A New Numerical Technique for Solving Fractional Integro-Differential Equations in Fuzzy Space

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Abstract: In this work, the generalized exponential spline methodt is studied to approximate the solution of fuzzy fractional integro differential equations under Caputo derivative. The existence and uniqueness theorems for these equations by considering the type of differentiability of solution are proved. The results shown that generalized exponential method is more accurate in terms of absolute error. Furthermore, the accuracy and efficiency of the proposed method are demonstrated is a series of numerical experiments.

Key-Words: Generalized exponential Spline, Fuzzy fractional integro-differential equation, Mittag-Leffler function, Caputo fractional derivative.

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1 Introduction

The applications of fractional calculus have attracted more attention in the past few years because of significant advantage of the fractional order models in comparison with integer order models. In addition, the fuzzy theory becomes a strong tool for modeling uncertain problems. A fuzzy differential equation may be viewed as a type of uncertain differential equations in which the uncertain values of parameters, coefficients, and/or boundary conditions are taken into account as fuzzy numbers. In fact, most problems in nature are indistinct and uncertain, therefore the models rule are important.

The concept of the fuzzy derivative was first introduced by Chang and Zadeh [6]. The fuzzy integral was introduced by Dubois, Prade [2], it was followed by many authors, both types fuzzy differential equations and integro differential equations have been studied extensively.

Over the years, there are many studies of different fractional systems have been introduced in various fields [27] Hajighasemi et.al. [4] investigated existence and uniqueness of solutions for fuzzy Volterra integrodifferntial equations with fuzzy kernel function. Ishak and Chaini [5] proposed the extended trapezoidal method to solve fuzzy initial value problem that has first order. Hasan & Hussien [14] applied the generalized spline technique and Caputo differential derivative to solve second kind of fractional integro-differential equations. They Compared of the applied method with exact solutions reveals that the method was tremendously effective. [13] introduced a new class of cubic spline function approach to solve fuzzy initial value problems efficiently. Also, the convergence of this method was shown. [26] Jaafar presents techniques of speech scrambling based-fractional order chaotic system which is used due to many properties.

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[20] studied the Legendre wavelet is to approximate the solution of fuzzy fractional integro-differential equation. They have considered the Caputo differentiability concept based on the differentiability to solve fuzzy fractional integro differential equations. *Hasan [24]* derives the formulae *of Sumudu transformation f*or linear fractional differential equations. *Hasan, & Nasif [25] determine a solution of nonlinear integral and integro-differential equations using two methods Laplace transform series decomposition method and Sumudu transform series decomposition method.*

A new class of fuzzy functions called fuzzy fractional integro differential equations is considered. Some numerical methods such as cubic spline and exponential spline have been used to determine the solutions of theses equations. We extend these numerical techniques to find the optimal solutions. Generalized exponential spline technique is used for this.

One of our intentions is to prove the uniqueness of the solution of a fuzzy fractional integro-differential equation.

The results shown that Exponential spline method is more accurate in terms of absolute error. Based on the parametric form of the fuzzy number, the integro differential equation is contacted in to two systems of the second kind. Illustrative examples are given to demonstrate the high precision and good performance of the new class. Graphical representations reveal the symmetry between lower and upper cut represent of fuzzy solutions and may be helpful for a better understanding of fuzzy model artificial, intelligence, medical science, quantum.

The study of fuzzy fractional integro differential equations is considered as a new branch of fuzzy mathematics. The analytical methods for finding the exact solutions of fuzzy fractional integro differential equations is very difficult, so the numerical technique is the best way to resort to it. The aims of this study to improve the accuracy of the numerical solutions of fuzzy fractional integro-differential equations. The exponential spline method has been to be able to be solve these equations but current practice has less accuracy with error in approximating the solution for large step size. We proposed extended cubic spline technique to solve fuzzy fractional integro-differential equations numerically. The results are expected to be more accurate as compared to be existing method.

The contributions of this paper as follows, we derive an efficient method for computing the approximate solutions of proposed model, and discover some properties which related between fuzzy theory and fractional integro-differential equations. Also, we show that proposed technique contributes effectively to determination of approximate solutions for fuzzy equations.

The paper is organized as follows: section 2. contains the Preliminaries. In section 4. methodology description for solving fuzzy fractional integro-differential equations is given. In section 4, example is presented. Conclusion of this paper is shown in Section 5.

2 Preliminaries and notations

In this section, we introduce notations, definitions, and preliminary results, which will be used throughout this work. we use the following notations: $y(t_n)$ and y_n are exact solution and approximate solution respectively in time t_n, the space $\mathrm{C}([\mathrm{a},\mathrm{b}]\times[0,1];\ \mathrm{L}^2(\mathrm{R}))$ the Banach space of all fuzzy real-valued continuous functions from $[\mathrm{a},\mathrm{b}]$ into $\mathrm{L}^2(\mathrm{R})$, the norm is defined as

$$\|(X(r,t)\|_{\mathrm{X}}^2 = \sup_{0\le r\le 1} \max\left\{\left(\int_a^b \left|\underline{X(r,t)}\right|^2 dt\right)^{0.5}, \left(\int_a^b \left|\overline{X(r,t)}\right|^2 dt\right)^{0.5}\right\}$$

. A fuzzy number $\boldsymbol{v}$ is a fuzzy set that normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line [12]. For $0 < r \le 1$

Any fuzzy number $\boldsymbol{v}$ can be represent by the following parametric forms $\left(\underline{\mathrm{v}}(r), \overline{\mathrm{v}}(r)\right), 0 \le r \le 1$. That satisfies $\underline{\mathrm{v}}(r)$ is non-decreasing and bounded left over $0 \le r \le 1$ and $\overline{\mathrm{v}}(r)$ is a bounded left continuous and non-increasing over $0 \le r \le 1$. Also, for each $\mathrm{r} \in [0,1]$ then $\underline{\mathrm{v}}(r) \le \overline{\mathrm{v}}(r)$. The r-level set is defend as $[u]^r = \{s; u(s) \ge r\}$, $0 \le r \le 1$. Consequently, $[u]^r$ can be written as close interval $[u]^r = \left[\underline{u}\,(r), \overline{u}(r)\right]$. The Hausdorff distance between fuzzy numbers $u\ and\ v$ is given by

$$d(u,v) = \left[\int_0^1 \left(\underline{\mathrm{u}}(r) - \underline{\mathrm{v}}(r)\right)^2 dr + \int_0^1 \left(\overline{\mathrm{u}}(r) - \overline{\mathrm{v}}(r)\right)^2 dr\right]^{0.5}$$

where d is the Hausdorff metric. If we denote $\mathcal{R}_{fu}$ is a set of all fuzzy numbers, then $(\mathcal{R}_{fu}\ , \mathrm{d})$ is complete, separable and locally compact metric space.

Proposition (2.1) [8]: If a fuzzy function $\mathcal{P}: [a,b] \times [0,1] \to X$, $\mathcal{P}(t,r) = \left(\underline{\mathcal{P}}(t,r), \overline{\mathcal{P}}(t,r)\right)$ is differentiable then $\underline{\mathcal{P}}(t,r)\ and\ \overline{\mathcal{P}}(t,r)$ are differentiable functions and $\mathcal{P}'(t,r) = \left(\underline{\mathcal{P}'}(t,r), \overline{\mathcal{P}}'(t,r)\right)$

Definition (2.1) [7]: For any partition $\mu = \{a = t_0, t_1, t_2, \dots, t_m = b\}$ and $\tau_i \in [t_i, t_{i+1}]$, $i = 0,1,2,\dots,n$ then the definite integral of a function $\mathcal{P}$ over $[a,b]$ is

$$\int_a^b \mathcal{P}\,(t)dt = \lim_{\vartheta \to 0} \mathcal{M}_\mu \tag{1}$$

Where, $\vartheta = \max\{|t_{i+1} - t_i|, i = 0,1,2,\dots,n\}$ and $\mathcal{M}_\mu = \sum_{i=1}^m \mathcal{P}(\tau_i)(t_{i+1} - t_i)$

In the case $\mathcal{P}$ is a fuzzy and continuous function then for each fuzzy parameter $0 \le r \le 1$, its definite integral exists and also [7]

$$\begin{cases} \underline{\left(\int_a^b \mathcal{P}\,(t,r)dt\right)} = \int_a^b \underline{\mathcal{P}}\,(t,r)dt \\ \overline{\left(\int_a^b \mathcal{P}\,(t,r)dt\right)} = \int_a^b \overline{\mathcal{P}}\,(t,r)dt \end{cases} \tag{2}$$

Definition (2.2) [17]: let $g \in C([a,b] \times [0,1]; R)$. The fuzzy Caputo fractional derivative (D_C^α) of a fuzzy-valued function g is defined as follows

$$D_C^\alpha g(t,r) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^{(n)}(s,r)}{(t-s)^{\alpha-n+1}}ds, \quad n-1<\alpha < \qquad (3)$$
$$g^{(n)}(t), \quad \alpha = n \in N, \quad 0 \le r \le 1$$

where Γ is the well-known gamma function.

Definition (2.3) [17]: The fuzzy Caputo fractional integral (I_C^α) of of a fuzzy-valued function $g \in C([a,b] \times [0,1]; R)$ is defined as follows

$$I_C^\alpha g(t,r) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g(s,r)}{(t-s)^{1-\alpha}}ds, \quad 0<\alpha$$
$$g(t,r), \quad \alpha = 0, 0 \le r \le 1 \qquad (4)$$

Proposition (2.2)[18]: let $g \in C([a,b] \times [0,1]; R)$. The fuzzy Laplace transform $\mathcal{L}$ of a fuzzy-valued function g is defined as follows

$$\mathcal{L} I_0^\alpha g(t,r) = \frac{1}{s^\alpha}\mathcal{L} g(s,r) \qquad (5)$$

And

$$\mathcal{L} D_0^\alpha g(t,r) = s^\alpha \mathcal{L} g(t,r)(s) - \sum_{k=1}^{n} s^{\alpha-k} g^{(k-1)}(0) \qquad (6)$$

Where, $\alpha > 0$ and $n = \lceil \alpha \rceil$.

Definition (2.4) [18]: The Mittag-Leffler function $E_{\alpha,\beta}(z)$ for any $\alpha, \beta > 0$ is defined as follows

$$E_{\alpha,\beta}(z) = \sum_{m=1}^{\infty} \frac{z^m}{\Gamma(m\alpha+\beta)} \qquad (7)$$

Proposition (2.3)[18]: The following relations are hold for any $\alpha, t, \delta, \beta > 0$ and $\lambda \in C$

1. If $y(t) = E_\alpha(-\delta t^\alpha)$ then $\mathcal{L}y(t) = \lambda^{\alpha-1}(\lambda^\alpha - \delta)^{-1}$
2. $\frac{d}{dr}E_\alpha(-\delta r^\alpha) = \sum_{m=1}^{\infty} \frac{(-\delta)^m r^{\alpha m-1}}{\Gamma(m\alpha)}$ (8)
3. If $y(t) = t^{\alpha-1} E_{\beta,\alpha}(at^\beta)$ then $\mathcal{L}y(t) = \lambda^{\beta-\alpha}(\lambda^\beta - a)^{-1}$

Definition (2.5) [21] Let X be a universal set. Then, the partial ordering "≤" on X such that $u = (\underline{u}(r), \overline{u}(r))$ and $\nu = (\underline{\nu}(r), \overline{\nu}(r)), 0 \le r \le 1$ are fuzzy numbers is defined as

$$u \le \nu \leftrightarrow \begin{cases} \underline{u}(r) \le \underline{\nu}(r) \\ \overline{u}(r) \le \overline{\nu}(r) \end{cases} \text{for all } 0 \le r \le 1 \qquad (9)$$

we use the notation $(X, \le)$ to refer to partially ordered set. It is proved that X is a complete metric space with the distance d.

Definition (2.6) [22] Let $(X, \le)$ be a partially ordered set and $\mathcal{P}: X \to X$. Then $\mathcal{P}$ is monotone non-decreasing if, it satisfies

$$u \le \nu \leftrightarrow \{\mathcal{P}(u) \le \mathcal{P}(\nu) \quad \text{For all } u, \nu \in X$$

Theorem (2.4) [23] Let $(X, \le)$ be a partially ordered set and (X, d) be a complete metric space. Suppose that $\mathcal{P}: X \to X$ is a monotone non-decreasing mapping and there exists $\text{Ꝃ} \in [0,1)$ such that

$$d(\mathcal{P}(u), \mathcal{P}(\nu)) \le \text{Ꝃ}\, d(u,\nu), \quad \text{for all } u, \nu \in X \text{ and } u \ge \nu$$

Then $\mathcal{P}$ has a unique fixed point $\omega \in X$ and $\{\mathcal{P}_n(u)\}_{n \in N} \to \omega$ for each $u \in X$.

3 PROBLEM DESCRIPTION

In this section, first we introduce a fuzzy fractional integro-differential equation and then we prove that this equation has a uniqueness solution. Consider the following fuzzy initial value problem:

$$\begin{cases} D_C^\alpha y(t,r) = g(t,r)\, y(t,r) + f(t,r) \\ \qquad + \int_a^b k(t,s) y(s,r) ds \\ y(a) = y_0(r) \end{cases} \qquad (10)$$

Where, D_C^α is a Caputo fractional derivative of order $0 < \alpha \le 1$ which defined on $[a,b]$ and is already given, $\beta > 0$, r is a fuzzy parameter with values in [0,1], $k(t,s)$ over $s,t \in [a,b]$ is the kernel of this equation.

In parametric form, equation (10) is represented as follows

$$\begin{cases} D_C^\alpha \underline{y}(t,r) = \underline{g(t,r)\, y(t,r)} + \underline{f}(t,r) \\ \qquad +\beta \int_a^b \underline{k(t,s) y(s,r)} ds \\ D_C^\alpha \overline{y}(t,r) = \overline{g(t,r)\, y(t,r)} + \overline{f}(t,r) \\ \qquad +\beta \int_a^b \overline{k(t,s) y(s,r)} ds \\ \underline{y}(a) = \underline{y_0}(r) \\ \overline{y}(a) = \overline{y_0}(r) \end{cases} \qquad (11)$$

In addition, $\underline{g(t,r)\, y(t,r)} = \underline{g}(t,r)\underline{y}(t,r)$, $\overline{g(t,r)\, y(t,r)} = \overline{g}(t,r)\overline{y}(t,r)$, $g(t,r) =$

$\underline{\left(\underline{g}(t,r), \overline{g}(t,r)\right)}$, $\underline{k(t,s)y(s,r)} = k(t,s)\underline{y}(s,r)$, $\overline{k(t,s)y(s,r)} = k(t,s)\overline{y}(s,r)$

Lemma (3.1) The fuzzy initial value problem (10) is equivalent to the following integral equation

$$y(t,r) = t^{\alpha-1} E_{\alpha,\alpha}(g(t,r)t^{\alpha})\, y_0(r) + t^{\alpha-1} E_{\alpha,\alpha}(g(r)t^{\alpha})\, \varphi\left(t,r,\int_a^b k(t,s)y(s,r)d\right) \quad (12)$$

Where,

$$\varphi\left(t,r,X(t,r),\int_a^b k(t,s)X(s,r)ds\right) = f(t,r) + \beta\int_a^b k(t,s)X(s,r)ds$$

Proof. We prove the case $g(r) = g(t,r)$ such that the function g not depend on t, leaving the other case for discussion by other researchers. Using Laplace operator $\mathcal{L}$ to both side of equation (10), provided g, f and k are integrable functions on [a,b], we have

$$\left(\lambda^{\alpha} - g(r)\right)\left(\mathcal{L}\, y(t,r)\right)(\lambda) = y_0(r) + \left(\mathcal{L}\,\varphi\left(t,r,\int_a^b k(t,s)X(s,r)ds\right)\right)(\lambda)$$

Thus by Multiplying both sides of above equation by $\left(\lambda^{\alpha} - g(r)\right)^{-1}$, here, we assume $\lambda^{\alpha} \in \rho\left(g(r)\right)$, to guarantee the existence of the inverse of $\left(\lambda^{\alpha} - g(r)\right)$, we get

$$\left(\mathcal{L}\, y(t,r)\right)(\lambda) = \left(\lambda^{\alpha} - g(r)\right)^{-1} y_0(r) + \left(\lambda^{\alpha} - g(r)\right)^{-1}\left(\mathcal{L}\,\varphi\left(t,r,\int_a^b k(t,s)X(s,r)ds\right)\right)(\lambda)$$

Using Proposition (2.3) and taking Laplace inverse transform, we will get the result of this Lemma.

Definition (3.1) An X- fuzzy valued function $y(t,r) \in C([a,b]\times[0,1]; L^2(R))$ is a solution of a fuzzy integro-differential equations with initial condition (12), if it satisfies the integral equation (14).

Now, for each $r \in [0,1]$, we define the operator **P** on the space $C([a,b]\times[0,1]; L^2(R))$ by the following form:

$$\mathbf{P}\left(y(r)\right)(t) = t^{\alpha-1} E_{\alpha,\alpha}(g(t,r)t^{\alpha})\, y_0(r) + t^{\alpha-1} E_{\alpha,\alpha}(g(r)t^{\alpha})\, \varphi\left(t,r,\int_a^b k(t,s)y(s,r)ds\right)$$

Assume that the following conditions hold:

(H_1) There exists a constant $Ç_1 > 0$ such that for each $y(r) \in X$, we have

$$\left\|\varphi\left(t,r,\int_a^b k(w,s)y(s,r)ds\right)\right\|_X^2 \le Ç_1\left(1 + \|y(r)\|_X^2\right)$$

(H_2) There exists a constant $Ç_2 > 0$ such that for each $x(r)$, $y(r) \in X$, we have

$$\left\|\left(\begin{array}{l}\varphi\left(t,r,\int_a^b k(t,s)x(s,r)ds\right)\\ -\varphi\left(t,r,\int_a^b k(t,s)y(s,r)ds\right)\end{array}\right)\right\|_X^2 \le Ç_2\|x(r)-y(r)\|_X^2$$

Lemma (3.2) For any $y \in C([a,b]\times[0,1]; L^2(R))$ and $r \in [0.1]$, the operator $\mathbf{P}\left(y(r)\right)(t)$ is continuous on [a, b] in the space $L^2(R)$.

Proof. Let $t_1, t_2 \in$ [a, b] such that $t_1 < t_2$. Then for any $y \in C([a,b]\times[0,1]; L^2(R))$, we have

$$\|(\mathbf{P}\, y(r))(t_2) - (\mathbf{P}\, y(r)(t_1))\|_X^2 = \left\|\left(t_2^{\alpha-1} E_{\alpha,\alpha}(g(r)t_2^{\alpha}) - t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\right) y_0(r) + t_2^{\alpha-1} E_{\alpha,\alpha}(g(r)t_2^{\alpha})\,\varphi\left(t_2,r,\int_a^b k(t_2,s)y(s,r)ds\right) - t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\,\varphi\left(t_1,r,\int_a^b k(t_1,s)y(s,r)ds\right)\right\|_X^2$$

Consequently, we obtain

$$\|(\mathbf{P}\, y(r))(t_2) - (\mathbf{P}\, y(r)(t_1))\|_X^2 \le \left|\left(t_2^{\alpha-1} E_{\alpha,\alpha}(g(r)t_2^{\alpha}) - t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\right)\right| \left\|\left(y_0(r)\right)\right\|_X^2 + \left|\left(t_2^{\alpha-1} E_{\alpha,\alpha}(g(r)t_2^{\alpha})\right)\right| \left\|\varphi\left(t_2,r,\int_a^b k(t_2,s)y(s,r)ds\right)\right\|_X^2 + \left|\left(t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\right)\right| \left\|\varphi\left(t_1,r,\int_a^b k(t_1,s)y(s,r)ds\right)\right\|_X^2$$

Using the condition (H_1), we have

$$\|(\mathbf{P}\, y(r))(t_2) - (\mathbf{P}\, y(r)(t_1))\|_X^2 \le \left|\left(t_2^{\alpha-1} E_{\alpha,\alpha}(g t_2^{\alpha}) - t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\right)\right| \left\|\left(y_0(r)\right)\right\|_X^2 + \left|\left(t_2^{\alpha-1} E_{\alpha,\alpha}(g(r)t_2^{\alpha})\right)\right| K_2\left(1 + \|y(r)\|_X^2\right) + \left|\left(t_1^{\alpha-1} E_{\alpha,\alpha}(g(r)t_1^{\alpha})\right)\right| K_2\left(1 + \|y(r)\|_X^2\right) \quad (15)$$

Hence, by using the strong continuity of the Mittag – Leffler function and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2 \to t_1$. Thus, we conclude **P** $y(r))(t)$ is a continuous on [a,b] in the space $L^2(R)$.

Theorem (3.3) Suppose that the assumptions (H_1), $(H_2)$are satisfied. Then for each $r \in [0,1]$, the fuzzy fractional integro-differential equations with its initial condition in (10) has a unique solution on [a,b], provided that

$$\gamma = Ç_2 \left|b^{\alpha-1} E_{\alpha,\alpha}(-P(r)b^{\alpha})\right| < 1$$

Proof:

To prove the existence of a fixed point of the operator **P** by using Banach's fixed point theorem. Fix $r \in [0,1]$, suppose that $x(r), y(r) \in C([a,b] \times [0,1];\ L^2(R))$, then

$$\|(\mathbf{P}\, x(r))(t) - (\mathbf{P}\, y(r)(t))\|_X^2$$
$$= \left\| \left(t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha}) X_0(r) + t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha})\, \varphi\left(t, r, \int_a^b k(t,s)x(s,r)ds\right)\right) - \left(t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha}) X_0(r) + t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha})\, \varphi\left(t, r, \int_a^b k(t,s)y(s,r)ds\right)\right) \right\|_X^2$$

Rewriting above equation as follows

$$\|(\mathbf{P}\, x(r))(t) - (\mathbf{P}\, y(r)(t))\|_X^2$$
$$= \left\| \left(t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha}) \left\{ \varphi\left(t, r, \int_a^b k(t,s)x(s,r)ds\right) - \varphi\left(t, r, \int_a^b k(t,s)y(s,r)ds\right)\right\}\right) \right\|_X^2$$

By using Cauchy–Schwarz inequality, we obtain

$$\|(\mathbf{P}\, x(r))(t) - (\mathbf{P}\, y(r)(t))\|_X^2$$
$$\le \left|t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha})\right| \left\| \left(\varphi\left(w, r, \int_a^b k(w,s)x(s,r)ds\right) - \varphi\left(w, r, \int_a^b k(w,s)y(s,r)ds\right)\right) \right\|_X^2$$

From the assumption (b) and Lemma (3.3.2), we obtain

$$\|(\mathbf{P}\, x(r))(t) - (\mathbf{P}\, y(r)(t))\|_X^2$$
$$\le Ç_2 \left|t^{\alpha-1} E_{\alpha,\alpha}(-g(r)t^{\alpha})\right| \|x(r) - y(r)\|_X^2$$

Consequently, we get

$$\|(\mathbf{P}\, x(r))(t) - (\mathbf{P}\, y(r)(t))\|_X^2$$
$$\le Ç_2 \left|b^{\alpha-1} E_{\alpha,\alpha}(-g(r)b^{\alpha})\right| \|x(r) - y(r)\|_X^2$$

Therefore, **P** is a contraction mapping on $C([a,b] \times [0,1];\ L^2(R))$. From contraction mapping principle theorem **P** has a unique fixed point, which is a solution of equation (12) on [a, b].

4 METHODOLOGY DESCRIPTION

Suppose that the $n+1$ data points, $t_i, i = 0,1,,2,,,,.n$ are the knots and increasing in order are given. Fuzzy generalized eyponential spline $S(t,r)$ through the above data points can be defined as follows (18)

$$S(t,r) = \sum_{i=1}^{m} a_i(r) e^{\beta i(t-t_0)} \tag{13}$$

Where, $S(t,r) = \left(\underline{S}(t,r), \overline{S}(t,r)\right)$,

$a_i(r) = \left(\underline{a_i}(r), \overline{a_i}(r)\right), i = 1,2,\ldots,m$ and β is arbitrary positive real values. By replacing t by t_0 in equation (13), we have

$$S(t_0, r) = \sum_{i=1}^{m} a_i(r)$$

Again, By replacing t by t_1 in equation (13), we have

$$S(t_1, r) = \sum_{i=1}^{m} a_i(r) e^{\beta i h_1}$$

By substituting $S(t.r)$ in the equation (13) into equation (10), we get

$$D_C^{\alpha} y(t,r) = g(t,r)\, y(t,r) + f(t,r) + \int_a^b k(t,s)y(s,r)ds$$

$$D_C^{\alpha}\left(\sum_{i=1}^{m} a_i(r) e^{\beta i(t-t_0)}\right) = g(t,r) \left(\sum_{i=1}^{m} a_i(r) e^{\beta i(t-t_0)}\right) + f(t,r) + \int_a^b k(t,s) \sum_{i=1}^{m} a_i(r) e^{\beta i(s-t_0)}\, ds$$

This implies

$$\sum_{i=1}^{m} a_i(r) D_C^{\alpha}\left(e^{\beta i(t-t_0)}\right) - \sum_{i=1}^{m} a_i(r)\ g(t,r)\left(e^{\beta i(t-t_0)}\right) - \sum_{i=1}^{m} a_i(r) \int_a^b k(t,s) e^{\beta i(s-t_0)} ds = f(t,r)$$

From the definition of fractional derivative in Section 2., we have

$$D_C^{\alpha}\left(e^{\beta t}\right) = \beta t^{1-\alpha} E_{1,2-\alpha}(\beta t) \tag{14}$$

Now, we use the following notations

$$\sum_{i=1}^{m} a_i(r)\left\{\beta i(\mathrm{t}-\mathrm{t}_0)^{1-\alpha}\mathrm{E}_{1,2-\alpha}\big(\beta i(\mathrm{t}-\mathrm{t}_0)\big) - g(t,r)\big(\mathrm{e}^{\beta i(\mathrm{t}-\mathrm{t}_0)}\big) - \int_a^b k(t,s)\mathrm{e}^{\beta i(\mathrm{s}-\mathrm{t}_0)}ds\right\} = f(t,r)$$

$$M_i(t) = \beta i(\mathrm{t}-\mathrm{t}_0)^{1-\alpha}\mathrm{E}_{1,2-\alpha}\big(\beta i(\mathrm{t}-\mathrm{t}_0)\big) - g(t,r)\big(\mathrm{e}^{\beta i(\mathrm{t}-\mathrm{t}_0)}\big) - \int_a^b k(t,s)\mathrm{e}^{\beta i(\mathrm{s}-\mathrm{t}_0)}ds \quad .\, i = 1,2,\dots \overset{0\le r\le 1}{m}$$

Adding the initial condition of equation (12), as a new raw in the following matrices

$$\mathcal{M} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ M_1(t_1) & M_2(t_1) & \dots & M_m(t_1) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ M_1(t_n) & M_2(t_n) & \dots & M_m(t_n) \end{bmatrix}$$

$$\mho(r) = \begin{bmatrix} a_1(r) \\ a_2(r) \\ \vdots \\ a_m(r) \end{bmatrix}$$

$$E(r) = \begin{bmatrix} y_0(r) \\ f(t_1,r) \\ \vdots \\ f(t_n,r) \end{bmatrix}$$

For each r , $\mathcal{M}$ and $E(r)$ are constant matrices with dimensions $(n+1)\times m$ and $(n+1)\times 1$ respectevely, $\mho(r)$ is unknown vector.
The system will construct has $n+1$ equations and m coefficients such that $n \ge 3$ therefore,

$$\mathcal{M}^{\tau}\mathcal{M}\mho(r) = \mathcal{M}^{\tau}E(r)$$

where, $\mathcal{M}^{\tau}$ is transpose matrix of $\mathcal{M}$.

5 ILLUSTRATIVE EXAMPLE

To show the efficiency and accuracy of the propose technique with various values of step size, we consider the following two examples.

Example (4.1): Consider the following integro-differential equation as

$$\begin{cases} \mathrm{D}_{\mathrm{C}}^{\alpha}X(t,r) + X(t,r) = \\ \big((r^5+2r)(1+\sinh(t)), (6-3r^3)(1+\mathrm{s} \\ \quad + \int_0^1 (t-s)X(s,r)ds \\ X(0,r) = \big((r^5+2r), (6-3r^3)\big) \\ \quad ,\; t \in [0,1], 0 \le r \le 1 \end{cases} \tag{15}$$

Where, $0 < \alpha \le 1$, the exact solution is given by

$$X(t,r) = \begin{pmatrix} (r^5+2r) \\ (t^{\alpha+1}\mathrm{E}_{2,\alpha+2}(t^2)+1), (6-3r^3)(t^{\alpha+1}\mathrm{E}_{2,\alpha+2}(t^2)+1) \end{pmatrix} \tag{16}$$

To compare we use the formula $d(X_n, X(t_n)) = \text{Sup}\ \max(\underline{X_n} - \underline{X(t_n)}, \overline{X_n} - \overline{X(t_n)})$ (17)

Let us compute The approximate solution of equation (15) by using generalized exponential spline method. Here, we take step size $h = 0.1$, $h = 0.01$ and $h = 0.001$
Consider equation (10), then

$$g(t,r) = -1, f(t,r) = \big((r^5+2r)(1+\sinh(t)), (6-3r^3)(1+\sinh(t))\big), a = 0, b = 1\ and\ k(t,s) = (t-s) \tag{18}$$

Approximate solutions $\underline{X_n}, \overline{X_n}$ can be found by solving equation in (16) (see Fig. 1., 2, 3) And Table 1, 2, 3,4) (19)

Table 1. The fuzzy coefficients of equation (19) are computed when $h = 0.01, t = 0.3$, $\alpha = 0.9$

r	$\underline{a_1}(r)$	$\overline{a_1}(r)$	$\underline{a_2}(r)$	$\overline{a_2}(r)$	$\underline{a_3}(r)$	$\overline{a_3}(r)$	$\underline{a_4}(r)$	$\overline{a_4}(r)$
0	0	13.6237	0	-12.0523	0	5.0201	0	-0.7202
0.1	0.4541	13.6169	-0.4018	-12.0463	0.1673	5.0176	-0.0240	-0.7199
0.2	0.9090	13.5692	-0.8041	-12.0041	0.3349	5.0000	-0.0481	-0.7174
0.3	1.3679	13.4397	-1.2101	-11.8896	0.5040	4.9523	-0.0723	-0.7105
0.4	1.8397	13.1877	-1.6275	-11.6666	0.6779	4.8594	-0.0973	-0.6972
0.5	2.3416	12.7722	-2.0715	-11.2990	0.8628	4.7063	-0.1238	-0.6752
0.6	2.9013	12.1523	-2.5667	-10.7506	1.0691	4.4779	-0.1534	-0.6425
0.7	3.5605	11.2872	-3.1498	-9.9853	1.3120	4.1591	-0.1882	-0.5967
0.8	4.3770	10.1360	-3.8722	-8.9669	1.6128	3.7349	-0.2314	-0.5359
0.9	5.4279	8.6578	-4.8018	-7.6592	2.0001	3.1903	-0.2870	-0.4577
1	6.8118	6.8118	-6.0261	-6.0261	2.5100	2.5100	-0.3601	-0.3601

(20)

Table 2. $h = 0.1$, $\alpha = \mathbf{0.9}$,

t	d
0	0
0.3	0.0036
0.5	8.7676×10^{-4}
0.7	0.0018
0.9	0.0047

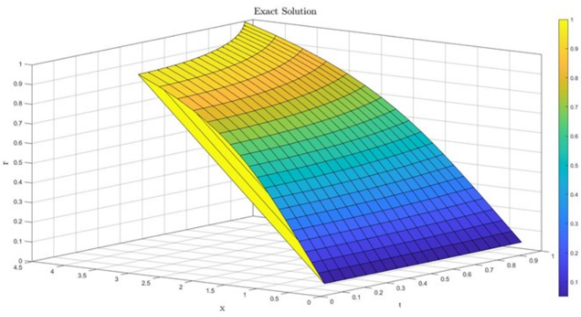

Fig. 1 Exact Solution

Table 3. $h = 0.01$, $\alpha = 0.9$

t	d
0	0
0.3	0.0024
0.5	5.1927×10^{-4}
0.7	8.9267×10^{-4}
0.9	0.0024

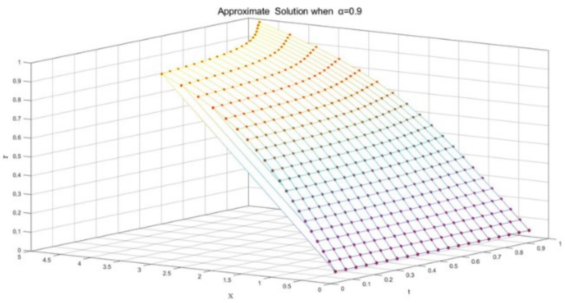

Fig. 2 Approximate Solution

Table 4. $h = 0.001$, $\alpha = 0.9$

t	d
0	0
0.3	0.0023
0.5	4.9968×10^{-4}
0.7	7.8689×10^{-4}
0.9	0.0021

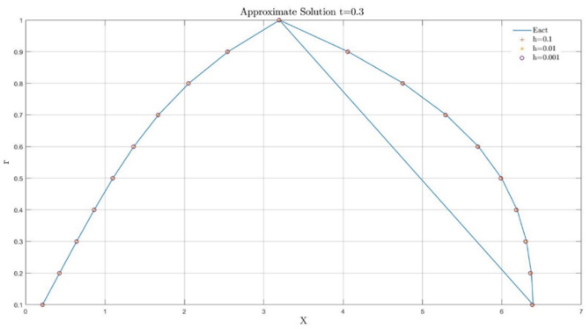

Fig. 3 Exact and Approximate Solution at t=0.3

6 Conclusion

In this article, a new class of exponential spline function method is introduced for solving fuzzy fractional integro-differential equations subject to fuzzy initial conditions. This technique proved its efficient and reliability in solving of these equations by providing the best approximate solutions. The numerical outputs obtained using the proposed technique are comparable to the exact solutions of our proposed model. This technique proved its efficient and reliability in solving of these equations by providing the best approximate solutions. The numerical outputs obtained using the proposed technique are comparable to the exact solutions of our proposed model. We showed that the step size h played fundamental and important role in reducing the error rate which resulting from the approximation of solutions for fuzzy integro-differential Equations. Thus, our work in this paper, one can extend this method to solve fractional-order greater than 1 of fuzzy initial value problems. Finally, we would like to refer that the proposed equation can be applied to real models and used for data analysis in various systems such as medicine, economy, engineering, biomedical, and environmental.

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