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Chaotic synchronization of irregular complex networks with multi-scroll attractors Genesio & Tesi 3-D

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***Abstract*—In this paper, the synchronization of N-coupled multi-scroll attractors Genesio & Tesi 3-D chaotic oscillators connected in irregular topology is presented. Synchronization of coupled Genesio & Tesi 3-D chaotic oscillator showing $2 \times 2 \times 2$, $2 \times 5 \times 5$ and $3 \times 6 \times 6$ scroll with and without master node coupling is achieved based on a coupling matrix from complex systems theory. The oscillators of the complex networks interact to each other only through one state of each system. Numerical simulations are provided to verify the effectiveness of this method.**

I. Introduction

Chaotic oscillators have received lots of attention in the last decade because of their potential application in private communications. Chaotic signals have been successfully used in data encryption due to their irregular forms. These systems have attracted the interest of the researchers who have studied extensively, probably the Chua's oscillator is the most well-know and implemented in the field of communications [1]. Among the variety of oscillators that have been reported, it is of special interest those who show scroll attractors, like Chua's oscillator (doble scroll) that has been generalized multiple times based on two cathegories: those in amending the function nonlinear and those that increase the system dimension [2]. In somewhat different systems, alternative models have been generated that have been shown to generate n-scroll attractors. These dynamic models are part of the family called *grid scroll attractors*. So far 3 types have been reported in which these attractors have qualified:

- scroll grid attractors 1-D
- scroll grid attractors 2-D
- scroll grid attractors 3-D

Chaotic synchronization has received increasing attention from researchers in the last decade. Since Pecora and Carroll synchronizing two identical chaotic systems with different initial conditions [4], chaotic synchronization was intensively studied. Many methods have been proposed to achieve synchronization between two chaotic systems, i.e. Pecora-Carroll (PC) method [4], Chaotic syncronization using observer [5], [6], Output synchronization problem (OSP) [7]. In this paper we synchronize N-coupled chaotic systems applying a law of control based on a coupling matrix only in one state.

In this paper, the oscillators whose name reported is *a new family of n-scroll attractors* are used, which for simplicity will be called Genesio & Tesi oscillators investigators to whom they owe their existence. R. Genesio and A. Tesi developed and proposed a chaotic system in order to examine the harmonic balance method to determine the existence and location of the chaotic behavior in 1992 [2], [3]. R. Genesio and A. Tesi successfully applied the method and proved that the model exhibited chaos [2], [3]. A generalization of the original model of Genesio & Tesi [3] to generate n-scroll was reported in [1]. Such generalization consisted in modifying the nonlinearity of the original model.

This paper is organized as follow: Section II a brief review on synchronization of complex dynamical networks is given. In Section III, the problem of synchronization in N-coupled chaotic systems in irregular networks is exposed as well as the model of multi-scroll attractors Genesio & Tesi 3-D system which will be used as fundamental nodes to compose the irregular networks; the corresponding simulation results are provided also in this section. Finally, some conclusions are given in Section IV.

II. Complex Networks

A complex network is defined as an interconnected set of nodes (two or more), where each node is a fundamental unit, with its dynamic depending of the nature of the network. Each node is defined as follows

$$\dot{x}_i = f(x_i) + u_{i1}, \qquad x_i(0) = c_i, \ i = 1, \ 2, \dots, \ N, \quad (1)$$

where $x_i = [x_{i1} \ x_{i2} \ \dots \ x_{in}] \in \Re^n$ are the state variables of the node i, c_i are the initial conditions and u_i establishes the synchronization between two or more nodes and is defined as follows [8]

$$u_{i1} = c \sum_{j=1}^{N} a_{ij} \Gamma x_j, \qquad i = 1, \ 2, \dots, \ N, \quad (2)$$

the constant c positive definite represents the coupling strength and Γ is a constant matrix linking coupled state variables. In this matrix, two nodes are linked through their ith state variables. Assume that $\Gamma = \text{diag}(r_1, \ r_2, \dots, \ r_n)$ is a diagonal matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $j \neq i$.

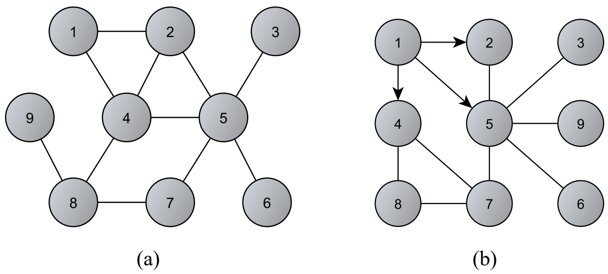

Fig. 1. Complex networks with irregular topologies: (a) Complex network with 9 oscillators without master oscillator, (b) Complex network with 9 oscillators with master oscillator (oscillator 1).

The matrix $A = (a_{ij}) \in \Re^{N\times N}$ is the coupling matrix which shows a connection between node i and j, then $a_{ij} = 1$, otherwise $a_{ij} = 0$ for $i \neq j$. The diagonal elements of A are def ned as

$$a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij} = -\sum_{j=1,j\neq i}^{N} a_{ji} \qquad i = 1, 2, ..., N. \quad (3)$$

The dynamical complex network (1) and (2) is said to achieve synchronization if $x_1(t) = x_2(t) = \ldots = x_n(t)$, as $t \to \infty$. The complexity of the network refers to the characteristics of the nodes or network topology.
It is considered a network with N identical multi-scroll attractors Genesio & Tesi 3-D as node. The topology of the complex network is def ned as a connection among each oscillator, with a regular or irregular pattern. In this paper, it is considered only the irregular topologies. In Fig. 1 it is shown two irregular networks with and without master oscillator.

III. SYNCHRONIZATION OF N-COUPLED MULTI-SCROLL ATTRACTOR GENESIO & TESI 3-D SYSTEMS VIA COUPLING MATRIX

In this section, synchronization of irregular complex networks constituted of N-coupled multi-scroll attractor Genesio & Tesi 3-D is achieved for diferent modalities of the attractor. First, it is shown the set of equations that describes the multi-scroll attractors Genesio & Tesi 3-D; then, necessary data corresponding to each complex irregular network to achieve synchronization is provided; f nally, synchronization results are shown.

A. Multi-scroll attractor Genesio & Tesi 3-D

The model multi-scroll attractor Genesio & Tesi 3-D is described by

$$\begin{aligned} \dot{x} &= y - f_1(y), \\ \dot{y} &= z - f_1(z), \\ \dot{z} &= -ax - ay - az + af_3(x), \end{aligned} \quad (4)$$

where

$$f_1(y) = \sum_{i=1}^{M_y} g_{\frac{(-2i+1)}{2}}(y) + \sum_{i=1}^{N_y} g_{\frac{(2i-1)}{2}}(y), \quad (5)$$

$$f_1(z) = \sum_{i=1}^{M_z} g_{\frac{(-2i+1)}{2}}(z) + \sum_{i=1}^{N_z} g_{\frac{(2i-1)}{2}}(z), \quad (6)$$

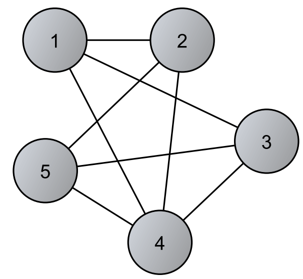

Fig. 2. Irregular complex network with 5 oscillators and without master oscillator reported in [9].

$$g_\theta(\cdot) = \begin{cases} 1, & \cdot \geq \theta, \theta > 0, \\ 0, & \cdot < \theta, \theta > 0, \\ 0, & \cdot \geq \theta, \theta < 0, \\ -1, & \cdot < \theta, \theta < 0, \end{cases} \quad (7)$$

$$f_3(x) = \sum_{l=1}^{k-1} \gamma g_{n_l}(x), \quad (8)$$

where

$$\begin{aligned} n_l &= \rho + 0.5 + (l-1)(\rho + \varsigma + 1), \\ \gamma &= \rho + \varsigma + 1, \end{aligned} \quad (9)$$

$$\begin{aligned} \rho &= |\min_{i,j}\{u_i^{eq,y} + u_j^{eq,z}\}|, \\ \varsigma &= |\max_{i,j}\{u_i^{eq,y} + u_j^{eq,z}\}|, \end{aligned} \quad (10)$$

and x, y, $z \in \Re$, $a = 0.8$, $u^{eq,y}$ and $u^{eq,z}$ are the vectors for the y and z variables related to the equilibrium points, the Eq. (7) is the core function. The equilibrium points satisfy

$$\begin{cases} x + y + z &= f_3(x), \\ y &= f_1(y), \\ z &= f_1(z), \end{cases} \quad (11)$$

where the points for the y, z variables are given by

$$\begin{aligned} u^{eq,y} &= \{-M_y, \ldots, -1, 0, 1, \ldots, N_y\}, \\ u^{eq,z} &= \{-M_z, \ldots, -1, 0, 1, \ldots, N_z\}. \end{aligned} \quad (12)$$

With this nonlinearities the system produces $k \times (M_y + N_y + 1) \times (M_z + N_z + 1)$ scroll grid attractors [3].

B. Synchronization of irregular complex networks

1) Case I: First, it is synchronize complex networks of identical multi-scroll attractor Genesio & Tesi 3-D. The coupled network topology is illustrated in Fig. 2, where every oscillator is described by Eqs. (4)-(10); considering a synchronization scheme N-coupled multi-scroll attractor Genesio & Tesi 3-D chaotic systems, the coupling matrix corresponding to irregular topology is given by

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 & 1 \\ 1 & 0 & -3 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 1 & -3 \end{bmatrix}. \quad (13)$$

This matrix A has its eigenvalues $\lambda(A) = \{0, -3, -3, -5, 5\}$. The Gamma matrix is defined as $\Gamma = \text{diag}(1, 0, 0)$ that means the synchronization is achieved by the first state. According to Eq. (1), the control laws u_{i1} for $i = 1, \ldots, 5$ are given by the A matrix as follow

$$\begin{aligned} u_{11} &= c(-3x_1 + x_2 + x_3 + x_4), \\ u_{21} &= c(x_1 - 3x_2 + x_4 + x_5), \\ u_{31} &= c(x_1 - 3x_3 + x_4 + x_5), \\ u_{41} &= c(x_1 + x_2 + x_3 - 4x_4 + x_5), \\ u_{51} &= c(x_2 + x_3 + x_4 - 3x_5). \end{aligned} \tag{14}$$

The five nodes are defined as follow

$$\begin{aligned} \dot{x}_1 &= y_1 - f_1(y_1) + u_{11}, \\ \dot{y}_1 &= z_1 - f_1(z_1), \\ \dot{z}_1 &= -ax_1 - ay_1 - az_1 + af_3(x_1), \\ &\vdots \\ \dot{x}_5 &= y_5 - f_1(y_5) + u_{51}, \\ \dot{y}_5 &= z_5 - f_1(z_5), \\ \dot{z}_5 &= -ax_5 - ay_5 - az_5 + af_3(x_5). \end{aligned} \tag{15}$$

The initial conditions for each oscillator are

$$\begin{aligned} (x_1, y_1, z_1)(0) &= (0.5, 0.5, 0.5), \\ (x_2, y_2, z_2)(0) &= (-0.1, 0.1, 0.1), \\ (x_3, y_3, z_3)(0) &= (-0.4, 0.4, 0.4), \\ (x_4, y_4, z_4)(0) &= (0.2, 0.2, 0.2), \\ (x_5, y_5, z_5)(0) &= (-0.3, 0.3, 0.3). \end{aligned}$$

The coupling strength is obtained using the next stability analysis that shows it is possible synchronizing a complex network using a much smaller value that the given by the Wang & Chen theorem [8].

Stability analysis: Due to the systems are identical, the synchronization error between any pair of them is the same, besides that, the control law u_{i1} is presented in every oscillator, therefore, it is only needed to obtain the coupling strength c to synchronize one pair of chaotic oscillators. Consider any pair of oscillators that are coupled to each other from the complex network in Fig. 2 and keep their configuration, the equations of the two coupled oscillators are given by (1)-(2) with $N = 2$, and the elements a_{ij} correspond to the new coupling matrix are obtained as explained in the previous section.

Now, the synchronization error is defined as $e_1 = x_1 - x_2$, $e_2 = y_1 - y_2$ and $e_3 = z_1 - z_2$, then, it is obtained the next synchronization error system

$$\begin{aligned} \dot{e}_1 &= -2ce_1 + e_2 - [f_1(y_1) - f_1(y_2)], \\ \dot{e}_2 &= e_3 - [f_1(z_1) - f_1(z_2)], \\ \dot{e}_3 &= -ae_1 - ae_2 - ae_3 + a[f_3(x_1) - f_3(x_2)]. \end{aligned} \tag{16}$$

When synchronization error system reaches equilibria the variables involved reach synchrony, this means, the equation (17) holds. At the equilibrium, nonlinearities $f_i(y)$, $f_i(z)$ and $f_i(x)$ $i = 1, 2$ are equal and the subtraction is zero.

$$\lim_{t\to\infty} \left\| [x_1\ y_1\ z_1]^T - [x_2\ y_2\ z_2]^T \right\| = 0. \tag{17}$$

The origin is the equilibrium point of the synchronization error system (16) and it can be found by setting $\dot{e}_1 = \dot{e}_2 = \dot{e}_3 = 0$. For analyzing stability of the equilibrium point, it is proposed the Lyapunov candidate function

$$V(e) = \frac{1}{2}(be_1^2 + 2e_1e_3 + be_2^2 + 4e_2e_3 + be_3^2). \tag{18}$$

where $b > \sqrt{5}$ so that $V(e) > 0$. The derivative of $V(e)$ evaluated along the trajectories of the synchronization error system (16) is given by

$$\begin{aligned} \dot{V}(e) = & -(2bc + a)e_1^2 + (b - 3a)e_1e_2 \\ & -(a + ba + 2c)e_1e_3 \\ & +(1 + b - 2a - ba)e_2e_3 \\ & -2ae_2^2 - (ba - 2)e_3^2. \end{aligned} \tag{19}$$

The matrix form of (19) allows us to show that $\dot{V}(e)$ is negative definite by showing that Q is positive definite, therefore

$$\dot{V}(e) = -\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} Q \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{20}$$

where

$$Q = \begin{bmatrix} (2bc + a) & -\frac{1}{2}(b - 3a) & q_1 \\ -\frac{1}{2}(b - 3a) & 2a & q_2, \\ q_1 & q_2 & (ba - 2) \end{bmatrix}. \tag{21}$$

and

$$\begin{aligned} q_1 &= \tfrac{1}{2}(a + ba + 2c), \\ q_2 &= -\tfrac{1}{2}(1 + b - 2a - ba). \end{aligned} \tag{22}$$

The determinant of principal minors of Q are given by

$$\begin{aligned} \det(1^{st}) &= (2bc + a), \\ \det(2^{nd}) &= 2a(2bc + a) - \frac{1}{4}(b - 3a)^2, \\ \det(3^{rd}) &= -2ac^2 + \frac{1}{2}(4a^2b^2 - a^2b^3 - 5a^2b \\ & \quad +2a^2 + 2ab^3 + 5ab^2 - 17ab - 3a \\ & \quad -b^2 - b^3)c + \frac{1}{4}(2a^2b^2 - a^2b^3 \\ & \quad -2a^2b + 3a^2 + ab^2 - 13ab - a + b^2). \end{aligned} \tag{23}$$

The set of equations (23) is function of system parameter, the free parameter b of the Lyapunov cadidate function and the coupling strength c. It can be seen that $(2bc + a) > 0$ and $2a(2bc + a) - \frac{1}{4}(b - 3a)^2 > 0$ for $c > 0$ so that, positive-definiteness of matrix Q depends on 3^{rd} principal minor. Replacing the system parameter $a = 0.8$, the 3^{rd} principal minor becomes

$$\begin{aligned} & -\left(\frac{8}{5}\right)c^2 - \left(\frac{1}{50}b^3 - \frac{139}{50}b^2 + \frac{42}{5}b + \frac{14}{25}\right)c \\ & + \left(-\frac{4}{25}b^3 + \frac{77}{100}b^2 - \frac{73}{25}b\right), \end{aligned} \tag{24}$$

that depends on the free parameter b of the Lyapunov candidate

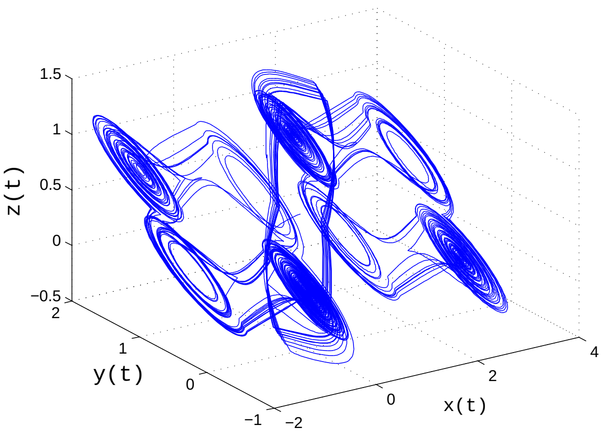

Fig. 3. View on 3-D state space $2 \times 2 \times 2$ Genesio & Tesi 3-D attractor obtained with $M_y = 0$, $N_y = 1$, $M_z = 0$, $N_z = 1$ and $k = 2$.

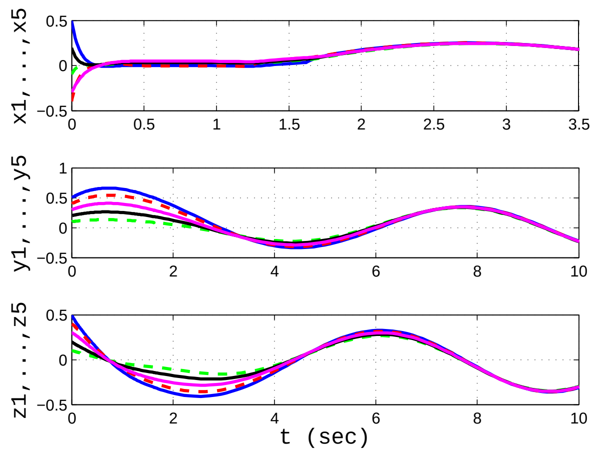

Fig. 4. Evolution of the synchronization of the three states x, y and z of each ascillator using $c = 5$.

function whose lower bound is $b > \sqrt{5}$ and the coupling strength c. For this particular case, it is set $b = 5$ and Eq. (24) becomes

$$-\left(\frac{8}{5}\right)c^2 + \left(\frac{611}{25}\right)c - \left(\frac{441}{50}\right), \tag{25}$$

where it was found that $Q > 0$ for $0.3698 < c < 14.9052$ so that $\dot{V}(e) < 0$ and the equilibrium point is asymptotically stable. Using the value of the coupling strength $c = 5$ for this case, synchronization of the variables involved and stability of the synchronization error system (16) are guaranteed. The Fig. 3 shows the multi-scroll attractor Genesio & Tesi 3-D system for 5 oscillators and Fig 4 shows the three states of each oscillator synchronizing. In Fig. 5 the phase portrait between the first state of each oscillator is shown, here, synchronization of the first states can be confirmed.

2) Case II: Then, the complex network with topology ilustrated in Fig 6 is synchronized, where each node is a multi-scroll attractor Genesio & Tesi 3-D oscillator described by Eqs.

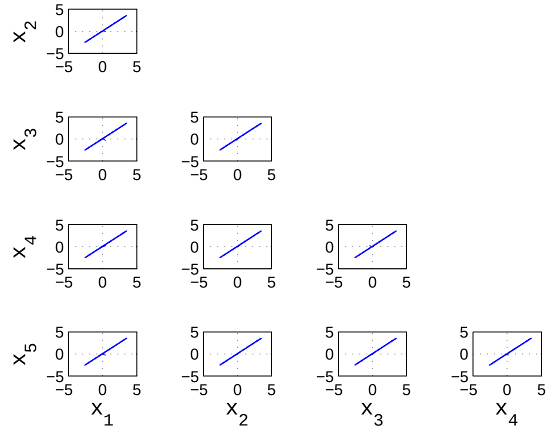

Fig. 5. Confirmed synchronization of first state of each oscillator x_i vs. x_j $i = 1, \ldots, 4$; $j = 2, \ldots, 5$.

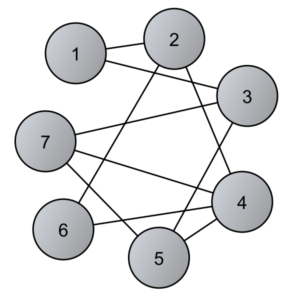

Fig. 6. Irregular complex network with 7 oscillators and without master oscillator.

(4)-(10), the corresponding coupling matrix A is defined as follow

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -3 \end{bmatrix}. \tag{26}$$

This matrix eigenvalues are $\lambda(A) = \{0, -1.1864, -1.5858, -3.4707, -4, -4.4142, -5.3429\}$. The Gamma matrix is defined as $\Gamma = \mathrm{diag}(0, 1, 0)$ that means the synchronization is achieved by the second state. According to Eq. (1), the control laws u_{i2}, for $i = 1, \ldots, 7$ are given by the A matrix as follow

$$
\begin{aligned}
u_{12} &= c(-2y_1 + y_2 + y_3), \\
u_{22} &= c(y_1 - 3y_2 + y_4 + y_6), \\
u_{32} &= c(y_1 - 3y_3 + y_5 + y_7), \\
u_{42} &= c(y_2 - 4y_4 + y_5 + y_6 + y_7), \\
u_{52} &= c(y_3 + y_4 - 3y_5 + y_7), \\
u_{62} &= c(y_2 + y_4 - 2y_6), \\
u_{72} &= c(y_3 + y_4 + y_5 - 3y_7).
\end{aligned} \tag{27}
$$

The seven oscillators using the control laws are defined as follow

$$
\begin{aligned}
\dot{x}_1 &= y_1 - f_1(y_1), \\
\dot{y}_1 &= z_1 - f_1(z_1) + u_{12}, \\
\dot{z}_1 &= -ax_1 - ay_1 - az_1 + af_3(x_1), \\
\vdots\ &\ \vdots \qquad\qquad \vdots \\
\dot{x}_7 &= y_7 - f_1(y_7), \\
\dot{y}_7 &= z_7 - f_1(z_7) + u_{72}, \\
\dot{z}_7 &= -ax_7 - ay_7 - az_7 + af_3(x_7).
\end{aligned} \tag{28}
$$

The initial conditions for each oscillator are

$$
\begin{aligned}
(x_1, y_1, z_1)(0) &= (0.1, -0.1, 0.1), \\
(x_2, y_2, z_2)(0) &= (0.2, -0.2, 0.2), \\
(x_3, y_3, z_3)(0) &= (0.3, -0.3, 0.3), \\
(x_4, y_4, z_4)(0) &= (0.4, -0.4, 0.4), \\
(x_5, y_5, z_5)(0) &= (0.5, -0.5, 0.5), \\
(x_6, y_6, z_6)(0) &= (0.6, -0.6, 0.6), \\
(x_7, y_7, z_7)(0) &= (0.7, -0.7, 0.7).
\end{aligned}
$$

The coupling strength used for this case is $c = 1$ and it was obtained by an analysis similar to the previous one. The Fig. 7 shows the multi-scroll attractor Genesio & Tesi 3-D system with 7 oscillators and Fig 8 shows the three states of each oscillator synchronizing. In Fig. 12 the phase portrait between the second state of each oscillator is shown, here, synchronization of the second states can be confirmed.

3) Case III: Finally, the last irregular complex network is synchronized, the topology of the networks is illustrated in Fig. 9, where each oscillator is a chaotic oscillator described by Eqs. (4)-(10); considering a synchronization scheme N-coupled multi-scroll attractor Genesio & Tesi 3-D chaotic oscillators, the coupling matrix corresponding is given by

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & -5 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & -5 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix}. \tag{29}
$$

This eigenvalues of the matrix are $\lambda_1 = 0$, $\lambda_2 = -0.1909$, $\lambda_3 = -0.8214$, $\lambda_4 = -1.3328$, $\lambda_5 = -1.8179$, $\lambda_6 = -3.6868$, $\lambda_7 = -3.9150$, $\lambda_8 = -5.6492$ and $\lambda_9 = -6.5859$.

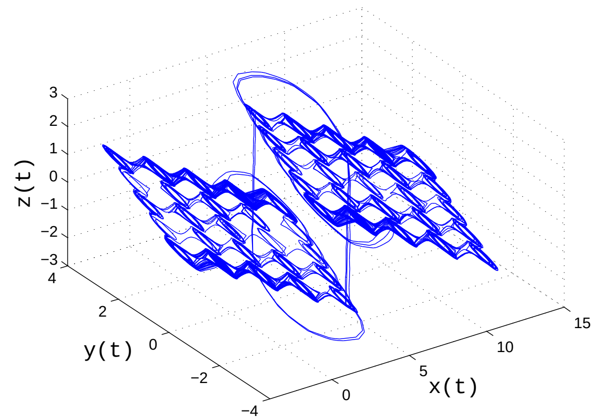

Fig. 7. View on 3-D state space $2 \times 5 \times 5$ Genesio & Tesi 3-D attractor obtained with $M_y = 2$, $N_y = 2$, $M_z = 2$, $N_z = 2$ and $k = 2$.

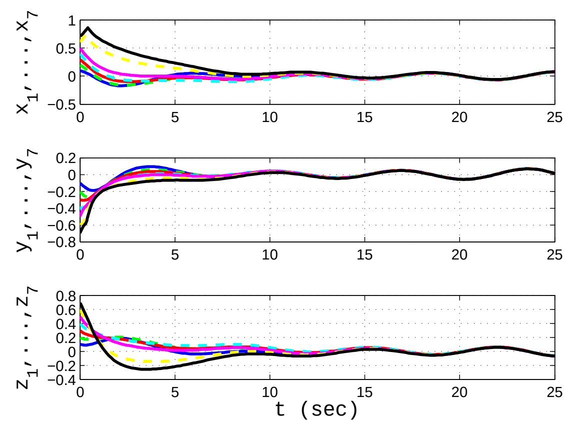

Fig. 8. Evolution of the synchronization of the three states x, y and z of each assillator using $c = 1$.

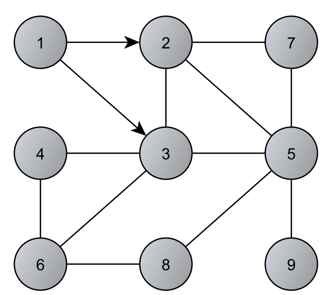

Fig. 9. Irregular complex network with 9 oscillators and with master oscillator.

The Gamma matrix as in the first synchronization, is defined as $\Gamma = \text{diag}(1, 0, 0)$ that means the synchronization is achieved by the first state. According to Eq. (1), the control laws u_{i1} for $i = 1, \ldots, 9$ are given by the A matrix as follow

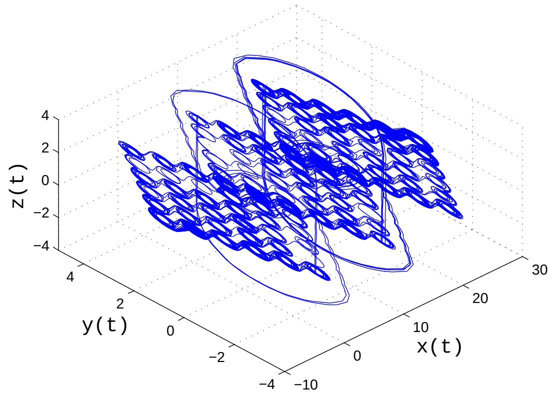

Fig. 10. View on 3-D state space $3 \times 6 \times 6$ Genesio & Tesi 3-D attractor obtained with $M_y = 2$, $N_y = 3$, $M_z = 2$, $N_z = 3$ and $k = 3$.

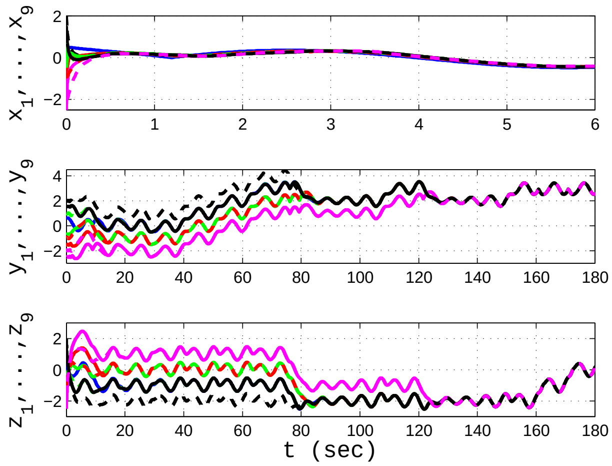

Fig. 11. Evolution of the synchronization of the three states x, y and z of each ascillator using $c = 9$.

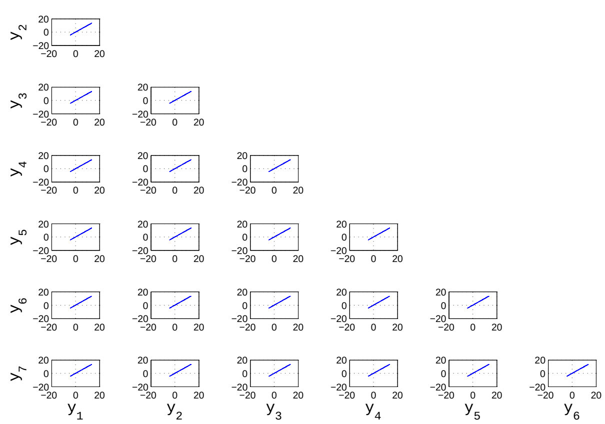

Fig. 12. Confirmed synchronization of second state of each oscillator y_i vs. y_j $i = 1, \ldots, 6$; $j = 2, \ldots, 7$.

$$
\begin{aligned}
u_{11} &= 0, \\
u_{21} &= c(x_1 - 4x_2 + x_3 + x_5 + x_7), \\
u_{31} &= c(x_1 + x_2 - 5x_3 + x_4 + x_5 + x_6), \\
u_{41} &= c(x_3 - 2x_4 + x_6), \\
u_{51} &= c(x_2 + x_3 - 5x_5 + x_7 + x_8 + x_9), \\
u_{61} &= c(x_3 + x_4 - 3x_6 + x_8), \\
u_{71} &= c(x_2 + x_5 - 2x_7), \\
u_{81} &= c(x_5 + x_6 - 2x_8), \\
u_{91} &= c(x_5 - x_9).
\end{aligned} \tag{30}
$$

The first oscillator that is the master node has no change because of its control law is zero and it is defined by

$$
\begin{aligned}
\dot{x}_1 &= y_1 - f_1(y_1), \\
\dot{y}_1 &= z_1 - f_1(z_1), \\
\dot{z}_1 &= -ax_1 - ay_1 - az_1 + af_3(x_1).
\end{aligned} \tag{31}
$$

The eight nodes left are defined as follow

$$
\begin{aligned}
\dot{x}_2 &= y_2 - f_1(y_2) + u_{21}, \\
\dot{y}_2 &= z_2 - f_1(z_2), \\
\dot{z}_2 &= -ax_2 - ay_2 - az_2 + af_3(x_2), \\
\vdots \;\; &\vdots \qquad\qquad \vdots \\
\dot{x}_9 &= y_9 - f_1(y_9) + u_{91}, \\
\dot{y}_9 &= z_9 - f_1(z_9), \\
\dot{z}_9 &= -ax_9 - ay_9 - az_9 + af_3(x_9).
\end{aligned} \tag{32}
$$

The initial conditions for each oscillator are

$$
\begin{aligned}
(x_1, y_1, z_1)(0) &= (-0.5, 0.50.5), \\
(x_2, y_2, z_2)(0) &= (-1, 1, 1), \\
(x_3, y_3, z_3)(0) &= (-1.5, 1.5, 1.5), \\
(x_4, y_4, z_4)(0) &= (-2, 2, 2), \\
(x_5, y_5, z_5)(0) &= (-2.5, -2.5, -2.5), \\
(x_6, y_6, z_6)(0) &= (0.5, -0.5, -0.5), \\
(x_7, y_7, z_7)(0) &= (1, -1, -1), \\
(x_8, y_8, z_8)(0) &= (1.5, -1.5, -1.5), \\
(x_9, y_9, z_9)(0) &= (2, -2, -2).
\end{aligned}
$$

The coupling strength used for this case is $c = 9$ and it was obtained as in previous cases. The Fig. 10 shows the multi-scroll attractor Genesio & Tesi 3-D system with 9 oscillators and Fig. 11 shows the three states of each oscillator synchronizing. The phase portrait between the first state of each oscillator is divided into three parts that are shown in Fig. 13, 14 and 15, here, synchronization of the first states can be confirmed.

IV. Conclusion

In this paper the synchronization of N-coupled irregular network is achieved using the complex network theory. It was used a coupling signal only in one state of the system and adjusting the coupling strength c the synchronization is

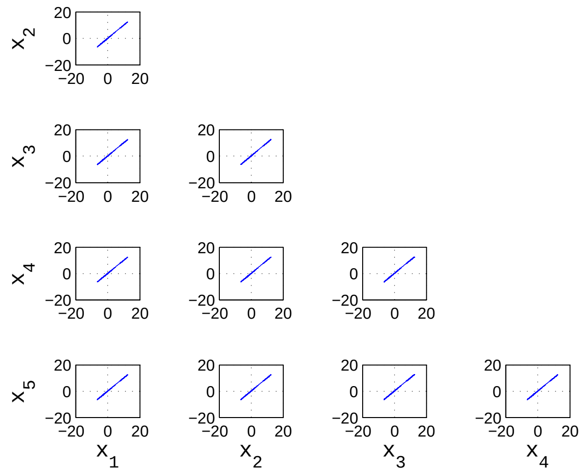

Fig. 13. Confirmed synchronization of first state x_i vs x_j $i = 1, \ldots, 4$; $j = 2, \ldots, 5$.

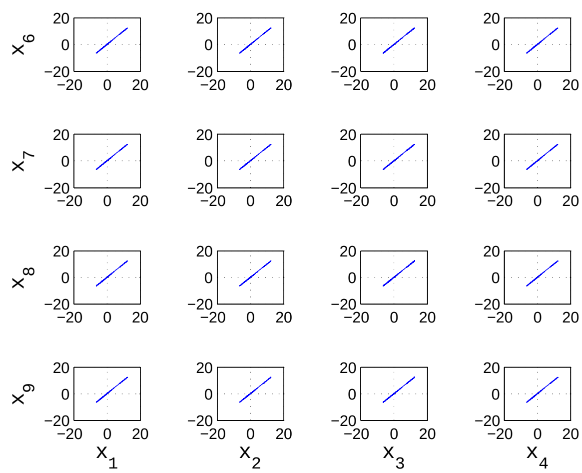

Fig. 14. Confirmed synchronization of first state x_i vs x_j $i = 1, \ldots, 4$; $j = 6, \ldots, 9$.

achieved in multi-scroll attractor Genesio & Tesi 3-D systems for different modality of the attractor. The proposed synchronization law has the advantage that is simply to develop and the network can be synchronized applying this law only in one state of the system. It can be observed the convergence in each state of the systems and the error system decay towards zero as $t \rightarrow \infty$. Simulations show the effectiveness of the proposed synchronization scheme. There were synchronized three different complex networks with irregular topology for different modality of the attractor of multi-scroll Genesio & Tesi 3-D system using a coupling strength obtained by an alternative stability analysis that is, for many cases, smaller.

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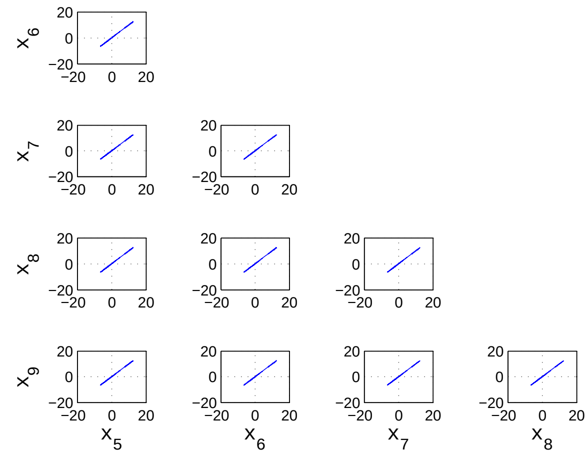

Fig. 15. Confirmed synchronization of first state x_i vs x_j $i = 5, \ldots, 8$; $j = 6, \ldots, 9$.

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