2),d_{n+1}\}}^{d}\Big[\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+$$
$$+\frac{a_2(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu\Big\}d\rho=$$
$$=\int_{d_{n+1}}^{h^{(1)}[d,g_1(x,t)]}\eta_2(\delta)\Big\{\frac{a_2(d,\delta)\mu_2(\delta)}{a_2(d,\delta)\alpha_2(\delta)+\beta_2(\delta)}+$$
$$+\frac{a_2(d,\delta)}{a_1(d,\delta)}\,\frac{a_2(d,\delta)\alpha_2(\delta)-\beta_2(\delta)}{a_2(d,\delta)\alpha_2(\delta)+\beta_2(\delta)}\times$$
$$\times\Big[\frac{a_2(d-\nu,d_{n+1})\varphi'_{n+1}(d-\nu)}{a_1(d-\nu,d_{n+1})+a_2(d-\nu,d_{n+1})}+$$
$$+\frac{\psi_{n+1}(d-\nu)}{a_1(d-\nu,d_{n+1})+a_2(d-\nu,d_{n+1})}\Big]$$
$$\Big|_{\nu=d-h_2\{g_2(d,\delta),d_{n+1}\}}\frac{\partial h_2\{g_2(d,\delta),d_{n+1}\}}{\partial\rho}-$$
$$-\frac{a_2(d,\delta)\gamma_2(\delta)}{a_2(d,\delta)\alpha_2(\delta)+\beta_2(\delta)}\times$$
$$\times\int_{h_2\{g_2(d,\delta),d_{n+1}\}}^{d}\Big[\frac{a_2(\nu,d_{n+1})\varphi'_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}+$$
$$+\frac{\psi_{n+1}(\nu)}{a_1(\nu,d_{n+1})+a_2(\nu,d_{n+1})}\Big]d\nu\Big\}d\delta.$$

Here we have made a change of integration variables $\delta=\rho+d_2$.

So, by changing $\breve{t}=t-d_2$ the solutions (49) turns into a solution u_{3n+3} in Δ_{3n+3} the form (26) under $k=n+1$ from Theorem 5. This implies the validity of formulas (24)–(26) for $k=n+1$ the classic solutions $u\in C^2(G_{n+1})$ of problem (15)–(17) on G_{n+1}.

By virtue of the assumption of mathematical induction on a rectangle Q_n, the correctness criterion for the mixed problem (41)–(43) for a rectangle $\breve{G}_n$ according to (18) consists of the smoothness requirements

$$\varphi\in C^2[0,d],\ \psi\in C^1[0,d],\ \breve{\mu}_1,\ \breve{\mu}_2\in C^2[d_n,d_{n+1}],\tag{50}$$

and matching conditions (22), (23). Since part of the transformations used above reduces smoothness requirements (51) by replacement $\breve{t}=t-d_2$ to smoothness requirements (18) for $k=n+1$ on G_{n+1} from Theorem 5, we proved the assertion of Theorem 5 on the rectangle G_{n+1} by induction on n.

It remains to verify the doubly continuous differentiability of the unique and stable solution of the main mixed problem (15)–(17) on the common side $x=d_{n+1}$ of the rectangles Q_n and G_{n+1}. Indeed, by the assumption of mathematical induction, classical solutions $\breve{u}_{3n-2}$, $\breve{u}_{3n-1}$, $\breve{u}_{3n}$ are twice continuously differentiable not only in triangles $\breve{\Delta}_{3n-2}$, $\breve{\Delta}_{3n-1}$, $\breve{\Delta}_{3n}$, but also on pieces of their common characteristics $\breve{g}_2(x,\breve{t})=\breve{g}_2(0,d_n)$, $\breve{g}_1(x,\breve{t})=\breve{g}_1(d,d_n)$ of Eq. (41). Since we derived the solutions u_{3n+1}, u_{3n+2}, u_{3n+3} from them by a twice continuously differentiable non-degenerate change $\breve{x}=x$, $\breve{t}=t-d_2$, therefore, the solutions u_{3n+1}, u_{3n+2}, u_{3n+3} are twice continuously differentiable everywhere in G_{n+1} and, in particular, on pieces of the characteristics $g_2(x,t)=g_2(0,d_{n+1})$, $g_1(x,t)=g_1(d,d_{n+1})$ of Eq. (15), since according to the induction hypothesis $u\in C^2(Q_n)$, the initial data are $\varphi_{n+1}\in C^2[0,d]$, $\psi_{n+1}\in C^1[0,d]$.

By construction, on the common side $t = d_{n+1}$ of these rectangles, the equalities obviously hold

$$u_{3n-j}(x, d_{n+1}) = \varphi_{n+1}(x) = u_{3n+1}(x, d_{n+1}), \tag{51}$$

$$\frac{\partial u_{3n-j}}{\partial t}\Big|_{t=d_{n+1}} = \psi_{n+1}(x) = \frac{\partial u_{3n+1}}{\partial t}\Big|_{t=d_{n+1}} \tag{52}$$

for all $x \in [jx_0, max\{x_0, jd\}]$, $j = 0,\ 1$. Differentiating equalities (51) once and twice with respect to x and t, using equation (15), we calculate the values of partial derivatives:

$$\frac{\partial u_{3n-j}(x, d_{n+1})}{\partial x} = \varphi'_{n+1}(x) = \frac{\partial u_{3n+1}(x, d_{n+1})}{\partial x},$$

$$\frac{\partial^2 u_{3n-j}(x, d_{n+1})}{\partial x^2} = \varphi''_{n+1}(x) = \frac{\partial^2 u_{3n+1}(x, d_{n+1})}{\partial x^2},$$

$$\frac{\partial^2 u_{3n-j}(x,t)}{\partial x \partial t}\Bigg|_{t=d_{n+1}} = \psi'_{n+1}(x) =$$

$$= \frac{\partial^2 u_{3n+1}(x,t)}{\partial x \partial t}\Bigg|_{t=d_{n+1}}, \ \frac{\partial^2 u_{3n-j}(x,t)}{\partial t^2}\Bigg|_{t=d_{n+1}} =$$

$$= [a_2(x, d_{n+1}) - a_1(x, d_{n+1})]\psi'_{n+1}(x) +$$

$$+ a_1(x, d_{n+1})a_2(x, d_{n+1})\varphi''_{n+1}(x) +$$

$$+ [(a_2)_t(x, d_{n+1})/a_2(x, d_{n+1})]\psi_{n+1}(x) +$$

$$+ a_1(x, d_{n+1})(a_2)_x(x, d_{n+1})\varphi'_{n+1}(x) =$$

$$= \frac{\partial^2 u_{3n+1}(x,t)}{\partial t^2}\Bigg|_{t=d_{n+1}},$$

$$x \in [jx_0, max\{x_0, jd\}],\ j = 0,\ 1. \tag{53}$$

Here, when deriving the last equality, we used all previous equalities from (53). From what was said in the previous paragraph and these equalities, it follows that the functions (24)–(26) are twice continuously differentiable for $k = \overline{1, n}$ and $k = n + 1$ at the junction $t = d_{n+1}$ of the rectangles Q_n and G_{n+1}. Theorem 5 is proved.

Remark 2. The results of Theorem 5 should be generalized to model and general non-homogeneous two-speed telegraph equations with right-hand sides $f(x, t) \neq 0$ and supplemented with two consequences about the possible minimal smoothness of their right-hand sides $f(x, t)$ depending only on x or t and simultaneously on x and t. For classic solutions of mixed problems with real coefficients, only in the works of the author of this article and his students in these corollaries the minimal possible necessary (obligatory) and sufficient (admissible) smoothness of the right-hand sides of the wave equations is strictly proved. In sets of classical (and generalized) solutions, such unimprovable results for mixed problems were obtained by us by recurrent continuations of the right-hand sides $f(x, t)$ of the wave equations and initial data $\varphi(x)$, $\psi(x)$ along the Ot axis, and not by traditional periodic and other continuations along the Ox axis.

4 Conclusion

In the half-band G of the plane new method of implicit characteristics is used to prove the global Hadamard correctness theorem to the mixed problem for the homogeneous model telegraph equation (15) with two variable velocities $a_1(x, t)$, $a_2(x, t)$ and non-characteristic non-stationary first oblique derivatives (17) at the ends of a bounded string. The half-strip G of the plane is exhausted in the limit as $n \to +\infty$ by expanding rectangles Q_n in which the Hadamard correctness criteria (18), (22), (23) of the non-characteristic mixed problem (15)–(17) and explicit formulas (24)–(26) of its classic solutions $u \in C^2(Q_n)$, $n = 1, 2, ...$, are derived. These results are obtained by F.E. Lomovtsev's new implicit characteristics method, which is based on twelve identities of the inversion of two implicit functions of the characteristics of the wave equations and their four inverse functions. Smoothness requirements (18) are necessary and sufficient for twice continuous differentiability of solutions (24)–(26) in triangles Δ_{3k-l}, $l = 0,\ 1,\ 2,\ k = 1, 2, ..., n$. Matching conditions (22), (23) are necessary and sufficient for twice continuous differentiability of these solutions on characteristics $g_2(x, t) = g_2(0, d_k)$, $g_1(x, t) = g_1(d, d_k)$, $k = 1, 2, ... n$. These correctness criteria and formulas of classical solutions in Theorem 5 are obtained from Theorem 3 by the Lomovtsev's method auxiliary mixed problems for semi-bounded string. In our Theorem 5 with complete, unimprovable and final results for classical solutions, the recurrent continuation of the original data $\varphi_k(x)$, $\psi_k(x)$ along the Ot axis on the rectangles G_k, $k = 1, 2, ..., n$, is preferable to the traditional periodic and other continuations $\varphi(x)$, $\psi(x)$ along the Ox axis in works of other authors.

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