Numerical solution of singularly perturbed delay differential equation with layer behavior by using local polynomial regression

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Abstract: In this study, we apply local polynomial regression (LPR) to the solution of singularly perturbed delay differential equation with layer behavior. The method was tested using a numerical example to demonstrate its usefulness. The method presented in this paper was also compared with those developed by Ghomanjani et al.[1] and was found to be better. The proposed method was simple and computationally advantageous. The computed results we are compared with exact solutions. Plotted graphs of the solutions are also available in this study.

Keywords: Singularly perturbed delay differential equation, Local polynomial regression, Layer behavior

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1 Introduction

Boundary value problems (BVPs) have been studied by several researchers [1,2,3,8,9,10]. In the present study, local polynomial regression (LPR) is used to develop a technique for solving singularly perturbed delay differential equation with layer behavior in the following form. These equations are important boundary value problems (BVPs) owing to their potential applications in science and engineering. A number of studies on the LPR method for solving differential equations exist in the literature (see Caglar et al. [4,5,6,7]).

It is well-known that singularly perturbed delay differential equation with layer behavior can be defined as follows:

$$\mathcal{E}y''(t) + a(t)y'(t-\delta) + b(t)y(t) = f(t), 0 < t < 1, \quad (1)$$
$$y(t) = \emptyset(t), \ -\delta \le t \le 0,$$
$$y(1) = \gamma \ ,$$

where $\mathcal{E}$ is a small parameter obeying $0 < \mathcal{E} \ll 1$ and δ is a small shifting parameter falling in the boundaries $0 < \delta \ll 1$ and $a(t)$, $b(t)$, $f(t)$ and $\emptyset(t)$ are assumed to be smooth while γ is a constant.

For $\delta = 0$, the problem is a boundary value problem for a singularly perturbed differential equation. As the singular perturbation parameter tends to zero, the order of the corresponding reduced problem decreases by one; thus, there will be one layer. Depending on the nature of the coefficient of the convection term, the layer can be either a boundary layer or an interior layer.

When the delay parameter is smaller than the perturbation parameter, approximating $y'(t-\delta)$ using Taylor's series approximation is valid [1,2, 3]. Since, we assumed that $\delta < \mathcal{E}$, the terms $y'(t-\delta)$ are approximated as,

$$y'(t-\delta) \approx y'(t) - \delta y''(t) + O(\delta^2) \quad (2)$$

Substituting (2) into (1) we get the following:

$$(\mathcal{E} - a(t)\delta)y''(t) + a(t)y'(t) + b(t)y(t) = f(t), \quad (3)$$

Equations of form (1) are relevant in many real-life applications such as heat transfer, viscoelastic flow, abrasive blasting, and control theory. Therefore, solving these problems is important for researchers.

The remainder of this paper is organized as follows. The theoretical background is presented in section 2. The LPR solutions are presented in section 3. A numerical example is presented in section 4 and a brief conclusion is provided in section 5.

2 Local polynomial regression

Suppose that the $(p+1)$-th derivative of $x(t)$ exists at point t_o exists. We approximated the unknown regression function $y(t)$ locally at $t_o$using a polynomial of order p. Theoretically, we can approximate, in the neighborhood of t_o, $y(t)$ using a Taylor expansion:

$$y(t) \approx \sum_{k=0}^{p} \beta_k (t_i - t_0)^k \;, \tag{4}$$

where,

$$\beta_k = \frac{t^{(k)}(t_0)}{k!} \,. \tag{5}$$

This polynomial is , used to approximate the unknown function locally at t_0 and is obtained by solving a locally weighted least squares regression problem, that is minimizing,

$$\sum_{i=1}^{n} \{Y_i - \sum_{k=0}^{p} \beta_k (t_i - t_0)^k\}^2 K \frac{(t_i - t_0)}{h}, \tag{6}$$

where h is the smoothing parameter and K is a Kernel function. We have selected the quartic Kernel as follows

$$K(t) = \begin{cases} \frac{15}{16}\,(1-t^2)^2 & if\ |t| \le 1 \\ 0 & otherwise \end{cases} \;.$$

The idea is to estimate the smoothing parameter by minimizing maksimum error $\hat{e}(h)$:

$\hat{e}(h) = \max(\hat{e}_i)\,, i = 1,2,\dots,n$.

The optimal bandwidth is defined as the minimum value of $\hat{e}(h)$, calculated based on different selections of h.

The local polynomial Kernel regression estimate β is given by

$\beta = (X^T W X)^{-1} X^T W Y$, where

$x = ((t_i - t_0)^{i-1})_{1 \le i \le n}$, $1 \le j \le p+1$, $W = diag\{K_h(t_i - t_0)\}$ the vectors

$Y = (Y_1, Y_2, \dots, Y_n)'$ and

$\beta = \beta_0, \beta_1, \dots, \beta_p)'$. A detailed description can be found in Ref. [4].

3 Illustration of method

In this section, the LPR method for solving Equation (3) is outlined. Let Equation (4) be an approximate solution of Equation (1):

$$y'(t-\delta) = y'(t) - \delta y''(t) \tag{7}$$

$$\mathcal{E} y''(t) + a(t)(y'(t) - \delta y''(t)) + b(t)y(t) = f(t) \tag{8}$$

$$(\mathcal{E} - a(t)\delta) y''(t) + a(t)y'(t) + b(t)y(t) = f(t) \tag{9}$$

$$(\mathcal{E} - a(t)\delta) \sum_{k=0}^{p} \beta_k (t_i - t_0)'' + a(t) \sum_{k=0}^{p} \beta_k (t_i - t_0)' + b(t) \sum_{k=0}^{p} \beta_k (t_i - t_0) \;\; = f(t) \tag{10}$$

This leads to a system,

$$i = 1;\quad a_{1,j} = \beta_j (t_1 - t_0)^j,\;\; j = 0,p, \qquad y(1) = \emptyset(1) \tag{11}$$

$$i = n;\;\; a_{nj} = \beta_j (t_n - t_0)^j \,, \quad j = 0,p, \quad y(i) = \gamma \tag{12}$$

$$i = 2, n-1;\qquad b_{ij} = \; j(j-1)\beta_j (t_i - t_0)^{j-2},\; j = 2,p \tag{13}$$

$$c_{ij} = a\, j\beta_j (t_i - t_0)^{j-1}, j = 1,p \tag{14}$$

$$d_{ij} = \; b\beta_j (t_i - t_0)^j,\; j = 0,p \tag{15}$$

$$y(i) = f((t_i) \tag{16}$$

The matrix form can be written as follows using (11-16).

$$X = \begin{bmatrix} a_{1,0} & a_{1,1} & \cdots & \cdots & \cdots & a_{1,p} \\ d_{2,0} & d_{2,1}+c_{2,1} & \cdots & d_{2,m+1}+c_{2,m+1} & \cdots & d_{2,p}+c_{2,p} \\ d_{3,0} & d_{3,1}+c_{3,1} & \cdots & d_{3,m+1}+c_{3,m+1} & \cdots & d_{3,p}+c_{3,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n-1,0} & d_{n-1,1} & \cdots & d_{n-1,m+1}+c_{n-1,m+1} & \cdots & d_{n-1,p}+c_{n-1,p} \\ a_{n,0} & a_{n,1} & \cdots & \cdots & \cdots & a_{n,n} \end{bmatrix} \quad (17)$$

$$Y = \begin{bmatrix} y(1) \\ \vdots \\ y(n) \end{bmatrix}.$$

By substituting (17) into

$$\beta = (X^T W X)^{-1} X^T W Y,$$

the coefficients can be estimated by solving the matrix system. Therefore, an approximate solution for Equation (4) is obtained.

4 Numerical results

Here, we test the methods discussed in sections 2 and 3 using the following examples from the literature [1] and calculate the absolute errors of the analytical solutions. We also compared this method with that of Ghomanjani et al. [1]. All computations were performed using the MATLAB software.

Example 1. Consider the following boundary value problem (see [1])

$$\mathcal{E}y''(t) + y'(t-\delta) - y(t) = 0 \;, 0 < t < 1, \quad (18)$$
$$y(t) = 1 \quad , -\delta \le t \le 0$$
$$y(1) = 1 \quad . \quad (19)$$

A boundary layer exists on the left -hand side of the interval. Note that the exact solution is,

$$y(t) = \frac{(1-e^{m_2})e^{m_1 t} + (e^{m_1}-1)e^{m_2 t}}{(e^{m_1} - e^{m_2})} \quad (20)$$

where,

$$m_1 = \frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}, \quad m_2 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \quad (21)$$

In addition , we plotted graphs of the exact and computed solution of the problem in Figure 1 (Appendix). The maximum errors are listed in Table 1 (Appendix).

Example 2. Next we consider the problem (see[1])

$$\mathcal{E}y''(t) + y'(t-\delta) - y(t) = 0, \quad 0 < t < 1, \quad (22)$$
$$y(t) = 1 \quad , -\delta \le t \le 0,$$
$$y(1) = -1 \quad . \quad (23)$$

A boundary layer exists on the left-hand side of the interval. Note that the exact solution is

$$y(t) = \frac{(1+e^{m_2})e^{m_1 t} - (1+e^{m_1})e^{m_2 t}}{(e^{m_2} - e^{m_1})} \quad (24)$$

where,

$$m_1 = \frac{1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)},$$
$$m_2 = \frac{1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \quad (25)$$

In addition , we plotted graphs of the exact and computed solution of the problem in Figure 2 (Appendix). The maximum errors are listed in Table 2 (Appendix).

5 Conclusions

In this study, the LPR was applied to the numerical solution of a singularly perturbed delay differential equation with layer behavior. The method was tested on examples and the maximum absolute errors were tabulated. The numerical results demonstrate that the LPR method approximates the exact solution very well. It has been observed that the error in the absolute values are better than those shown by Ghomanjani et al. [1] . It may be concluded that the LPR is a very powerful and efficient method for finding solutions to these types of problems. Moreover, it can be seen from Table 1 (Appendix) and Table 2 (Appendix) that the error decreases asymptotically at n different nodes. This method can also be extended to different bandwidths and polynomial degree functions.

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Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.

APPENDIX

Table 1: The maximum absolute errors for $\mathcal{E} = 0.1$ with different δ and $p = 13$ (polynomial degree) for Example 1.

δ	$n = 21$	$n = 41$	$n = 121$	$Max\ error\ in$ [1]
0.01	1.57485e-04	5.087794e-05	5.550333e-06	0.0045
0.03	5.92967e-04	1.313718e-04	1.946914e-04	0.0090
0.06	0.001973	0.0144996	0.1037118	0.0070
0.08	0.019014	0.4223131	0.5326387	0.0300

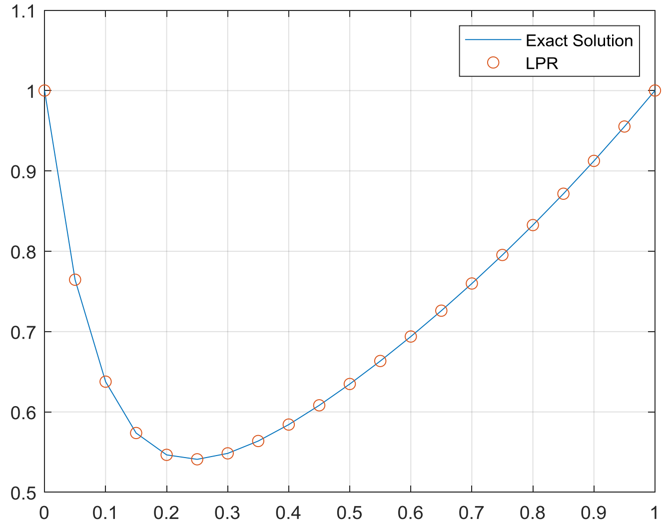

Fig. 1: Graphs of the exact and computed solution of BVP with $\mathcal{E} = 0.1\ and\ \delta = 0.01$ for Example 1.

Table 2: The maximum absolute errors for $\mathcal{E} = 0.1$ with different δ and $p = 13$ (polynomial degree) for Example 2.

δ	$n = 21$	$n = 41$	$n = 121$	$Max\ error\ in$ [1]
0.01	3.364445e-04	6.318114e-05	9.1682607e-05	0.007
0.03	3.364445e-04	6.318114e-05	9.1682607e-05	0.022
0.06	3.364445e-04	6.318114e-05	9.1682607e-05	0.023
0.08	3.364445e-04	6.318114e-05	9.1682607e-05	0.025

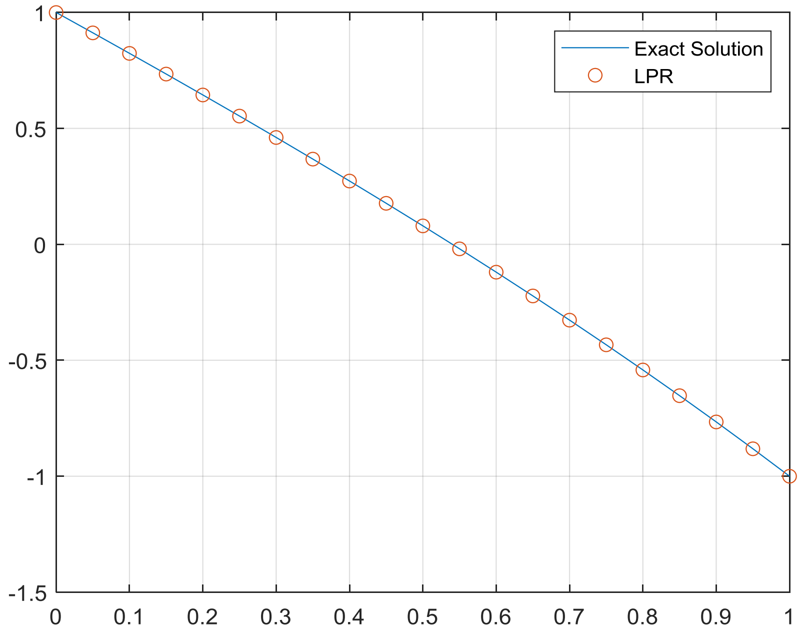

Fig. 2: Graphs of the exact and computed solution of BVP with $\mathcal{E} = 0.1$ *and* $\delta = 0.01$ for Example 2.