

Computer Experimentation with Dirichlet Functions

ANDREI-FLORIN ALBIŞORU¹, DORIN GHIŞU²

¹Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, ROMANIA

²Department of Mathematics, Glendon College, York University, Toronto, CANADA

Abstract: There is a vast amount of literature about Dirichlet series, starting with the works of Cahen and followed by the works of Hardy and Riesz, Valiron, Landau, Bohr, Kojima, etc. These series are generalizations of the famous Euler series. Using his functional equation, Riemann extended the Euler series across the convergence line. The problem of extending general Dirichlet series using Riemann's method appeared, and it has been successfully dealt with in the particular case of Dirichlet L-series, obtaining functions with properties similar to those of the Riemann Zeta function. However, until recently, no other class of Dirichlet series has been known, that can be continued as a meromorphic function in the whole complex plane. Moreover, the chance that Dirichlet series might exist, such that their continuation has several poles, appeared to be very small. Our discovery of Dirichlet functions generated by Blaschke products by a change of variable completely reversed this point of view. Now, it is known not only that a whole class of Dirichlet series exists with continuations, series that have infinitely many poles but also that they can have some essential singular points. In this paper, the behavior of a Dirichlet function in a neighborhood of an essential singular point is revealed, and the behavior is really surprising. The Dirichlet functions generated by finite Blaschke products are fit for computer experimentation since they are given by formulas that can be implemented with ease in computer programs. In this paper, we are dealing with such Dirichlet functions in a general context and indicate their zeros, poles, and branch points. We are looking for global mapping properties of these functions, describing in detail their fundamental domains. Computer graphics are offered, adding a new chapter to the study of Dirichlet functions, as well as in that of Blaschke products. Computer programs have been created that can deal with infinite Dirichlet series and with the remarkable properties of Dirichlet functions generated by them.

Key-Words: fundamental domains, Blaschke product, Dirichlet functions, conformal mapping, pre-images of lines and circles, computer experimentation.

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1 Introduction

We have studied in [1] Dirichlet functions generated by some second-degree Blaschke products $B(z)$ having a double zero. The two branch points of $B(z)$ coincide, with the zero and the other with the pole of the function. The Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ generated by $B(z)$ has infinitely many branch points at which $\zeta_{A,\mathbb{N}}(s) = 0$ or $\zeta_{A,\mathbb{N}}(s) = \infty$ and which projects by e^{-s} , the first into the zero and the second into the pole of $B(z)$.

To every zero of $\zeta_{A,\mathbb{N}}(s)$ corresponds a pole located onto the same horizontal line; all zeros have the same abscissa, and all poles have the same abscissa. The distance between two consecutive zeros, and the distance between two consecutive poles, is always 2π .

This configuration allowed a precise description of the fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, and their conformal mapping.

In this paper, we deal with Dirichlet functions generated by Blaschke products with distinct zeros.

In this case, we are faced with the challenge of finding the branch points of those Blaschke products and proving that the points representing their pre-image by e^{-s} are still located two by two on some horizontal lines. The case of the second and third-degree Blaschke products is presented in detail, allowing one to draw conclusions in the general case.

For this purpose, we need to study the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis. Speiser first used this technique in 1934, [2], for the Riemann Zeta function. By using it, Speiser succeeded in proving an equivalent form of the Riemann Hypothesis.

This was a remarkable accomplishment, and it is cited in many number theory papers, yet, as Arias-de-Reina, [3], noticed, nobody reproduced Speiser's result. In our opinion, this has been because a key piece of his method was missing, namely, the theory of fundamental domains of analytic functions. We refined the Speiser technique in several papers dealing with general Dirichlet series and used it in the study of the Riemann Hypothesis. As it

appeared still at the time of Speiser's work, the Riemann Hypothesis is related to some Riemann type of functional equation, which is satisfied by Dirichlet L-functions, [4], [5], and also to the fact that the L-series can be expressed as Euler products. We dealt in [1] with the general Dirichlet series of the form

$$\zeta_{A,\Lambda}(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

where $A = \{a_n\}$ is an arbitrary infinite sequence of complex numbers and $\Lambda = \{0 = \lambda_0 \leq \lambda_1 \leq \dots\}$ is a non-decreasing sequence of nonnegative numbers such that $\lim_{n \rightarrow \infty} \lambda_n = +\infty$. When $a_0 = 1$ we say that the series is normalized. It is easy to see that for normalized series, we have

$$\lim_{\sigma \rightarrow +\infty} \zeta_{A,\Lambda}(\sigma + it) = 1$$

This simple equality, which appears in most of the works dealing with general Dirichlet series, [6], [7], [8], [9], [10], [11], [12], [13] has significant consequences regarding the landscape of the pre-image by $\zeta_{A,\Lambda}(s)$ of the real axis. We succeeded in making this landscape clear in [14] by proving that for every normalized general Dirichlet L-series, the components of the pre-image of the real axis are of three distinct types:

a). There are infinitely many unbounded curves Γ'_n , $n \in \mathbb{Z}$ which are mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the interval $(1, +\infty)$ of the real axis. These curves do not intersect each other, and two consecutive ones bound infinite strips S_n in which σ varies from $-\infty$ to $+\infty$. The strips S_n cover the whole complex plane. We count them from $n = -\infty$ to $n = +\infty$ such that $0 \in S_0$.

b). Every strip S_n contains a unique curve $\Gamma_{n,0}$, which does not intersect any Γ'_n and is mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the interval $(-\infty, 1)$ of the real axis.

c). Every strip S_n can contain several curves $\Gamma_{n,k}$, $k \neq 0$ mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the whole real axis. The curves $\Gamma_{n,k}$ cannot intersect any Γ'_n , but they can intersect each other in a multiple zero or a multiple pole of $\zeta_{A,\Lambda}(s)$.

Every strip S_n contains a unique unbounded component of the pre-image of the unit circle, which is a parabola-shaped curve with branches tending asymptotically to Γ'_n and Γ'_{n+1} .

In this paper, we are using this knowledge in the study of Dirichlet functions generated by arbitrary Blaschke products.

Finally, we develop a clear image of the geometry of conformal mapping by these functions. This reveals, for the first time, the complexity of the

class of functions obtained by analytic continuation of Dirichlet series.

2 Dirichlet Functions defined by Second-Degree Blaschke Products

In this section, the study of Dirichlet functions defined by second-degree Blaschke products is undertaken. We prove theorems about these functions under general conditions and illustrate the facts using computer-generated graphics. We make obvious the connection between the properties of these functions and those of the Blaschke products that have generated them and we reveal new facts about those Blaschke products in this way. These facts concern the location of zeros of their derivative, the fundamental domains of these Blaschke products, as well as their conformal mapping onto the whole complex plane with some slits. The conclusion is that the Dirichlet functions generated by the second-degree Blaschke products are periodic functions of period $2\pi i$. This is the first known example of periodic Dirichlet functions.

Let

$$B(z) = \frac{z - z_1}{1 - \bar{z}_1 z} \frac{z - z_2}{1 - \bar{z}_2 z}, \quad z_k = r_k e^{i\alpha_k}, \quad (1)$$

where $0 < r_1 \leq r_2 < 1$, $\alpha_k \in \mathbb{R}$, $k = 1, 2$, be a second-degree Blaschke product with distinct zeros z_1 and z_2 . The meromorphic function $B(z)$ has the poles $\frac{1}{\bar{z}_1}$ and $\frac{1}{\bar{z}_2}$.

It is known, [15], [16], [17], that for every real number φ , the equation $B(z) = e^{i\varphi}$ has two distinct roots $e^{i\gamma_k}$, $k = 1, 2$, which determine two disjoint arcs on the unit circle. These arcs, with one end removed, are mapped each one bijectively by $B(z)$ onto the unit circle. The equation $B'(z) = 0$ has no root on the unit circle. The following theorem is elementary, and some of its affirmations are probably known facts in the theory of Blaschke products.

Theorem 1. The roots of the equation

$$B'(z) = 0$$

are of the form

$$\zeta_k = \rho_k e^{i\gamma}, \quad k = 1, 2$$

for some real value γ and

$$0 < r_1 \leq \rho_1 \leq r_2 < 1 < \frac{1}{r_2} \leq \rho_2 \leq \frac{1}{r_1}$$

with strict inequalities if $r_1 < r_2$. Moreover, we have $\rho_2 = \frac{1}{\rho_1}$, and that all these roots are aligned if and only if $0, z_1$ and z_2 are aligned.

Proof: An easy computation shows that the equation $B'(z) = 0$ is equivalent to

$$Az^2 - 2(1 - r_1^2 r_2^2)z + \bar{A} = 0, \quad (2)$$

where

$$A = (1 - r_1^2)\bar{z}_2 + (1 - r_2^2)\bar{z}_1, \quad (3)$$

hence

$$|A|^2 = (1 - r_1^2)^2 r_2^2 + (1 - r_2^2)^2 r_1^2 + 2(1 - r_1^2)(1 - r_2^2)\Re z_1 \bar{z}_2. \quad (4)$$

The roots of the equation (2) are

$$\zeta_{1,2} = \frac{1}{A} \left[1 - r_1^2 r_2^2 \pm \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2} \right] \quad (5)$$

Let us show that

$$(1 - r_1^2 r_2^2)^2 - |A|^2 > 0.$$

Indeed,

$$\begin{aligned} & (1 - r_1^2 r_2^2)^2 - |A|^2 = \\ & = 1 - 2r_1^2 r_2^2 + r_1^4 r_2^4 - r_1^4 r_2^2 + 2r_1^2 r_2^2 - r_2^2 - r_1^2 r_2^4 \\ & \quad + 2r_1^2 r_2^2 - r_1^2 - 2(1 - r_1^2)(1 - r_2^2)\Re z_1 \bar{z}_2 \\ & = (1 + r_1^2 r_2^2)^2 - (r_1^2 + r_2^2)(1 + r_1^2 r_2^2) \\ & \quad - 2(1 - r_1^2)(1 - r_2^2)\Re z_1 \bar{z}_2 \\ & = (1 + r_1^2 r_2^2)(1 + r_1^2 r_2^2 - r_1^2 - r_2^2) \\ & \quad - 2(1 - r_1^2)(1 - r_2^2)\Re z_1 \bar{z}_2 \\ & = (1 - r_1^2)(1 - r_2^2)(r_1^2 r_2^2 + 1 - 2\Re z_1 \bar{z}_2) \\ & > (1 - r_1^2)(1 - r_2^2)(1 - r_1 r_2)^2 \\ & > 0 \end{aligned}$$

It is obvious that

$$1 - r_1^2 r_2^2 > 0$$

and

$$1 - r_1^2 r_2^2 > \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2}.$$

Therefore the two numbers inside the brackets in (5) are positive. Thus, with

$$A = |A|e^{-i\gamma},$$

we have

$$\zeta_{1,2} = \rho_{1,2}e^{i\gamma},$$

where

$$\rho_{1,2} = \frac{1}{|A|} \left[1 - r_1^2 r_2^2 \pm \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2} \right].$$

To check that $\rho_1 < 1 < \rho_2$, we need to show that

$$\begin{aligned} & 1 - r_1^2 r_2^2 - \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2} < |A| \\ & < 1 - r_1^2 r_2^2 + \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2}, \end{aligned}$$

or

$$\begin{aligned} & |1 - r_1^2 r_2^2 - |A|| < \\ & < \sqrt{(1 - r_1^2 r_2^2 - |A|)(1 - r_1^2 r_2^2 + |A|)} \end{aligned}$$

which is equivalent to

$$\sqrt{1 - r_1^2 r_2^2 - |A|} < \sqrt{1 - r_1^2 r_2^2 + |A|}$$

and this is obvious. Thus,

$$\ln \rho_1 < 0 < \ln \rho_2.$$

Moreover, we have

$$\zeta_1 \zeta_2 = \frac{\bar{A}}{A},$$

hence

$$|\zeta_1 \zeta_2| = \rho_1 \rho_2 = 1,$$

which implies

$$\rho_2 = \frac{1}{\rho_1},$$

and then

$$\zeta_2 = \frac{1}{\rho_1} e^{i\gamma} = \frac{1}{\rho_1 e^{-i\gamma}} = \frac{1}{\bar{\zeta}_1}.$$

Thus,

$$B(\zeta_2) = B\left(\frac{1}{\bar{\zeta}_1}\right) = \frac{1}{B(\zeta_1)}.$$

Suppose now that 0, z_1 and z_2 are aligned, i.e.,

$$z_1 = r_1 e^{i\alpha} \text{ and } z_2 = r_2 e^{i\alpha}$$

or

$$z_1 = r_1 e^{i\alpha} \text{ and } z_2 = -r_2 e^{i\alpha}.$$

In the first case

$$\Re z_1 \bar{z}_2 = r_1 r_2$$

and

$$A = (r_1 + r_2)(1 - r_1 r_2)e^{-i\alpha},$$

thus

$$\zeta_1 = \frac{1 + r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)}}{r_1 + r_2} e^{i\alpha},$$

hence ζ_1 is aligned with 0, z_1 and z_2 .

Moreover,

$$r_1 \leq \rho_1 \leq r_2$$

if and only if

$$r_1 \leq \frac{1 + r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)}}{r_1 + r_2} \leq r_2,$$

or

$$\begin{aligned} r_1^2 + r_1 r_2 &\leq 1 + r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)} \\ &\leq r_2^2 + r_1 r_2, \end{aligned}$$

or

$$1 - r_2^2 \leq \sqrt{(1 - r_1^2)(1 - r_2^2)} \leq 1 - r_1^2,$$

which is

$$1 - r_2^2 \leq 1 - r_1^2,$$

i.e.,

$$r_1 \leq r_2$$

and this has been postulated from the start. This is equivalent to

$$\frac{1}{r_2} \leq \rho_2 \leq \frac{1}{r_1}.$$

In the second case

$$\Re z_1 \bar{z}_2 = -r_1 r_2$$

and

$$A = (r_1 - r_2)(1 + r_1 r_2)e^{-i\alpha},$$

thus

$$\zeta_1 = \frac{1 - r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)}}{r_1 - r_2} e^{i\alpha} = -\rho_1 e^{i\alpha},$$

hence again ζ_1 is aligned with $0, z_1$ and z_2 .

Moreover,

$$r_1 \leq \rho_1 \leq r_2$$

if and only if

$$-r_2 \leq -\frac{1 - r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)}}{r_1 - r_2} \leq -r_1,$$

or

$$\begin{aligned} r_2^2 - r_1 r_2 &\geq 1 - r_1 r_2 - \sqrt{(1 - r_1^2)(1 - r_2^2)} \\ &\geq -r_1^2 + r_1 r_2 \end{aligned}$$

or

$$\begin{aligned} 1 - r_2^2 &\leq \sqrt{(1 - r_1^2)(1 - r_2^2)} \\ &\leq 1 - 2r_1 r_2 + r_1^2 \\ &\leq 1 - r_1^2, \end{aligned}$$

i.e.,

$$r_1 \leq r_2. \quad \square$$

Fig. 1 below illustrates these facts. We kept

$$z_1 = 0.4 + 0.2i = \sqrt{0.2}e^{i\alpha}, \alpha = \arctan 0.5$$

and $\frac{1}{z_1}$ fixed (red points) and let z_2 vary on a circle centered at the origin and of radius $r = 0.894427$. We plotted several values of z_2 and $\frac{1}{z_2}$ (green points) and the corresponding ζ_1 (orange points) and ζ_2 (blue points). Let us notice that when the argument of z_2 is α or $\pi + \alpha$, the seven points

$$0, z_1, z_2, \zeta_1, \zeta_2, \frac{1}{z_1} \text{ and } \frac{1}{z_2}$$

are aligned. The figure also illustrates the inequalities involving r_1, r_2, ρ_1 and ρ_2 when the seven points are aligned. Let us prove that these inequalities are always true. Suppose that instead of discrete positions of z_2 , as in Fig. 1, we let z_2 move continuously on the circle $|z| = r_2$. Then, the root $\zeta_1 = \rho_1 e^{i\gamma}$ will also move continuously on a curve η . If η intersects the circle $|z| = r_2$ at a point z_0 , then we have

$$\zeta_1 = z_2 = z_0,$$

hence z_0 is a double zero of $B(z)$, in other words,

$$z_1 = z_2 = z_0,$$

hence $r_1 = r_2$. Otherwise, if r_1 remains less than r_2 , then η cannot intersect the circle $|z| = r_2$ and ρ_1 remains less than r_2 . A similar argument can be employed in order to prove that $r_1 < r_2$ implies $r_1 < \rho_1$.

Fig. 2 illustrates the geometry of conformal mapping by a second-degree Blaschke product $B(z)$ as in (1) with

$$z_1 = 0.4 + 0.2i,$$

$$z_2 = 0.5257311121191337 - 0.723606797749979i.$$

The unit circle (black) is mapped two-to-one onto itself. The two arcs, with one end removed, determined on it by the intersection with the pre-image of the positive real half-axis (in blue), are each one mapped one to one onto the unit circle. The same is true for the arcs determined by the intersection with the pre-image of the negative real half-axis (in red). The two colors meet at the zeros z_1 and z_2 and at the poles $\frac{1}{z_1}$ and $\frac{1}{z_2}$ of $B(z)$. The points ζ_1 and ζ_2 are the zeros of $B'(z)$, hence the branch points of $B(z)$. The curves connecting them with the zeros, respectively with the poles of $B(z)$, are the pre-images of the interval I_1 connecting $w = 1$ with

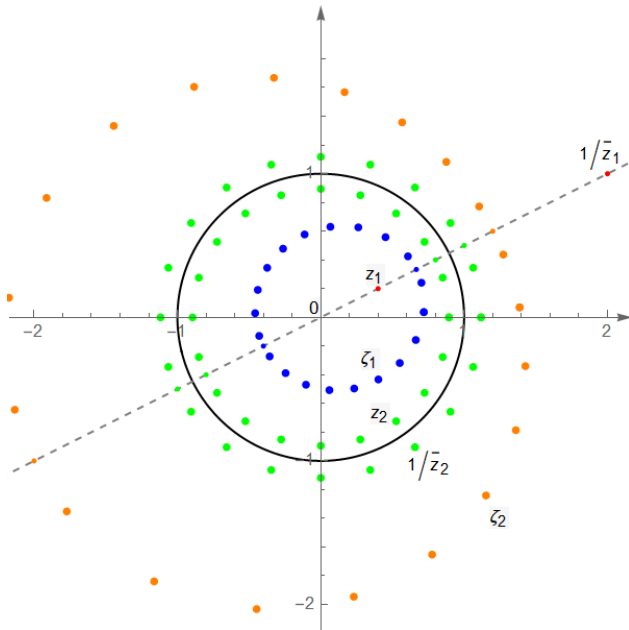


Figure 1: An illustration of the location of the roots of $B'(z)$

$w = B(\zeta_1)$, respectively of the ray I_2 starting at $w = B(\zeta_2)$. They are mapped two to one by $B(z)$ onto these curves.

By the conformal mapping theorem, the domains bounded by the blue curves and these two curves are mapped conformally by $B(z)$ onto the complex plane with a slit alongside the positive real half axis and the intervals I_1 and I_2 .

The function $B(z)$ is analytic in the disk $|z| < \frac{1}{r_2}$ and it admits the Taylor expansion

$$B(z) = \sum_{n=0}^{\infty} \frac{1}{n!} B^{(n)}(0) z^n, \quad (6)$$

convergent for $|z| < \frac{1}{r_2}$. This series can be converted into a Dirichlet series by the change of variable $z = e^{-s}$, where $s = \sigma + it$. Since $B(0) = z_1 z_2 \neq 0$, to obtain a normalized Dirichlet series

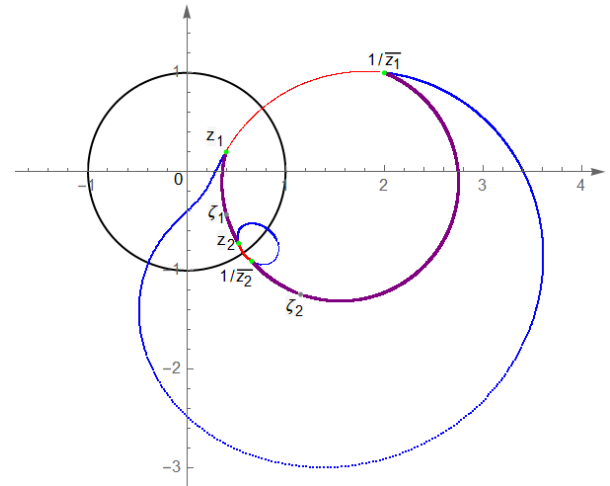
$$\sum_{n=0}^{\infty} a_n e^{-ns}$$

with $a_0 = 1$ we need to divide $B(z)$ by $z_1 z_2$. Then, we get

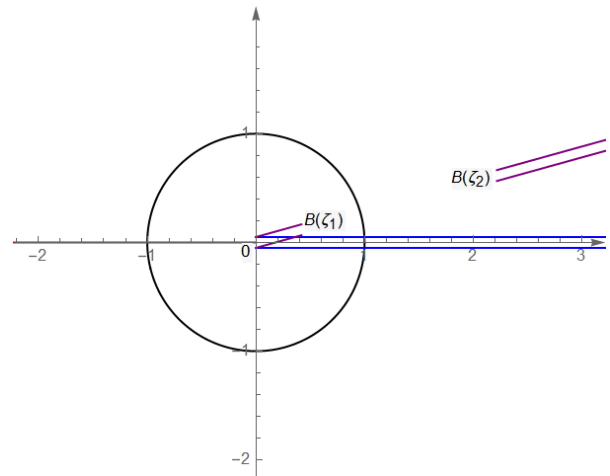
$$\zeta_{A,\mathbb{N}}(s) = \frac{1}{z_1 z_2} B(e^{-s}) = \sum_{n=0}^{\infty} a_n e^{-ns}, \quad (7)$$

where

$$a_n = \frac{1}{z_1 z_2} \frac{B^{(n)}(0)}{n!}.$$



(a)



(b)

Figure 2: An illustration of the fundamental domains of $B(z)$ and their conformal mapping

Obviously,

$$a_0 = \frac{1}{z_1 z_2} \frac{B(0)}{0!} = 1.$$

This series converges for

$$|e^{-s}| = |e^{-\sigma-it}| = e^{-\sigma} < \frac{1}{r_2},$$

i.e., for $\sigma > \ln r_2$ and diverges for $\sigma < \ln r_2$, which means that $\ln r_2$ is the abscissa of convergence of the series (7). Here,

$$A = \{a_n \mid n \in \mathbb{N}\},$$

and the subscript \mathbb{N} in relation (7) indicates that the exponents of the series (7) are the natural numbers. The series can be analytically continued across the convergence line to a meromorphic function in the whole complex plane since $B(e^{-s})$ is defined everywhere and is analytic, except for its poles. We keep the notation $\zeta_{A,\mathbb{N}}(s)$ for this function. Since $B(z)$ has the poles

$$\frac{1}{z_1} = \frac{1}{r_1} e^{i\alpha_1} \text{ and } \frac{1}{z_2} = \frac{1}{r_2} e^{i\alpha_2},$$

the Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ will have infinitely many poles

$$\ln r_1 + (2m\pi - \alpha_1)i \text{ and } \ln r_2 + (2m\pi - \alpha_2)i,$$

for $m \in \mathbb{Z}$. Also, this function has infinitely many zeros, which are

$$-\ln r_1 + (2m\pi - \alpha_1)i \text{ and } -\ln r_2 + (2m\pi - \alpha_2)i.$$

There are similarities and dissimilarities between the functions $\zeta_{A,\mathbb{N}}(s)$ and the Dirichlet L-functions $L(s; \chi)$, the functions in the latter being defined by Dirichlet characters χ . A similarity is that the nontrivial zeros of $L(s; \chi)$ are all located on a vertical line since $L(s; \chi)$ is an Euler product. The same is true for the zeros of $\zeta_{A,\mathbb{N}}(s)$ if $r_1 = r_2$. Otherwise, those zeros are located on two vertical lines. A dissimilarity between the two classes of Dirichlet functions is the fact that while the Dirichlet L-functions have just one pole, the functions $\zeta_{A,\mathbb{N}}(s)$ have infinitely many poles.

Theorem 2. The branch points of the function $w = \zeta_{A,\mathbb{N}}(s)$ are

$$-\ln \rho_1 + (2m\pi - \gamma)i \text{ and } -\ln \rho_2 + (2m\pi - \gamma)i,$$

$m \in \mathbb{Z}$ for some $\gamma \in \mathbb{R}$, $\alpha_1 < \gamma < \alpha_2$, when

$$\alpha_1 < \alpha_2 < \alpha_1 + \pi,$$

and $\alpha_2 < \gamma < \alpha_1 + 2\pi$ when

$$\alpha_1 + \pi < \alpha_2 < \alpha_1 + 2\pi.$$

We have $\gamma = \alpha_1$ when $\alpha_1 = \alpha_2$ and $\gamma = \alpha_2$ when $\alpha_2 = \alpha_1 + \pi$.

Proof: Indeed, the branch points of $\zeta_{A,\mathbb{N}}(s)$ are the roots of the equation $\zeta'_{A,\mathbb{N}}(s) = 0$, or

$$\frac{d}{ds} \frac{1}{z_1 z_2} B(e^{-s}) = 0,$$

or

$$\frac{1}{z_1 z_2} B'(e^{-s})(-e^{-s}) = 0.$$

By Theorem 1, such a root s should satisfy the equality

$$e^{-s} = \rho_k e^{i\gamma} = e^{\ln \rho_k + i\gamma}, \quad k = 1, 2,$$

thus $\zeta'_{A,\mathbb{N}}(s) = 0$ if and only if

$$s = -\ln \rho_k + (2m\pi - \gamma)i, \quad k = 1, 2.$$

The analytic function $\zeta_{A,\mathbb{N}}(s)$ is locally injective except for the branch points. Then, the components of the pre-image by it of a disk $|w| < r$, for r small enough, are disjoint closed sets containing each one a zero

$$-\ln r_k + (2m\pi - \alpha_k)i, \quad k = 1, 2,$$

$m \in \mathbb{Z}$ of $\zeta_{A,\mathbb{N}}(s)$.

Having in view that

$$\begin{aligned} \gamma &= -\text{Arg } A \\ &= \arctan \frac{r_2(1-r_1^2) \sin \alpha_2 + r_1(1-r_2^2) \sin \alpha_1}{r_2(1-r_1^2) \cos \alpha_2 + r_1(1-r_2^2) \cos \alpha_1}, \end{aligned}$$

we obtain the inequalities for γ after an elementary computation. \square

Theorem 3. There is a value $r = r_0$ such that the intersection of adjacent components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle

$$C_{r_0} : |w| = r_0$$

is exactly

$$-\ln \rho_1 + (2m\pi - \gamma)i$$

for every $m \in \mathbb{Z}$.

Proof: Indeed, two such components must intersect for some value $r = r_0$, since otherwise, a part of the complex plane would be mapped conformally onto the whole complex plane, contradicting the Riemann mapping theorem.

The point where two adjacent components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle C_{r_0} meet each other is a branch point of $\zeta_{A,\mathbb{N}}(s)$, since in any neighborhood of this point the function assumes twice the same value, once on each one of the components. Yet, the only branch points of $\zeta_{A,\mathbb{N}}(s)$ are

$$-\ln \rho_k + (2m\pi - \gamma)i.$$

Since those components are in the right half-plane, these branch points are

$$-\ln \rho_1 + (2m\pi - \gamma)i. \quad \square$$

We notice that

$$\zeta_{A,\mathbb{N}}(-\ln \rho_1 + (2m\pi - \gamma)i) = \frac{1}{z_1 z_2} B(\rho_1 e^{i\gamma})$$

does not depend on m , hence those components meet simultaneously at all branch points of $\zeta_{A,\mathbb{N}}(s)$ from the right half-plane.

When $r = 1$, we have the circle C_1 . Using the notation $\Upsilon_{m,r}$ for the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle C_r , the symbol for the pre-image of the unit circle is $\Upsilon_{m,1}$. If $r_0 \geq 1$, this pre-image is formed by two curves around each zero of the function $\zeta_{A,\mathbb{N}}(s)$. One of them is a closed curve, and the other one must be unbounded due to the equality

$$\lim_{\sigma \rightarrow +\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1. \quad (8)$$

If $r_0 < 1$, the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle is just one unbounded curve containing both zeros of $\zeta_{A,\mathbb{N}}(s)$.

These curves will help us later to illustrate the conformal mapping by $\zeta_{A,\mathbb{N}}(s)$ of the fundamental domains of this function. The unbounded component of $\Upsilon_{m,1}$ intersects the pre-image of the real axis in three points, two on the pre-image of the negative real half-axis and one on the pre-image of the positive real half-axis.

Let us notice that the circles C_r and the rays issued from the origin form an orthogonal mesh whose pre-image by $\zeta_{A,\mathbb{N}}(s)$ is also orthogonal. Once we know this last mesh, we can accurately describe the conformal mapping by $\zeta_{A,\mathbb{N}}(s)$ of its fundamental domains.

If $r > 1$, the curves $\Upsilon_{m,r}$ fuse into a unique unbounded curve traversing all the strips S_m and intersecting orthogonally Γ'_m in points $s'_{m,r}$ such that $\zeta_{A,\mathbb{N}}(s'_{m,r}) = r$. When $r = \frac{1}{r_1 r_2}$ the curve $\Upsilon_{m,r}$ is the imaginary axis. It intersects orthogonally the curve Γ'_m at a point $i\tau_m$, $\tau_m \in \mathbb{R}$, and

$$\zeta_{A,\mathbb{N}}(i\tau_m) = \zeta_{A,\mathbb{N}}(i\tau_{m+1}) = \frac{1}{r_1 r_2}.$$

There is another point it_m , $t_m \in \mathbb{R}$ between $i\tau_m$ and $i\tau_{m+1}$ on the imaginary axis for which we have

$$\zeta_{A,\mathbb{N}}(it_m) = \frac{1}{r_1 r_2}.$$

This point is at the intersection of $\Gamma_{m,1}$ with the imaginary axis. One can easily check that the half-closed intervals of the imaginary axis between $i\tau_m$ and it_m and between it_m and $i\tau_{m+1}$ are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the circle centered in origin and having the radius $\frac{1}{r_1 r_2}$, as illustrated by Fig. 3 below.

When

$$\frac{1}{r_1 r_2} < r < \frac{1}{r_1^2 r_2^2},$$

the curves $\Upsilon_{m,r}$ are located in the left half-plane, and still traverse every strip S_m . As

$$\lim_{\sigma \rightarrow -\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = \frac{1}{r_1^2 r_2^2},$$

the curves $\Upsilon_{m,r}$ become unbounded for $r = \frac{1}{r_1^2 r_2^2}$. For $r > \frac{1}{r_1^2 r_2^2}$ they become again closed curves containing both poles

$$\ln r_1 + (2m\pi - \alpha_1)i \text{ and } \ln r_2 + (2m\pi - \alpha_2)i.$$

When they reach the points $-\ln \rho_2 + (2m\pi - \gamma)$, they split each one into two closed curves containing one the pole $\ln r_1 + (2m\pi - \alpha_1)i$ and the other the pole $\ln r_2 + (2m\pi - \alpha_2)i$. When $r \rightarrow +\infty$ these curves squeeze to the respective poles. A dynamic of the variation of these curves would nicely illustrate this phenomenon. We notice that every curve $\Upsilon_{m,r}$ intersects consecutively components of the pre-image of the positive real half-axis and the negative real half-axis, since the circles C_r do the same with the two half axes. This simple topological fact, which we coined in other papers as *the color alternation rule*, is important for describing the global mapping properties of Dirichlet functions. It explains the necessity of the existence of two types of curves $\Gamma_{m,0}$ and $\Gamma_{m,k}$, $k \neq 0$. However, it does not explain the position of these curves, one with respect to the other. Certainly, this position has something to do with the relative position of the zeros z_k of $B(z)$. Indeed, computer experimentation reveals this fact; see Fig. 4a and Fig. 5a.

We need to remember that any analytic function's branch points are located on the boundaries of its fundamental domains. Therefore,

$$-\ln \rho_k + (2m\pi - \gamma)i, \quad k = 1, 2$$

should be on those unknown boundaries. One way to obtain them is the following. We have seen

that the components of the pre-image of C_r meet simultaneously at

$$-\ln \rho_1 + (2m\pi - \gamma)i,$$

which means that for all $m \in \mathbb{Z}$, we have

$$\zeta_{A,\mathbb{N}}(-\ln \rho_1 + (2m\pi - \gamma)i) = w_1,$$

for some $w_1 \in \mathbb{C}$. Let I_1 be the segment connecting w_1 with the point $w = 1$. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of I_1 is formed with unbounded curves $\Psi_{m,1}$ starting at the intersection point $s_{m,1}$ of $\Upsilon_{m,1}$ and $\Gamma_{m,1}$ and passing through

$$-\ln \rho_1 + (2m\pi - \gamma)i.$$

To every point $s = \sigma + it$ on $\Psi_{m,1}$ between $s_{m,1}$ and $-\ln \rho_1 + (2m\pi - \gamma)i$ corresponds a point

$$s' = \sigma' + it'$$

on $\Psi_{m,1}$ with

$$\sigma' > -\ln \rho_1 \text{ such that } \zeta_{A,\mathbb{N}}(s) = \zeta_{A,\mathbb{N}}(s')$$

and when $\sigma \rightarrow 1$, we have $\sigma' \rightarrow +\infty$. This shows that, indeed, every curve $\Psi_{m,1}$ is unbounded, and mapped two-to-one by $\zeta_{A,\mathbb{N}}(s)$ onto I_1 . Now we can apply the conformal mapping theorem [19] and conclude that each one of the two domains bounded by $\Psi_{m,1}$ and $\Upsilon_{m,1}$ is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the unit disk with the slit I_1 . By the same theorem, each one of the two domains bounded, one by the imaginary axis and the curves $\Gamma_{m,1}$, $\Upsilon_{m,1}$, Γ'_m and the other one by the imaginary axis, and the curves $\Gamma_{m,1}$, $\Upsilon_{m,1}$, and Γ'_{m+1} are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the ring domain

$$1 < |w| < \frac{1}{r_1 r_2}$$

with a slit alongside the interval $(1, \frac{1}{r_1 r_2})$ of the real axis. Analogously, each one of the domains bounded one by the imaginary axis, and the curves $\Gamma_{m,1}$, $\Upsilon_{m,r}$, Γ'_m and the other by the imaginary axis and the curves $\Gamma_{m,1}$, $\Upsilon_{m,r}$, Γ'_{m+1} , where $r = \frac{1}{r_1^2 r_2^2}$, is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the ring domain

$$\frac{1}{r_1 r_2} < |w| < \frac{1}{r_1^2 r_2^2}$$

with a slit alongside the interval $(\frac{1}{r_1 r_2}, \frac{1}{r_1^2 r_2^2})$ of the real axis.

Finally, the two domains bounded by the preimage of the positive real half-axis, $\Upsilon_{m,r}$, for $r = \frac{1}{r_1^2 r_2^2}$

and $\Psi_{m,2}$ are conformally mapped by $\zeta_{A,\mathbb{N}}(s)$ onto the exterior of the circle $|w| = \frac{1}{r_1^2 r_2^2}$ with the slit I_2 alongside a ray starting from $\zeta_{A,\mathbb{N}}(\zeta_2)$ and the interval $(\frac{1}{r_1^2 r_2^2}, \infty)$ of the real axis.

To sum up, we get the following result.

Theorem 4. Every strip S_m is divided by the curves $\Psi_{m,1}$, $\Psi_{m,2}$ and $\Gamma_{m,1}$ into two sub strips $\Omega_{m,1}$ and $\Omega_{m,2}$, which are mapped each one conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit L alongside the interval $(1, +\infty)$ of the real axis and the intervals I_1 and I_2 . These are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$.

Fig. 3 below portrays the fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, as well as their conformal mapping onto the w -plane with some slits. We have taken

$$z_1 = 0.06 + 0.05i,$$

$$\alpha_1 = \arctan \frac{5}{6} = 39.8056^\circ,$$

$$z_2 = -0.1 + 0.02i,$$

$$\alpha_2 = -\arctan 0.2 = -11.310^\circ.$$

Correspondingly, we have

$$r_1 = 0.0781025,$$

$$r_2 = 0.10198,$$

$$\zeta_1 = -12.4608 + 21.5989i,$$

$$\zeta_2 = -0.0200404 + 0.034737i,$$

$$\rho_1 = 24.9356,$$

$$\rho_2 = 0.0401033,$$

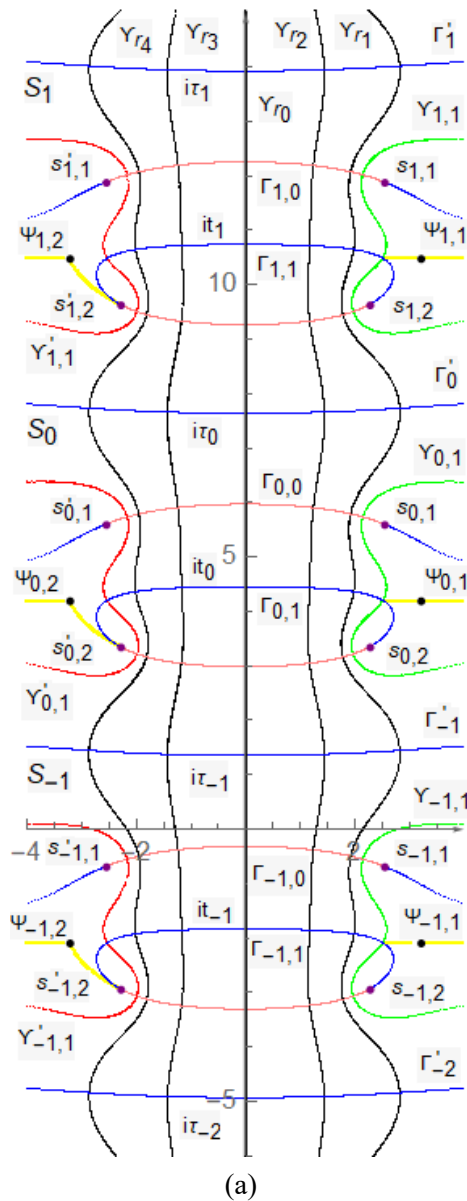
$$\gamma = 119.98^\circ.$$

We notice that the value of γ is in the range predicted by Theorem 2.

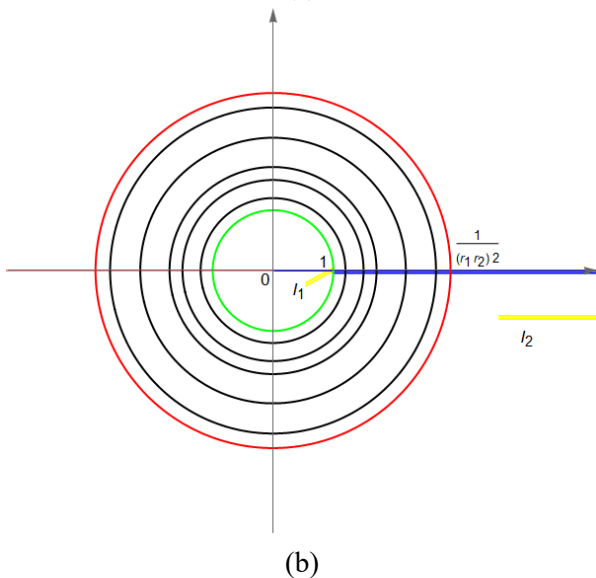
We have taken three strips, S_1, S_0 and S_{-1} to illustrate the periodicity of the function $\zeta_{A,\mathbb{N}}(s)$.

Let us notice that since the mapping is bijective in every fundamental domain $\Omega_{m,k}$, the function $\varphi_{m,k}$ which is $\zeta_{A,\mathbb{N}}(s)$ restricted to $\Omega_{m,k}$ has an inverse function $\varphi_{m,k}^{-1}$, which maps conformally the plane w with the slit L onto $\Omega_{m,k}$. Then, $\varphi_{m,2}^{-1} \circ \varphi_{m,1}$ is a conformal mapping of $\Omega_{m,1}$ onto $\Omega_{m,2}$. We can extend by continuity this mapping to the boundaries of the domains and by the Riemann-Caratheodory theorem, the boundaries are mapped one to the other. It can be easily seen that the zero of $\zeta_{A,\mathbb{N}}(s)$ from $\Omega_{m,1}$ is mapped into the zero of $\zeta_{A,\mathbb{N}}(s)$ from $\Omega_{m,2}$. The same is true for the poles. Moreover, every curve $\Upsilon_{m,r}$, $r < r_0$ from $\Omega_{m,1}$ is mapped one to one onto the corresponding curve from $\Omega_{m,2}$. The curve $\Upsilon_{m,1} \cap \Omega_{m,1}$ is mapped one to one onto $\Upsilon_{m,1} \cap \Omega_{m,2}$ such that

$$\varphi_{m,2}^{-1} \circ \varphi_{m,1}(s_{m,1}) = \infty$$



(a)



(b)

Figure 3: The fundamental domains of $\zeta_{A,\mathbb{N}}(s)$ and their conformal mapping onto the w -plane with some slits

and

$$\lim_{\sigma \rightarrow +\infty} \varphi_{m,2}^{-1} \circ \varphi_{m,1}(\sigma + it) = s_{m,1}$$

when $\sigma + it \in \Upsilon_{m,1}$. The curve $\Psi_{m,1}$ is mapped one to one by $\varphi_{m,2}^{-1} \circ \varphi_{m,1}$ onto itself having the fixed point $-\ln \rho_1 + (2m\pi - \gamma)i$ and the curve $\Psi_{m,2}$ is mapped one to one by $\varphi_{m,2}^{-1} \circ \varphi_{m,1}$ onto itself, having the fixed point $-\ln \rho_2 + (2m\pi - \gamma)i$. The curve Γ'_m is mapped one to one by $\varphi_{m,2}^{-1} \circ \varphi_{m,1}$ onto the arc of $\Gamma_{m,1}$ between $s_{m,1}$ and $s_{m,2}$ such that

$$\lim_{\sigma \rightarrow +\infty} \varphi_{m,2}^{-1} \circ \varphi_{m,1}(\sigma + it) = s_{m,1},$$

where $\sigma + it \in \Gamma'_m$ and

$$\lim_{\sigma \rightarrow -\infty} \varphi_{m,2}^{-1} \circ \varphi_{m,1}(\sigma + it) = s_{m,2}$$

where $\sigma + it \in \Gamma'_m$. Also, the arc of $\Gamma_{m,1}$ between $s_{m,1}$ and $s_{m,2}$ is mapped one to one by $\varphi_{m,2}^{-1} \circ \varphi_{m,1}$ onto the curve $\Gamma'_{m+1,1}$. This shows that the function $\varphi_{m+1,1}^{-1} \circ \varphi_{m,1}$ maps Γ'_m onto $\Gamma'_{m+1,1}$, $\Upsilon_{m,1}$ onto $\Upsilon_{m+1,1}$, etc., and one can easily show that it is a translation of $\Omega_{m,1}$ into $\Omega_{m+1,1}$ of vector $2\pi i$. This implies that every strip S_m can be translated into the strip S_{m+1} such that the components of the pre-image of the real axis, in them, are matched by translation. The same is true for the components of the pre-image of circles $|w| = r$ for any r . Thus, we can formulate the following result.

Theorem 5. The components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the complex plane with the slit L are identical in the sense that they match by a translation of one into the other. The components of the same type of the pre-image of the real axis, and of circles centered at the origin coincide by the translation. The function $\zeta_{A,\mathbb{N}}(s)$ is periodic of period $2\pi i$.

3 The Case of Dirichlet Functions defined by Blaschke Products of Degree 3

In this section, we are looking for properties of Dirichlet functions defined by the third-degree Blaschke products similar to those from the previous section and which can be generalized to Dirichlet functions defined by Blaschke products of arbitrary degree. These properties concern the pre-image of the real axis and the pre-images of some circles centered at the origin, the zeros of the derivative and the fundamental domains. We illustrate them by

computer-generated graphics which reveal a variety of possible situations.

Let

$$B(z) = \frac{z - z_1}{1 - \bar{z}_1 z} \frac{z - z_2}{1 - \bar{z}_2 z} \frac{z - z_3}{1 - \bar{z}_3 z} \quad (9)$$

be a third-degree Blaschke product, where

$$z_k = r_k e^{i\alpha_k}, 0 < r_k < 1, k = 1, 2, 3.$$

The function $B(z)$ is an analytic function in the complex plane, except for the poles $\frac{1}{\bar{z}_k}$, $k = 1, 2, 3$. It can be written under the form

$$B(z) = \frac{z^3 - s_1 z^2 + s_2 z - s_3}{1 - \bar{s}_1 z + \bar{s}_2 z^2 - \bar{s}_3 z^3} \quad (10)$$

where

$$\begin{aligned} s_1 &= z_1 + z_2 + z_3, \\ s_2 &= z_1 z_2 + z_1 z_3 + z_2 z_3, \\ s_3 &= z_1 z_2 z_3. \end{aligned}$$

The equation

$$\begin{aligned} (3z^2 - 2s_1 z + s_2)(1 - \bar{s}_1 z + \bar{s}_2 z^2 - \bar{s}_3 z^3) &= \\ = (-3\bar{s}_3 z^2 + 2\bar{s}_2 z - \bar{s}_1)(z^3 - s_1 z^2 + s_2 z - s_3) & \quad (11) \end{aligned}$$

gives the zeros of $B'(z)$ and is also equivalent to

$$h(z) = az^4 + bz^3 + \alpha z^2 + \bar{b}z + \bar{a} = 0 \quad (12)$$

where

$$\begin{aligned} a &= \bar{s}_2 - s_1 \bar{s}_3, \\ b &= -2(\bar{s}_1 - s_2 \bar{s}_3), \\ \alpha &= |s_1|^2 - |s_2|^2 + 3(1 - |s_3|^2). \end{aligned}$$

One can use Ferrari's method to solve this equation, which will provide four roots, two inside the unit disk and two outside it.

The function

$$\zeta_{A,\mathbb{N}}(s) = -\frac{1}{z_1 z_2 z_3} B(e^{-s}) \quad (13)$$

is a normalized Dirichlet function having the zeros

$$-\ln r_k + (2m\pi - \alpha_k)i$$

and the poles

$$\ln r_k + (2m\pi - \alpha_k)i, k = 1, 2, 3, m \in \mathbb{Z}.$$

It can be easily checked that if $h(z_0) = 0$, then we have also $h\left(\frac{1}{\bar{z}_0}\right) = 0$, thus, the four solutions of the equation (12) are two by two symmetric with respect to the unit circle.

This implies that the branch points of $\zeta_{A,\mathbb{N}}(s)$ are two by two symmetric with respect to the imaginary axis.

Theorem 6. There is a partition of the complex plane into infinitely many horizontal strips S_m with boundaries which are components Γ'_m of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the interval $(1, +\infty)$ of the real axis. Every strip S_m contains three zeros in the right half-plane, three poles symmetric with those zeros with respect to the imaginary axis, and four branch points of $\zeta_{A,\mathbb{N}}(s)$, two by two symmetric with respect to the imaginary axis. Every strip S_m contains also a unique component $\Upsilon_{m,1}$ of the pre-image of the unit circle, and a unique component $\Upsilon_{m,2}$ of the pre-image of the circle

$$|w| = \frac{1}{r_1^2 r_2^2 r_3^2},$$

which are parabola-like curves with branches tending asymptotically to the curves Γ'_m and Γ'_{m+1} and containing first the zeros and two branch points of $\zeta_{A,\mathbb{N}}(s)$ located in the right half-plane and the second the poles and two branch points located in the left half-plane. Every strip S_m can be divided into three sub-strips, which are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$.

Proof: We have

$$\lim_{\sigma \rightarrow +\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1,$$

hence the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of $w = 1$ contains the point ∞ on the Riemann sphere, which implies that the pre-image of the unit circle should also contain that point; in other words, a component Υ of the pre-image of the unit circle must be unbounded.

Let us look for the pre-image of a ray L_α in the w -plane making a small angle α with the positive real half-axis. Let Γ_α be a component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of L_α and

$$s_\alpha = \sigma_\alpha + it_\alpha \in \Gamma_\alpha \cap \Upsilon.$$

For $s = \sigma + it \in \Gamma_\alpha$, we have that $\sigma \rightarrow +\infty$ and Γ_α is the union of an arc Γ''_α interior to Υ and an unbounded curve Γ'_α exterior to Υ , on which $\sigma \rightarrow +\infty$. When $\alpha \rightarrow 0$, we have $\sigma_\alpha \rightarrow +\infty$ and then Γ''_α becomes unbounded, and Γ'_α extends for $\sigma = +\infty$ to $\sigma = -\infty$. For a positive angle α the curve Γ'_α is above Υ , and for a negative angle α , it is below Υ . Thus, the curve Υ is contained in an infinite strip S_m formed by two curves, which we denote now Γ'_m and Γ'_{m+1} , which are components of the pre-image of the interval $(1, +\infty)$ of the real axis.

Let us show that there are infinitely many strips S_m containing each one an unbounded component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle. Indeed, suppose that there is no other strip above S_m . Let us take two points $s_1 \in S_m$ and s_2 above S_m and let

$$w_k = \zeta_{A,\mathbb{N}}(s_k), \quad k = 1, 2.$$

We can connect w_1 and w_2 by an arc η , which does not intersect the positive real half-axis. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of η should contain an arc γ connecting s_1 and s_2 which does not intersect Γ'_{m+1} . Yet, this is impossible. Thus, there are infinitely many strips S_k above S_m . We find similarly that there are infinitely many such strips below S_m . We count them from $-\infty$ to $+\infty$ such that $0 \in S_0$. They cover the whole complex plane.

It is obvious that every strip S_m contains a unique unbounded component $\Upsilon_{m,1}$ of the pre-image of the unit circle, which in turn contains at least one zero of $\zeta_{A,\mathbb{N}}(s)$. Also, it contains a unique component $\Upsilon_{m,r}$ of the pre-image of the circle C_r with $r = \frac{1}{r_1^2 r_2^2 r_3^2}$, which in turn contains at least one pole of $\zeta_{A,\mathbb{N}}(s)$, see Fig. 4a and Fig. 5a. To obtain the fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, we need the segments of line I_1 and I_2 connecting $w = 1$ with the images by $\zeta_{A,\mathbb{N}}(s)$ of the branch points from the right half-plane and also the rays I'_1 and I'_2 starting at the images by $\zeta_{A,\mathbb{N}}(s)$ of the branch points from the left half-plane. The pre-images of I_1, I_2, I'_1, I'_2 are infinite curves $\Upsilon_{m,1}, \Upsilon_{m,2}, \Upsilon'_{m,1}, \Upsilon'_{m,2}$, which are mapped two to one by $\zeta_{A,\mathbb{N}}(s)$ onto I_1, I_2, I'_1, I'_2 respectively. Together with $\Gamma_{m,1}$ and $\Gamma_{m,-1}$, they divide the strip S_m into three sub-strips $\Omega_k, k = 1, 2, 3$, which are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$. It maps conformally these domains onto the complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis and some of the intervals I_1, I_2, I'_1, I'_2 . \square

Fig. 4a and Fig. 5a below illustrate these domains and their conformal mapping.

We have taken in Fig. 4 the following

$$\begin{aligned} z_1 &= 0.06 + 0.05i, \\ z_2 &= -0.1 + 0.02i, \\ z_3 &= 0.3 - 0.25i. \end{aligned}$$

Correspondingly, we have

$$\begin{aligned} r_1 &= 0.0781025, \\ r_2 &= 0.10198, \\ r_3 &= 0.390512, \\ \zeta_1 &= -16.7985 + 21.0894i, \\ \zeta_2 &= -0.0231081 + 0.0290108i, \\ \zeta_3 &= 0.199866 - 0.152249i, \\ \zeta_4 &= 3.16613 - 2.41182i, \\ \rho_1 &= 26.962, \\ \rho_2 &= 0.0370892, \\ \rho_3 &= 0.25125, \\ \rho_4 &= 3.9801, \\ \zeta_{A,\mathbb{N}}(\zeta_1) &= 103363 - 0.0202024i, \\ \zeta_{A,\mathbb{N}}(\zeta_2) &= 279.689 + 208.177i, \\ \zeta_{A,\mathbb{N}}(\zeta_3) &= 48.1882 + 159.492i, \\ \zeta_{A,\mathbb{N}}(\zeta_4) &= 0.881684 - 0.13i, \\ \frac{1}{r_1 r_2 r_3} &= 321.502, \\ \frac{1}{(r_1 r_2 r_3)^2} &= 103363, \\ |\zeta_{A,\mathbb{N}}(\zeta_1)| &= 103363, \\ |\zeta_{A,\mathbb{N}}(\zeta_2)| &= 348.66, \\ |\zeta_{A,\mathbb{N}}(\zeta_3)| &= 166.613, \\ |\zeta_{A,\mathbb{N}}(\zeta_4)| &= 0.891216. \end{aligned}$$

The numbers for Fig. 5 are the following

$$\begin{aligned} z_1 &= 0.3 + 0.2i, \\ z_2 &= -0.1 + 0.02i, \\ z_3 &= 0.3 - 0.25i. \end{aligned}$$

Correspondingly, we have

$$\begin{aligned}
 r_1 &= 0.360555, \\
 r_2 &= 0.10198, \\
 r_3 &= 0.390512, \\
 \zeta_1 &= 0.135479 + 0.0478703i, \\
 \zeta_2 &= 0.198254 - 0.0693903i, \\
 \zeta_3 &= 4.49355 - 1.57277i, \\
 \zeta_4 &= 6.56195 + 2.3186i, \\
 \rho_1 &= 0.143688, \\
 \rho_2 &= 0.210047, \\
 \rho_3 &= 4.76084, \\
 \rho_4 &= 6.95953, \\
 \zeta_{A,\mathbb{N}}(\zeta_1) &= 39.6551 + 8.34811i, \\
 \zeta_{A,\mathbb{N}}(\zeta_2) &= 24.6944 + 19.7634i, \\
 \zeta_{A,\mathbb{N}}(\zeta_3) &= 0.983065 + 0.0661221i, \\
 \zeta_{A,\mathbb{N}}(\zeta_4) &= 0.996326 - 0.00789193i, \\
 \frac{1}{r_1 r_2 r_3} &= 69.6429, \\
 \frac{1}{(r_1 r_2 r_3)^2} &= 4850.13, \\
 |\zeta_{A,\mathbb{N}}(\zeta_1)| &= 40.5243, \\
 |\zeta_{A,\mathbb{N}}(\zeta_2)| &= 31.6292, \\
 |\zeta_{A,\mathbb{N}}(\zeta_3)| &= 0.985286, \\
 |\zeta_{A,\mathbb{N}}(\zeta_4)| &= 0.996357.
 \end{aligned}$$

The purpose of taking two different sets of data for a Dirichlet function generated by a third-degree Blaschke product was to illustrate the diversity of the positions of the zeros of $\zeta'_{A,\mathbb{N}}(s)$ with respect to the components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis and as a consequence of the slits in the image plane. It can be noticed that in Fig. 4, there is one unbounded component of the pre-image of the unit circle and a bounded one, while in Fig. 5, there is still one unbounded component of the pre-image of the unit circle and two bounded ones. The same is true for the pre-images of the circle $|w| = \frac{1}{|B(0)|^2}$.

As in the case of a Dirichlet function defined by a second-degree Blaschke product, every fundamental domain $\Omega_{m,k}$ is mapped conformally onto the domain $\Omega_{m+1,k}$ by $\varphi_{m+1,k}^{-1} \circ \varphi_{m,k}(s)$ where $\varphi_{m+1,k}(s)$ is the restriction of $\zeta_{A,\mathbb{N}}(s)$ to $\Omega_{m+1,k}$. One can easily see that this mapping is a translation of vector $2\pi i$ hence every strip S_m can be translated into the strip S_{m+1} such that the components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of lines and circles match by that translation. The function $\zeta_{A,\mathbb{N}}(s)$ is periodic of period $2\pi i$.

We notice that for this particular choice of $z_1, z_2,$ and z_3 , the pre-image of the unit circle has one unbounded component $l_{m,2}$ and a bounded one $l_{m,1}$

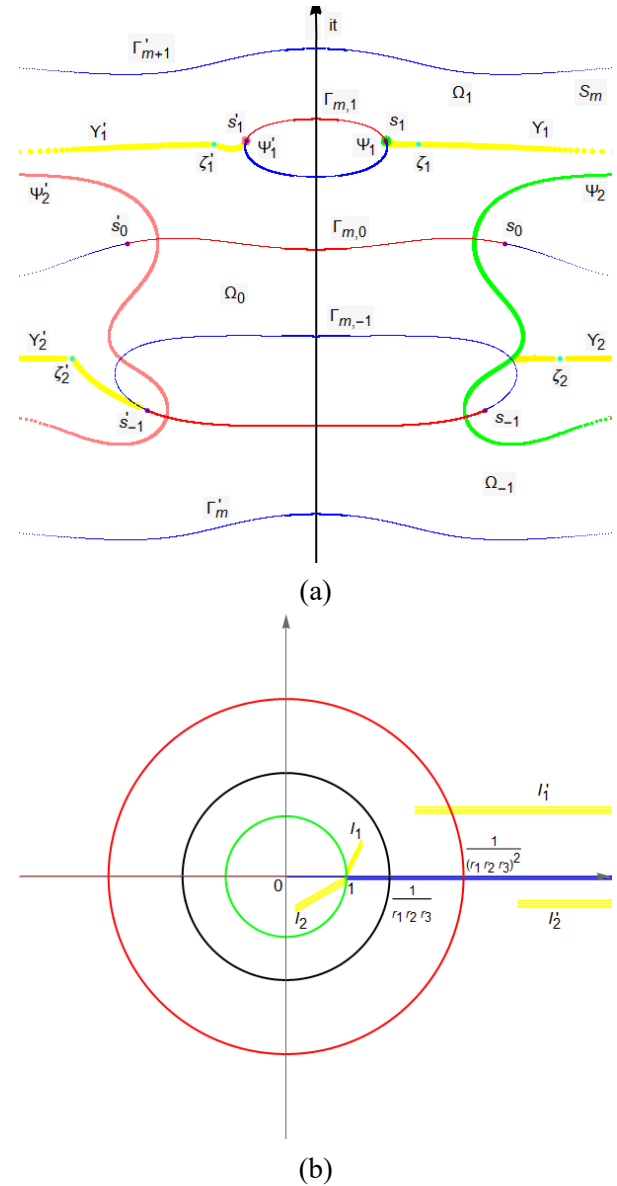


Figure 4: The fundamental domains of the Dirichlet function defined by a third-degree Blaschke product and their conformal mapping. The pre-image of the unit circle has one unbounded component and a bounded one in every strip S_m .

in every strip S_m . The component $\iota_{m,1}$ is very small in Fig. 4. The same is true for the pre-image of the circle of radius $\frac{1}{|B(0)|^2}$.

4 The Case of a Dirichlet Function Defined by an Arbitrary Finite Blaschke Product

Let

$$B(z) = \frac{z_1 - z}{1 - \bar{z}_1 z} \frac{z_2 - z}{1 - \bar{z}_2 z} \dots \frac{z_n - z}{1 - \bar{z}_n z} \quad (14)$$

be a Blaschke product of degree n , which does not cancel at the origin, and the zeros $z_k = r_k e^{i\alpha_k}$ are counted with multiplicities such that

$$0 < r_1 \leq r_2 \leq \dots \leq r_n < 1.$$

The function $B(z)$ is analytic in the whole complex plane except for the poles

$$\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}, \dots, \frac{1}{\bar{z}_n}.$$

We have $B(0) = z_1 z_2 \dots z_n$ and

$$\lim_{z \rightarrow \infty} B(z) = \frac{1}{\bar{z}_1} \frac{1}{\bar{z}_2} \dots \frac{1}{\bar{z}_n} = \frac{1}{B(0)}.$$

The function $B(z)$ admits a Taylor expansion

$$B(z) = \sum_{k=0}^{\infty} \frac{B^{(k)}(0)}{k!} z^k,$$

which is convergent in the disk $|z| < \frac{1}{r_n}$. The series

$$\zeta_{A,\mathbb{N}}(s) = \frac{1}{z_1 z_2 \dots z_n} \sum_{k=0}^{\infty} \frac{B^{(k)}(0)}{k!} e^{-ks} \quad (15)$$

is a normalized Dirichlet series having the abscissa of convergence $\sigma_c = \ln r_n < 0$ and which admits an analytic continuation in the whole complex plane except for the poles

$$\ln r_k + (2m\pi - \alpha_k)i, k = 1, 2, \dots, n, m \in \mathbb{Z}.$$

We keep the notation $\zeta_{A,\mathbb{N}}(s)$ for the function obtained by analytic continuation of the series (15).

When some of the zeros of $\zeta_{A,\mathbb{N}}(s)$ are multiple zeros, the geometry of the conformal mapping by $\zeta_{A,\mathbb{N}}(s)$ in their neighborhood is similar to that described in [18].

As in the case of $n = 2$ and $n = 3$, there is a partition of the complex plane into infinitely many

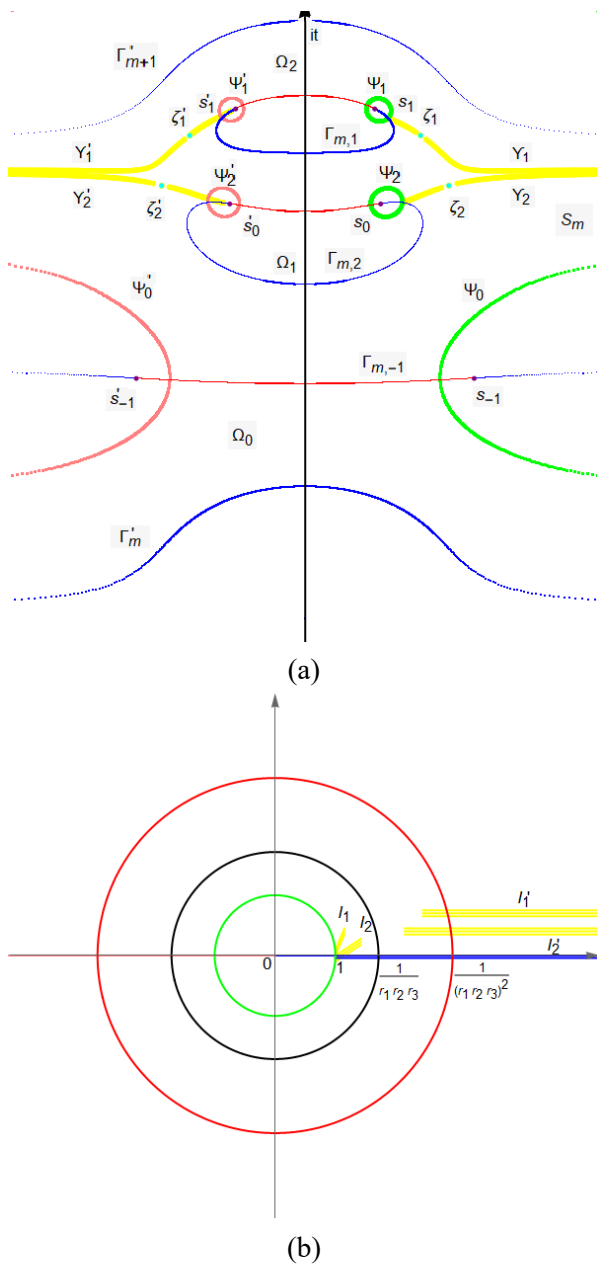


Figure 5: The fundamental domains of the Dirichlet function defined by a third-degree Blaschke product and its conformal mapping. The pre-image of the unit circle has one unbounded component and two bounded ones in every strip S_m .

horizontal strips S_m , $m \in \mathbb{Z}$ bounded each one by consecutive components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the interval $(1, +\infty)$ of the real axis.

A straightforward computation shows that the equation $B'(z) = 0$ is a $2n - 2$ degree equation whose $n - 1$ roots are inside the unit circle and $n - 1$ are outside the unit circle. They are two by two symmetric with respect to the unit circle. To the roots of $B'(z)$ inside the unit circle correspond roots of $\zeta'_{A,\mathbb{N}}(s)$ on the right half-plane and to those outside the unit circle correspond roots of $\zeta'_{A,\mathbb{N}}(s)$ on the left half-plane. The image by $w = \zeta_{A,\mathbb{N}}(s)$ of the roots of $\zeta'_{A,\mathbb{N}}(s)$ from the right half-plane and the point $w = 1$ determine intervals I_k whose pre-image by $\zeta_{A,\mathbb{N}}(s)$ is formed with infinite curves $\Psi_{m,k}$, which are mapped two-to-one by $\zeta_{A,\mathbb{N}}(s)$ onto I_k . Similarly, the image by $w = \zeta_{A,\mathbb{N}}(s)$ of the roots of $\zeta'_{A,\mathbb{N}}(s)$ from the right half-plane determine rays I'_k whose pre-image by $\zeta_{A,\mathbb{N}}(s)$ are infinite curves $\Psi'_{m,k}$ which are mapped two to one by $\zeta_{A,\mathbb{N}}(s)$ onto I'_k .

The pre-image of the real axis by $\zeta_{A,\mathbb{N}}(s)$ is formed with components described in Section 1, and again, we have strips S_m covering the whole complex plane. Every strip S_m contains $n - 1$ curves $\Upsilon_{m,k}$, and $n - 1$ curves $\Upsilon'_{m,k}$. Together with the components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis, they divide the strip S_m into n fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, which are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis and alongside the intervals I_k and I'_k .

Theorem 7. To the solutions of the equation $B(z) = 1$ correspond $n - 1$ points $s_{m,k}$, $k = 2, 3, \dots, n$, counted counterclockwise, in every strip S_m on the component $\Upsilon_{m,1}$ included in S_m of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle determining on that curve $n - 2$ arcs and two unbounded curves, such that every arc with one end removed and every unbounded curve is mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the unit circle.

Proof: There are n components $\Gamma_{m,k}$ of the pre-image of the real axis included in S_m , corresponding to the n zeros z_k of $B(z)$. They are disjoint curves if z_k are simple zeros and if z_k is multiple of order p , then p such curves meet at the point z_k . Each one of them, except $\Gamma_{m,1}$, intersects twice the pre-image of the unit circle at the points corresponding respectively to $w = 1$ and $w = -1$. The curve $\Gamma_{m,1}$ is intersected just once at the point corresponding to $w = -1$, since the point on $\Gamma_{m,1}$ corresponding to $w = 1$ is at infinity. If we denote by $s_{m,k}$ the intersection points corresponding to $w = 1$ and take into account the fact that $\zeta_{A,\mathbb{N}}(s)$ is injective

on each one of those arcs and unbounded curves, the affirmation of the theorem becomes obvious. The property to be injective of $\zeta_{A,\mathbb{N}}(s)$ is assured by the fact that no zero of $B'(z)$ is located on the unit circle, thus no zero of $\zeta'_{A,\mathbb{N}}(s)$ is located on $\Upsilon_{m,1}$. \square

Let us notice that the affirmation of Theorem 7 remains valid if we replace $\Upsilon_{m,1}$ by an arbitrary curve $\Upsilon_{m,r}$ belonging to the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle $|w| = r$ with the condition that the arcs are those of $\Upsilon_{m,r}$, and we disregard the fact that $\Upsilon_{m,r}$ can be disconnected for some values of r . Thus, for any $r > 0$, the arcs with one end removed determined on $\Upsilon_{m,r}$ by the intersection with the pre-image of the positive real half-axis, are mapped by $w = \zeta_{A,\mathbb{N}}(s)$ one to one onto the circle $|w| = r$. Varying r from 0 to infinity, it can be seen how the image of S_m covers n times the (w) -complex plane.

Based on these facts, we can state the following result.

Theorem 8. Any Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ defined by an n degree Blaschke product (14) allows a partition of the complex plane into infinitely many strips S_m , which can be moved one onto the other by a translation of vector $2j\pi$, $j \in \mathbb{Z}$. Every strip S_m can be partitioned into n sets whose interior are fundamental domains Ω_k of $\zeta_{A,\mathbb{N}}(s)$. The domains Ω_k are mapped conformally by $w = \zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis and some intervals I_k connecting $w = 1$ with $w = \zeta_{A,\mathbb{N}}(\zeta_k)$ and some rays I'_k starting at $w = \zeta_{A,\mathbb{N}}(\zeta'_k)$.

This analysis allows us to extrapolate for Dirichlet functions defined by infinite Blaschke products.

5 Dirichlet Functions defined by Infinite Blaschke Products

We deal here with Dirichlet functions defined by Blaschke products of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z} \quad (16)$$

where

$$z_n = r_n e^{i\alpha_n}, \quad 0 < r_n < 1$$

and

$$\sum_{n=1}^{\infty} (1 - r_n) < \infty.$$

This last inequality, known as the Blaschke condition, is a necessary and sufficient condition for the product

(16) to converge in the disk $|z| < 1$. Then, $\prod_{n=1}^{\infty} r_n$ makes sense, and it is $B(0)$. It also shows that for a convergent Blaschke product, all cluster points of zeros need to be located on the unit circle. Let us denote by E the set of cluster points of zeros, which is also the set of cluster points of poles of $B(z)$. We have shown in [18] that if $\mathbb{C} \setminus E$ includes an arc of the unit circle, then the product $B(z)$ can be continued analytically to a meromorphic function in $\mathbb{C} \setminus E$ having the poles $\frac{1}{\bar{z}_n}$.

We continue to use the notation $B(z)$ for this function. For every $z \in \mathbb{C} \setminus E$, we have

$$B(z) = \frac{1}{B\left(\frac{1}{\bar{z}}\right)}. \quad (17)$$

The function $B(z)$ is analytic in a neighborhood of $z = 0$, and therefore it coincides with its Taylor expansion

$$B(z) = \sum_{k=0}^{\infty} \frac{B^{(k)}(0)}{k!} z^k,$$

which is convergent for $|z| < 1$. This allows us to define, as previously, the normalized Dirichlet series

$$\zeta_{A,\mathbb{N}}(s) = \sum_{k=0}^{\infty} a_k e^{-ks} \quad (18)$$

where

$$a_k = \frac{1}{B(0)} \frac{B^{(k)}(0)}{k!},$$

$a_0 = 1$ and $s = \sigma + it$.

Every point of E is an essential non-isolated singular point of $B(z)$ since it is the limit of a sequence of poles of $B(z)$. If $e^{i\theta} \in E$, then the equation $e^{-\sigma-it} = e^{i\theta}$ with the solution $\sigma = 0$ and $t_m = 2m\pi - \theta$, $m \in \mathbb{Z}$ shows that $it_m = (2m\pi - \theta)i$ are essential non isolated singular points of $\zeta_{A,\mathbb{N}}(s)$. Indeed, it_m is the limit of a sequence of poles of the form $\ln r_{n_k} + (2m\pi - \alpha_{n_k})i$, as $n_k \rightarrow \infty$.

For the Blaschke product (16), we have $E \neq \emptyset$ since the infinite set $\{z_n\}$ of zeros of $B(z)$ is bounded, and then it must have a cluster point. If $e^{i\theta_0} \in E$ for $\theta_0 \in \mathbb{R}$, we have

$$e^{i\theta_0} = \lim_{n \rightarrow \infty} z_{n_k} = \lim_{n \rightarrow \infty} \frac{1}{\bar{z}_{n_k}},$$

for a subsequence (z_{n_k}) of zeros. We have seen also [18] that $e^{i\theta_0}$ is an isolated point of E , then

$$e^{i\theta_0} = \lim_{k \rightarrow \infty} e^{i\varphi_k},$$

where $B(e^{i\varphi_k}) = 1$.

Since $B(z)$ has infinitely many zeros, $z_n = r_n e^{i\alpha_n}$ and the equation $B(z) = 0$ is equivalent to the equation $\zeta_{A,\mathbb{N}}(s) = 0$, and to every zero $z_n = r_n e^{i\alpha_n}$ of $B(z)$ correspond infinitely many points $s_{n,m}$, solutions of the equations

$$e^{-s} = r_n e^{i\alpha_n} = e^{\ln r_n + i\alpha_n},$$

we get

$$s_{n,m} = -\ln r_n + (2m\pi - \alpha_n)i, \quad m \in \mathbb{Z}.$$

In a similar way, to every pole $z'_n = \frac{1}{r_n} e^{i\alpha_n}$ of $B(z)$ correspond infinitely many points $s'_{n,m}$, solutions of the equations

$$e^{-s} = \frac{1}{r_n} e^{i\alpha_n} = e^{-\ln r_n + i\alpha_n},$$

hence

$$s'_{n,m} = \ln r_n + (2m\pi - \alpha_n)i, \quad m \in \mathbb{Z}$$

These are poles of $\zeta_{A,\mathbb{N}}(s)$ of the same order as the poles z'_n of $B(z)$. Let us denote by E_s the set of cluster points of these poles. Obviously, $s_0 \in E_s$ if and only if $z_0 = e^{-s_0} \in E$.

Theorem 9. When the strip S_m of $\zeta_{A,\mathbb{N}}(s)$ contains an isolated point it_m of E_s , then in every neighborhood of it_m , there are infinitely many fundamental domains of $\zeta_{A,\mathbb{N}}(s)$.

The affirmation of this theorem might look counter-intuitive since the fundamental domains of functions $\zeta_{A,\mathbb{N}}(s)$ we have seen up to now were all infinite strips. However, we should remember that the partition of the complex plane into fundamental domains is not unique. That partition with infinite strips was just a convenient choice. In Fig. 3 and Fig. 4, the domain bounded by some $\Gamma_{m,k}$ is mapped conformally onto the upper or onto the lower half-plane. When a zero is multiple, then the domain bounded by two consecutive components of the pre-image of the positive real half-axis issuing from that zero is a fundamental domain that is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the positive real half-axis. Then, given an arbitrary neighborhood of the essential singular point it_m of $\zeta_{A,\mathbb{N}}(s)$, if a multiple zero and the corresponding multiple pole of $\zeta_{A,\mathbb{N}}(s)$ are in that neighborhood, so can be the respective fundamental domains. Now, the affirmation of the theorem looks plausible.

Proof: As in the case of Dirichlet functions defined by finite Blaschke products, there is a

partition of the complex plane into infinitely many strips S_m bounded by consecutive components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis, components which are mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto the interval $(1, +\infty)$. Again, every strip S_m contains a unique component of the pre-image of the real axis, which is mapped one-to-one by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(-\infty, 1)$ of the real axis and a unique component of the pre-image of the unit circle, which happens to be an unbounded curve.

Let $V \subset S_m$ be an arbitrary neighborhood of it_m . Since it_m is a limit of zeros and poles of $\zeta_{A,\mathbb{N}}(s)$, the neighborhood V contains infinitely many zeros and infinitely many poles of $\zeta_{A,\mathbb{N}}(s)$. For every couple

$$\ln r_n + (2m\pi - \alpha_n)i \text{ and } -\ln r_n + (2m\pi - \alpha_n)i,$$

of a pole and a zero, a component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis is a closed curve $\Gamma_{m,n}$ passing through the two points. If $\ln r_n$ is small enough, we have $\Gamma_{m,n} \subset V$. Since

$$\lim_{n \rightarrow \infty} [\ln r_n + (2m\pi - \alpha_n)i] = it_m,$$

hence

$$\lim_{n \rightarrow \infty} \ln r_n = 0,$$

we can take $\ln r_n$ to be as small as we want. The sequence (r_{n_k}) can always be considered non-decreasing, and then there is n_0 such that $\Gamma_{m,n_k} \subset V$ for $n_k \geq n_0$.

Consecutive components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the positive real half-axis determine on the imaginary axis disjoint intervals which, with one end removed, are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the circle $|w| = \frac{1}{|B(0)|}$. The arcs $\gamma_{m,1}$ and $\gamma_{m,2}$ on the pre-image of the real axis between the imaginary axis, and the two zeros on those components are each one mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto the interval $\left[0, \frac{1}{|B(0)|}\right]$. Then, there must be a branch point $\xi_{m,n}$ and two arcs $\eta_{m,1}$ and $\eta_{m,2}$ connecting $\xi_{m,n}$ with the two zeros, arcs which are mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto a segment I_m ending at $\frac{1}{|B(0)|}$ and located inside the circle $|w| = \frac{1}{|B(0)|}$. We can take $\xi_{m,n}$ such that no other solution of the equation $\zeta_{A,\mathbb{N}}(s) = 0$ belongs to the domain $\Omega'_{m,n}$ bounded by the imaginary axis, and the arcs $\gamma_{m,1}, \gamma_{m,2}, \eta_{m,1}$ and $\eta_{m,2}$. Thus, the domain $\Omega'_{m,n}$ is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the disk $|w| < \frac{1}{|B(0)|}$ with a slit alongside the interval $\left[0, \frac{1}{|B(0)|}\right]$, and alongside the interval I_m .

Similarly, the arcs $\gamma'_{m,1}$ and $\gamma'_{m,2}$ on the pre-image of the real axis between the imaginary axis and the two poles are each one mapped by $\zeta_{A,\mathbb{N}}(s)$ one to

one onto the interval $\left(\frac{1}{|B(0)|}, +\infty\right)$. Then, there must be a branch point $\xi'_{m,n}$ and two arcs $\eta'_{m,1}$ and $\eta'_{m,2}$ connecting $\xi'_{m,n}$ with the two poles which are mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto the ray I'_m starting at $\zeta_{A,\mathbb{N}}(\xi'_{m,n})$. Again, we can choose $\xi'_{m,n}$ such that no other zero of $\zeta'_{A,\mathbb{N}}(s)$ belongs to the domain $\Omega''_{m,n}$ bounded by the imaginary axis and the arcs $\gamma'_{m,1}, \gamma'_{m,2}, \eta'_{m,1}$ and $\eta'_{m,2}$. Thus $\Omega''_{m,n}$ is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the domain exterior to the circle $|w| = \frac{1}{|B(0)|}$ with a slit alongside the interval $\left(\frac{1}{|B(0)|}, +\infty\right)$, and alongside I'_m .

Summing up, the domain $\Omega_{m,n}$ bounded by $\gamma_{m,1}, \gamma_{m,2}, \eta_{m,1}, \eta_{m,2}, \gamma'_{m,1}, \gamma'_{m,2}, \eta'_{m,1}$ and $\eta'_{m,2}$ is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the positive real half-axis and alongside the intervals I_m and I'_m . There is n_0 such that $\Omega_{m,n} \subset V$ for $n \geq n_0$.

Since there are infinitely many disjoint domains $\Omega_{m,n}$ included in V , they accumulate to it_m . \square

Every strip S_m of $\zeta_{A,\mathbb{N}}(s)$ contains infinitely many curves $\Gamma_{m,k}, k \neq 0$, components of the pre-image of the real axis, which are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the real axis, and one component $\Gamma_{m,0}$ which is mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(-\infty, 1)$. Every zero $-\ln r_n + (2m\pi - \alpha_n)i$ of $\zeta_{A,\mathbb{N}}(s)$ belongs to at least one of these components. If the zero is multiple of order p , it belongs to p , such components which meet at that zero and at the corresponding pole. The curves $\Gamma_{m,k}$ are intersected by the pre-image of every circle $|w| = r$ in points $s_{m,k}$ such that the color alternation rule is satisfied.

Theorem 10. The pre-image of any circle

$$|w| = r \neq \frac{1}{|B(0)|}$$

intersects only a finite number of curves $\Gamma_{m,k}$.

Proof: The existence of infinitely many curves $\Gamma_{m,k}$ is guaranteed by the fact that there are infinitely many zeros of $\zeta_{A,\mathbb{N}}(s)$ in every strip S_m and to each zero, counted with multiplicities corresponding to one curve $\Gamma_{m,k}$. If a sequence of intersection points $(s_{m,k})$ of the pre-image of the circle $|w| = r$ with $\Gamma_{m,k}$ were infinite, then it would have an accumulation point $s_0 = \sigma_0 + it_0, \sigma_0 \neq 0$, on the pre-image of the circle $|w| = r$ and $\zeta_{A,\mathbb{N}}(s)$ would take the same value infinitely many times on the arc included in S_m of that pre-image. By the uniqueness theorem of analytic functions, [19], [20], s_0 cannot be interior to that arc. Then, s_0 belongs to Γ'_m or to Γ'_{m+1} . Due to the color alternation rule, to every point $s_{m,k}$ with

$\zeta_{A,\mathbb{N}}(s_{m,k}) = r$ corresponds a point $s'_{m,k}$ on the same component of the pre-image of the circle $|w| = r$ with $\zeta_{A,\mathbb{N}}(s'_{m,k}) = -r$ and the sequence $(s'_{m,k})$ has the same accumulation point s_0 , thus by the continuity of $\zeta_{A,\mathbb{N}}(s)$ we would have

$$\zeta_{A,\mathbb{N}}(s_0) = r \text{ and } \zeta_{A,\mathbb{N}}(s_0) = -r$$

which is absurd. \square

Theorem 11. The strips S_m of $\zeta_{A,\mathbb{N}}(s)$ are identical in the sense that any two of them, S_m and S_{m+p} , $p \in \mathbb{Z}$, can be brought into coincidence by a translation of vector $2p\pi i$. The pre-images of lines and circles match by that translation.

Proof: Every strip S_m can be partitioned into infinitely many sets whose interiors $\Omega_{m,k}$ are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$. Indeed, on every curve $\Gamma_{m,k}$, $k \neq 0$, there is a unique point $s_{m,k}$ such that $\zeta_{A,\mathbb{N}}(s_{m,k}) = 1$. We can use $\xi_{m,k}$ as in Theorem 9 to define the curves $\Psi_{m,k}$. Likewise, on the same $\Gamma_{m,k}$, there is a unique point $s'_{m,k}$ such that

$$\zeta_{A,\mathbb{N}}(s'_{m,k}) = \frac{1}{|B(0)|^2}$$

and we can define the curve $\Psi'_{m,k}$. The domains $\Omega_{m,k}$ bounded by consecutive components of the pre-image of the interval $(1, +\infty)$ and $\Psi_{m,k}$ and $\Psi'_{m,k}$ are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, which are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the interval $(1, +\infty)$ and some segments of line connecting $w = 1$ with $w = \zeta_{A,\mathbb{N}}(\zeta_k)$ and some rays starting at $w = \zeta_{A,\mathbb{N}}(\zeta'_k)$.

The domains $\Omega_{m,k}$ and $\Omega_{m+p,k}$, $p \in \mathbb{Z}$ are identical in the sense that they coincide by a translation of vector $2p\pi i$. Indeed, for every point $s \in \Omega_{m,k} \cap \Upsilon_{m,1}$, the point $T(s) = s + 2p\pi i$ belongs to $\Omega_{m+p,k} \cap \Upsilon_{m+p,1}$, since

$$\zeta_{A,\mathbb{N}}(s) = \zeta_{A,\mathbb{N}}(s + 2p\pi i),$$

hence the function $\varphi_1^{-1} \circ T \circ \varphi_2(s)$, where φ_1 and φ_2 are the restrictions of $\zeta_{A,\mathbb{N}}(s)$ to $\Omega_{m,k}$ and to $\Omega_{m+p,k}$ is the identity on the continuum $\Omega_{m,k} \cap \Upsilon_{m,1}$. By the uniqueness theorem of analytic functions, it should be the identity in $\Omega_{m,k}$, i.e., $T \circ \varphi_2(s) = \varphi_1(s)$ for every $s \in \Omega_{m,k}$, where T is a translation of vector $2p\pi i$ of $\Omega_{m,k}$ into $\Omega_{m+p,k}$. The translation T is the same for every k therefore we have a translation of S_m onto S_{m+p} of vector $2p\pi i$. \square

This theorem is important because we don't need to study $\zeta_{A,\mathbb{N}}(s)$ in the whole plane, but only in a particular strip, S_m , since all the features of this function in other strips are the same.

6 Denseness Theorem for Dirichlet Functions defined by Infinite Blaschke Products

We have proved in [18] a denseness theorem for analytic functions with natural boundary. Now, we can rephrase that theorem for Dirichlet functions defined by infinite Blaschke products. As stated in the previous section, it is enough to deal with the restriction of such a function to one of its strips.

Theorem 12. Let $w = \zeta_{A,\mathbb{N}}(s)$ be a Dirichlet function defined by an infinite Blaschke product $B(z)$ and let (w_k) be a sequence of points spread throughout the disk $w = \frac{1}{|B(0)|}$ such that the distance between two neighboring points is less than an arbitrarily small $\epsilon > 0$. Then, for any strip S_m of $\zeta_{A,\mathbb{N}}(s)$ and for every $k \in \mathbb{N}$, there is $\sigma > 0$ such that the image by $\zeta_{A,\mathbb{N}}(s)$ of the interval

$$I_\sigma = \{s \mid s = \sigma + it \in S_m\}$$

passes at a distance of less than ϵ for every w_k .

Proof: By Theorem 9, we can choose $\Omega_n \subset S_m$ such that they accumulate to the essential singular points of $\zeta_{A,\mathbb{N}}(s)$ from S_m . Then, I_σ will intersect only a finite number of these fundamental domains. Let us draw a mesh in the (w) -plane with lines parallel to the coordinate axes at a distance equal to $\frac{\epsilon}{3}$ between consecutive ones. We count the n_1 eyes of the mesh which intersect the circle $|w| = \frac{1}{|B(0)|}$ starting from the eye containing the point $\frac{1}{|B(0)|}$ and moving counterclockwise. Then, we count in a similar way the eyes adjacent to the first ones and included in the disk $|w| < \frac{1}{|B(0)|}$, etc. We get n sets D_j covering the disk $|w| \leq \frac{1}{|B(0)|}$, whose components of the pre-image are included one in every fundamental domain of $\zeta_{A,\mathbb{N}}(s)$. We can suppose, with no restriction, that $w_j \in D_j$. There are infinitely many fundamental domains $\Omega_{m,k}$ in every strip S_m , and each one of them contains a component of the pre-image of the disk $|w| \leq \frac{1}{|B(0)|}$, therefore a component $\Omega_{m,k,j}$ of the pre-image of every set D_j . The components of the pre-image of the first n_1 sets D_j intersect the imaginary axis, and therefore, there is a fundamental domain Ω_{m,k_1} and a $\sigma_1 > 0$ such that I_σ intersects all the components $\Omega_{m,k_1,j}$ for $j \leq n_1$ and every σ , $0 < \sigma \leq \sigma_1$. Then, there is a fundamental domain Ω_{m,k_2} , and $\sigma_2 > 0$, $\sigma_2 < \sigma_1$ such that I_{σ_2} intersects $n_2 \geq 1$ components $\Omega_{m,k_2,j}$ for $n_1 < j \leq n_1 + n_2$. Furthermore, I_{σ_2} intersects all the components $\Omega_{m,k_2,j}$ for $j \leq n_1 + n_2$, etc. Finally, for any n_0 , there is a domain Ω_{m,k_0} and $\sigma_0 > 0$ such

that I_{σ_0} intersects all the components $\Omega_{m,k_0,j}$, $j \leq n_0$. This means that the image by $\zeta_{A,\mathbb{N}}(s)$ of I_{σ_0} passes at a distance less than ϵ of every point w_k , where $|w_k| \leq \frac{1}{|B(0)|}$. \square

In Fig. 6 below, we have taken the zeros

$$z_k = 1 - \frac{1}{2^k}, \quad k \in \mathbb{N},$$

for an infinite Blaschke product $B(z)$ generating the Dirichlet function

$$w = \zeta_{A,\mathbb{N}}(s) = \sum_{n=0}^{\infty} a_n e^{-ns},$$

where

$$a_n = \frac{B^{(n)}(0)}{n!} B(0), \quad n = 0, 1, 2, \dots$$

The points

$$s_m = 2m\pi i, \quad m \in \mathbb{Z}$$

are essential non-isolated singular points of $\zeta_{A,\mathbb{N}}(s)$. For every $\sigma > 0$, the interval

$$I_\sigma = \{\sigma + it, t \in \mathbb{R}\} \subset S_m$$

is mapped by $\zeta_{A,\mathbb{N}}(s)$ into the disk $|w| < \frac{1}{|B(0)|}$. According to Theorem 12, if σ is small enough, the image of I_σ by $\zeta_{A,\mathbb{N}}(s)$ is almost dense in that disk, as Fig. 6 suggests.

The case where $\sigma < 0$ can be treated similarly, except that this time, we have to deal with infinitely many points w_k . Let us count again the n_1 eyes of the mesh covering the circle $|w| = \frac{1}{|B(0)|}$ and continue with n_2 adjacent eyes exterior to this circle and so on. There is a fundamental domain Ω_{m,k_1} and $\sigma_1 < 0$ such that for every σ , $\sigma_1 < \sigma < 0$ the segment I_σ intersects all the components $\Omega_{m,k_1,j}$ for $j \leq n_1$. Then, there is a fundamental domain Ω_{m,k_2} and $\sigma_2 < 0$, $\sigma_2 > \sigma_1$ such that I_{σ_2} intersects $n_2 \geq 1$ components $\Omega_{m,k_2,j}$ for $n_1 < j \leq n_1 + n_2$. Obviously, I_{σ_2} intersects all the components $\Omega_{m,k_2,j}$ for $j \leq n_1 + n_2$, and so on. We infer that for any r ,

$$\frac{1}{|B(0)|} < r < \frac{1}{|B(0)|^2}$$

there is there is $\sigma_0 < 0$ such that for every σ , $\sigma_0 < \sigma < 0$, the image of I_σ passes at a distance less than ϵ of every point w_k of the sequence, $|w_k| \leq r$.

The codes that generated the graphics that appear in this paper can be consulted at https://github.com/florin-albisoru/Computer_Experimentation_Dirichlet_Functions.git.

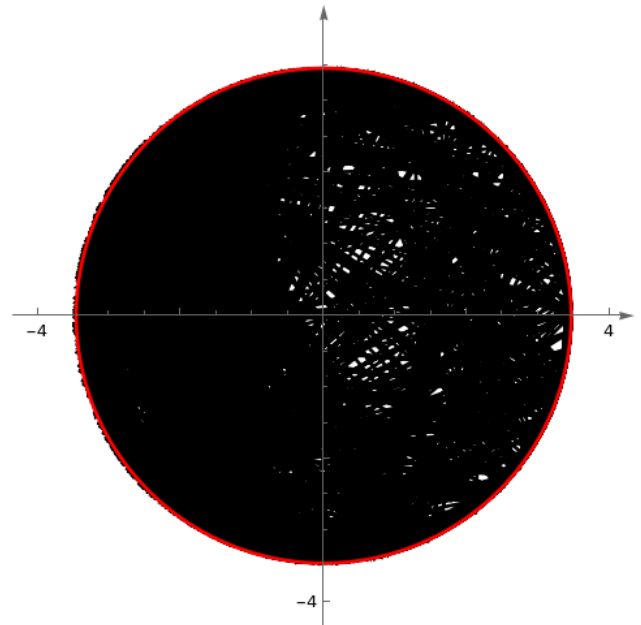


Figure 6: Illustrating the denseness property of a vertical line close on the imaginary axis $s = 0.001 + it$

7 Inclusiveness

Theorem 13. For any Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ defined by a Blaschke product $B(z)$ and any $r > 0$, there is at least one zero of $\zeta_{A,\mathbb{N}}(s)$ in every component $\Upsilon_{m,r}$ of the pre-image of the circle $|w| = r$, where $r \leq 1$. There is a unique unbounded component $\Upsilon_{m,1}$ in every strip S_m of $\zeta_{A,\mathbb{N}}(s)$, and there can be several bounded ones. Each of them can contain several zeros of $\zeta_{A,\mathbb{N}}(s)$. If

$$1 < r < \frac{1}{|B(0)|},$$

only one component $\Upsilon_{m,r}$ intersects Γ'_m and Γ'_{m+1} . Several zeros of $\zeta_{A,\mathbb{N}}(s)$ can be in the domain bounded by them.

Proof: Indeed, 0 belongs to the disk $|w| < r$ and then any component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of this disk should contain a point s_0 such that $\zeta_{A,\mathbb{N}}(s_0) = 0$. If two components $\Upsilon_{m,r}$ have fused, the closed curve obtained contains the zeros from both of them; therefore, it can contain several zeros of $\zeta_{A,\mathbb{N}}(s)$.

Since

$$\lim_{\sigma \rightarrow +\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1,$$

there must be an unbounded component $\Upsilon_{m,1}$ of the unit circle. For $\sigma_\alpha + it_\alpha$ belonging to the intersection of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of a ray,

which makes an angle $\alpha \neq 0$ with the positive real half-axis and the unbounded component $\Upsilon_{m,1}$, we have $\sigma_\alpha < +\infty$, since $\zeta_{A,\mathbb{N}}(\sigma_\alpha + it_\alpha)$ cannot reach the value 1. However, when $\alpha \rightarrow 0$ we have $\sigma_\alpha \rightarrow +\infty$; thus, the pre-image of the interval $(0, 1)$ should be formed with infinite curves and so must be the pre-image of the interval $(1, +\infty)$. Consecutive components of the pre-image of the interval $(1, +\infty)$ contain the unbounded curve $\Upsilon_{m,1}$ which generated them and they form the strips S_m . Therefore there is a unique unbounded $\Upsilon_{m,1}$ in every strip S_m , yet there can be several bounded ones, as it has been documented for the Dirichlet L -functions. These bounded components cannot intersect any Γ'_m since on $\Upsilon_{m,1}$ we have $|\zeta_{A,\mathbb{N}}(s)| = 1$ while on Γ'_m we have $|\zeta_{A,\mathbb{N}}(s)| > 1$. Obviously, there can be several zeros of $\zeta_{A,\mathbb{N}}(s)$ in the domain bounded by the unbounded $\Upsilon_{m,r}$, Γ'_m and Γ'_{m+1} . \square

Theorem 14. For a Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ defined by an infinite Blaschke product $B(z)$, the number of zeros of $\zeta_{A,\mathbb{N}}(s)$ inside every component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle is finite.

Proof: The components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle cannot intersect the imaginary axis since $|\zeta_{A,\mathbb{N}}(s)| = 1$ on such a component and

$$|\zeta_{A,\mathbb{N}}(s)| = \frac{1}{|B(0)|} > 1$$

on the imaginary axis at the points where it is defined. Then, if there were infinitely many zeros of $\zeta_{A,\mathbb{N}}(s)$ inside such a component, they would have an accumulation point s_0 in it with $\zeta_{A,\mathbb{N}}(s_0) = 0$. This point cannot be on the pre-image of the unit circle, since there $|\zeta_{A,\mathbb{N}}(s)| = 1$. Therefore, it belongs to the same open set as the zeros, and by the uniqueness theorem of analytic functions, we would have $\zeta_{A,\mathbb{N}}(s) \equiv 0$ in that component, which is absurd. \square

What exactly this number of zeros is remains a mystery for the moment. What we know for sure is the fact that in every strip S_m of $\zeta_{A,\mathbb{N}}(s)$, there are infinitely many bounded components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle. Increasing the radius r of the circle up to $\frac{1}{|B(0)|}$, they are all absorbed by the component from S_m of the pre-image $\Upsilon_{m,r}$ of the circle $|w| = r$. If there are k zeros of $\zeta_{A,\mathbb{N}}(s)$ in the domain bounded by $\Upsilon_{m,r}$, Γ'_m and Γ'_{m+1} , then $\Upsilon_{m,r}$ can be divided into $k - 2$ arcs and the unbounded curves which, with an end removed, are mapped one to one onto the circle $|w| = r$. At the limit, when E_s is not dense on the imaginary axis, we

get infinitely many half-closed intervals in the strip S_m on the imaginary axis, which are mapped one to one onto the circle $|w| = \frac{1}{|B(0)|}$.

Corollary 15. For a Dirichlet function $\zeta_{A,\mathbb{N}}(s)$ defined by a Blaschke product $B(z)$ of degree n , the number of poles of the function included in the pre-image of the circle $|w| = \frac{1}{|B(0)|^2}$ is at least 1 and less than or equal to $n - 1$.

Proof: Indeed, every pole of $\zeta_{A,\mathbb{N}}(s)$ is the symmetric with respect to the imaginary axis of a zero of $\zeta_{A,\mathbb{N}}(s)$, and the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle is symmetric with respect to the imaginary axis with the circle $|w| = \frac{1}{|B(0)|^2}$ and the affirmation of this corollary is obvious. \square

Corollary 16. For a Dirichlet function defined by an infinite Blaschke product, the number of poles included in the pre-image of the circle $|w| = \frac{1}{|B(0)|^2}$ is finite.

Proof: The arguments of the proof are similar to those of Corollary 14. \square

8 Conclusions

This paper aimed to deepen the study of Dirichlet functions defined by Blaschke products and to illustrate the findings with computer-generated graphics. If the Blaschke products are finite, those functions are given by formulas that are easy to implement into computer programs. The graphics we obtained illustrate unexpected aspects of Dirichlet functions, which were unknown until now. Computer experimentation with Blaschke products and the Dirichlet functions they generated revealed unknown properties of these functions that have been rigorously proved in the end. In the infinite case, we succeeded in extrapolating the technique of pre-images of lines and circles and getting an idea of the geometry of the conformal mappings by those Dirichlet functions, culminating with a denseness theorem. We proved that the denseness property, attributed up to now only to functions with natural boundary, is common to any Dirichlet function defined by an infinite Blaschke product. These results open the way to new research directions and potentially to physics applications.

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