Various implications of fractional calculus of arbitrary order for Mittag-Leffler functions with complex parameters

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Abstract - In this scientific document, fundamental information about the renowned Mittag-Leffler functions, which play significant roles in science and engineering, particularly in mathematics, will be presented. Essential background on the fractional calculus of arbitrary order associated with these functions will then be provided. This investigation will subsequently focus on the complex-parameter forms of these functions, followed by the presentation of specific results related to their fractional-order computations.

Key-Words: Analytic functions, Operators, the Mittag-Leffler functions, Fractional calculus of arbitrary order, Complex-type power series.

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1 Introduction

In this introductory section, only certain information assocating with special functions and fractional calculus of arbitrary order will be recalled, which deal with the scope of this particular investigation.

Special functions (SFs), identified by established names and specific notations, play a crucial role in numerous scientific fields, especially in mathematics and engineering. These functions, defined over both real and complex domains, belong to a broad family with a wide range of applications. Among them, the Mittag-Leffler functions (MLFs) stand out due to their mathematical significance and are the primary focus of this research. Defined for both real and complex arguments, MLFs are particularly effective in modeling phenomena such as the behavior of fractional-order viscoelastic materials. In particular, experimental studies on the time-dependent relaxation behavior of viscoelastic materials have shown that stress decreases rapidly at the beginning of the relaxation process and then diminishes very slowly over time. It may take a long period for the stress to stabilize at an asymptotic value. Modeling this relaxation behavior typically requires a large number of Maxwell elements, leading to complex optimization problems when determining the associated material parameters.

The introduction of fractional derivatives into viscoelasticity theory has led to the development of efficient models, such as the fractional-type Zener model. This model has proven effective in predicting the dynamic behavior of rubber-like materials while requiring only a small number of material parameters. Solving the corresponding constitutive equation yields a relaxation function of the MLF type, characterized by a power series with negative arguments. This function accurately captures all essential features of the relaxation process under an arbitrary continuous input signal that includes a discontinuity at the origin.

The foundational references cited in this document provide extensive insights into the theories, applications, implications, and background of special functions, particularly the Mittag-Leffler functions and related topics, [1], [2], [3], [4], [5], [6], [7], [8].

This scientific investigation also is pertinent to fractional-order computations (FOCs), with particular emphasis on fractional-order integrals (FOIs) and fractional-order derivatives (FODs). FODs generalize classical integer-order derivatives and have a long history, originating in the works of pioneering mathematicians such as G. W. Leibniz and L. Euler. Shortly after their inception, these concepts attracted the interest of many prominent mathematicians and scientists due to their vast potential for applications. FOCs involve various differentiation and integration operators of arbitrary real or complex order, significantly expanding the modeling capabilities of traditional calculus and differential equations. Their applications span a wide array of scientific disciplines, including physics, mechanics, engineering, economics, finance, biology, and chemistry, as outlined in references, [9], [10], [11], [12], [13].

At the same time, mathematical equations involving fractional differential operators (FDOs) are known as fractional differential equations (FDEs). Various types of FDEs can be considered, such as heat and diffusion-type equations where time derivatives of fractional order replace classical derivatives, or more general evolution systems involving fractional operators in both temporal and spatial variables—for instance, the fractional (sub-)Laplacian. Notably, our recent research has focused on second-order partial differential equations that involve Caputo fractional derivatives (CFDs) in the time variable and Bessel operators in the spatial domain. We have also studied multi-term generalizations of time-fractional diffusion-wave equations in abstract settings, including the corresponding initial and boundary value problems, which relates both direct and inverse. To explore the regularity properties of solutions in specific cases of the fractional wave equation, we have conducted microlocal analysis, aiming to improve upon existing results and expand the use of pseudo-differential and microlocal techniques in the study of various FDEs.

One of the most significant applications of FDEs is in linear viscoelasticity, where they provide a more precise and natural description of the material behavior. Models of wave propagation in viscoelastic media are commonly formulated as fractional differential equations, incorporating appropriate initial and boundary conditions, and derived through rheological analogies based on fundamental physical principles.

The inclusion of fractional derivatives in these models accurately reflects the viscoelastic nature of such materials. These materials are also often modeled using different constitutive equations involving derivatives of both real and complex orders, which relate the fractional generalized Zener, Maxwell, or Kelvin-Voigt models. Wave propagation in these media has been studied in both finite and infinite spatial domains.

For various interesting references on fractional calculus (FC), fractional-order computations (FOCs), integrals (FOIs), derivatives (FODs), differential equations (FDEs), and Caputo fractional derivatives (CFDs), along with their comprehensive applications, the reader is referred to the foundational studies listed in the mentionend references, [14], [15], [16], [17], [18], [19].

Lastly, in the following sections of this special research, some well-known special sets will be also needed. The mathematical notations:

$$\mathbb{N}\ ,\ \mathbb{R}\ \text{ and }\ \mathbb{C}$$

represent the familiar sets of natural numbers, real numbers, and complex numbers, respectively.

Now, let us move on to the next sections to introduce the necessary information that may be relevant to the main topics of this research: FOCs, MLFs, and related topics.

2 Certain information on MLFs and some of related topics

In this section, only MLFs and some fundamental information and applications related to them will be included.

In the mathematical literature, as is indicated in the relevant references, [20], [21], [22], there are two main types of MLFs. They are the one-parameter Mittag-Leffler function and the two-parameter Mittag-Leffler function. Later, two generalized forms were defined, which include these two special functions and are known as the three-parameter Mittag-Leffler type functions. As emphasized in the first section, our study will focus on MLFs with complex parameters (variables). Let us now revisit them.

The one-parameter Mittag-Leffler function and the two-parameter Mittag-Leffler function, *which* are complex-valued functions of a complex variable (or argument) z and moreover depend on one or two complex parameters, are denoted by the symbols

$$\mathrm{E}_{\gamma}(z) \quad \text{and} \quad \mathrm{E}_{\gamma,\beta}(z) \quad (z \in \mathbb{C})$$

and also defined by

$$\mathrm{E}_{\gamma}(z) = \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\gamma\ell+1)} z^{\ell} \tag{1}$$

and

$$\mathrm{E}_{\gamma,\beta}(z) = \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\gamma\ell+\beta)} z^{\ell}, \tag{2}$$

respectively, *where* $\gamma \in \mathbb{C}$ and $\beta \in \mathbb{C}$ with $\Re e(\gamma) > 0$ and $\Re e(\beta) > 0$, and the (classical) function

$$\Gamma(\xi) = \int_0^{\infty} q^{\xi-1} e^{-q} dq, \tag{3}$$

which is the classical Gamma function with complex parameter ξ ($\xi \in \mathbb{C}$ and $\Re e(\xi) > 0$), [23].

As one of various important assertions in relation with the special information in (1) and (2), and in light of the definition (3) together with its applications, by taking $|z| \to \infty$ and then changing $q = Q^2$ for the improper integral given by

$$\int_0^z e^{-Q^2} dQ,$$

the following special result:

$$\int_0^{\infty} e^{-Q^2} dQ = \frac{\sqrt{\pi}}{2} \tag{4}$$

can be easily derived, [24].

By help the particular result (4), one of numerous relationships between the (complex) error function $erf(z)$ and the complementary (complex) error function $erfc(z)$ can be easily received in the form:

$$erf(z) + erfc(z) = 1 \quad (z \in \mathbb{C}), \tag{5}$$

as an essential relations between these (complex) functions, [25].

As is well known, for all complex values z, these complex functions play special roles in various fields of science and engineering, and they are defined as:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-Q^2} dQ \tag{6}$$

and

$$erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-Q^2} dQ, \tag{7}$$

respectively, [26].

Moreover, by taking cognizance of the Taylor-MacLaurin series expansion of the exponential-type function given by

$$w(Q) = e^{-Q^2},$$

the equivalent forms of these complex-type functions introduced as in (6) and (7) can be also received as the familiar series expansions of them given by

$$erf(z) = \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} z^{2\ell+1} \tag{8}$$

and

$$erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} z^{2\ell+1}, \tag{9}$$

respectively.

We particularly note that *when* γ and β are positive real numbers, each of the functions defined by (1) and (2) converges for all values of the argument z, making MLFs entire functions. In this regard, they are highly advantageous special functions in mathematical theories and various applications. For more detailed information on the concepts introduced between (1) and (9), readers may refer to the information in the works, [27], [28].

As a result of a simple focus, it can be easily seen that there is a special relationship between those special functions given by (1) and (2), such as

$$\mathrm{E}_{\gamma}(z) \equiv \mathrm{E}_{\gamma,1}(z) \quad (z \in \mathbb{C}).$$

Moreover, *when* all the information in (1)-(9) is taken into account, and by appropriately selecting the relevant parameters, a wide range of both fundamental results and relationships between these complex functions and other special functions can be easily observed. Here, we aim to focus on only a few of them. As an extensive example, through the instrumentality of the mentioned functions:

$$\mathrm{E}_{\gamma,\beta}(z) \quad \text{and} \quad \mathrm{E}_{\gamma}(z)$$

in the forms given by

$$\mathrm{E}_{\gamma,\beta}(-\mu^2) \quad \text{and} \quad \mathrm{E}_{\gamma}(-\mu^2)$$

the following assertions:

$$\begin{aligned} &\int_0^z \mathrm{E}_{\gamma,\beta}(-\mu^2) d\mu \\ &\quad = \int_0^z \left(\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\gamma\ell+\beta)} \mu^{2\ell} \right) d\mu \\ &\quad = \sum_{\ell=0}^{\infty} \left(\frac{(-1)^{\ell}}{\Gamma(\gamma\ell+\beta)} \int_0^z \mu^{2\ell} d\mu \right) \\ &\quad = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\Gamma(\gamma\ell+\beta)} z^{2\ell+1} \end{aligned} \tag{10}$$

and also

$$\begin{aligned} &\int_0^z \mathrm{E}_{\gamma,1}(-\mu^2) d\mu \\ &\quad \equiv \int_0^z \mathrm{E}_{\gamma}(-\mu^2) d\mu \end{aligned}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\Gamma(\gamma\ell+1)} z^{2\ell+1} \quad (11)$$

can be easily determined, of course, *under* the indicated conditions on their mentioned definitions.

At the same time, many of their relationships with various elementary functions are, of course, also possible. As just a few specific examples, it immediately follows that

$$\begin{aligned} \mathrm{E}_{0,1}(\pm z) &\equiv \mathrm{E}_0(\pm z) \\ &= \sum_{\ell=0}^{\infty} \frac{(\pm 1)^{\ell}}{\Gamma(1)} z^{\ell} \\ &= \sum_{\ell=0}^{\infty} \frac{(\pm 1)^{\ell}}{0!} z^{\ell} \\ &= \frac{1}{1 \mp z} \quad (|z| < 1), \qquad (12) \end{aligned}$$

$$\begin{aligned} \mathrm{E}_{1,1}(\pm z) &\equiv \mathrm{E}_1(\pm z) \\ &= \sum_{\ell=0}^{\infty} \frac{(\pm 1)^{\ell}}{\Gamma(\ell+1)} z^{\ell} \\ &= \sum_{\ell=0}^{\infty} \frac{(\pm 1)^{\ell}}{\ell!} z^{\ell} \\ &= e^{\pm z} \quad (z \in \mathbb{C}), \qquad (13) \end{aligned}$$

$$\begin{aligned} &\int_0^z \mathrm{E}_{2,1}(-\mu^2) d\mu \\ &\equiv \int_0^z \mathrm{E}_2(-\mu^2) d\mu \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)!} z^{2\ell+1} \qquad (14) \\ &= Sin(z) \quad (z \in \mathbb{C}), \end{aligned}$$

and

$$\begin{aligned} &\int_0^z \mathrm{E}_{1,1}(-\mu^2) d\mu \\ &\equiv \int_0^z \mathrm{E}_1(-\mu^2) d\mu \\ &= \int_0^z \left\{ \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \mu^{2\ell} \right\} d\mu \\ &= \sum_{\ell=0}^{\infty} \left\{ \frac{(-1)^{\ell}}{\ell!} \int_0^z \mu^{2\ell} \, d\mu \right\} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} z^{2\ell+1} \\ &= \frac{\sqrt{\pi}}{2} erf(z) \qquad (15) \\ &= \frac{\sqrt{\pi}}{2} \left(1 - erfc(z)\right), \qquad (16) \end{aligned}$$

where $z \in \mathbb{C}$. Accordingly, the assertion (16) immediately leads to

$$erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{E}_{1,1}(-\mu^2) d\mu, \quad (17)$$

which is the special assertion given in (9).

Finally, each of the results presented between equations (10) and (16) contains highly useful and important calculations in terms of special functions. The only way to determine each of them is by identifying the special relationships between MLFs, the (complex) error functions, and other special (elementary) functions. In this regard, one can center on the main studies relating with special functions at the end of this special document.

3 Certain information on FOCs and some of related topics

In this third section, we will focus on some basic definitions related to FOCs, some special relationships, and specific calculations related to SFs and MLFs.

Firstly, for a continuous function like $\lambda := \lambda(z)$, the fractional integral of order ς is denoted by

$$\mathfrak{D}_z^{-\varsigma}[\lambda(z)] \equiv \mathfrak{D}_z^{-\varsigma}[\lambda](z),$$

and also defined by

$$\mathfrak{D}_z^{-\varsigma}[\lambda](z) = \frac{1}{\Gamma(\varsigma)} \int_0^z \frac{\lambda(\rho)}{(z-\rho)^{1-\varsigma}} d\rho, \qquad (18)$$

where $\varsigma > 0$ and the complex-valued function is analytic in any simply connected region of the complex plane comprising the origin, and the multiplicity of the complex power $(z-\rho)^{\varsigma-1}$ is also raised by necessitating $log(z-\rho)$ to be a real number *when* $z - \rho > 0$.

For a continuous function like $\lambda := \lambda(z)$, the fractional derivative(s) of order ς is denoted by

$$\mathfrak{D}_z^{\varsigma}[\lambda(z)] \equiv \mathfrak{D}_z^{\varsigma}[\lambda](z)$$

and, also defined by

$$\mathfrak{D}_z^{\varsigma}[\lambda](z) = \begin{cases} \frac{1}{\Gamma(1-\varsigma)} \frac{d}{dz} \int_0^z \frac{\lambda(\rho)}{(z-\rho)^{\varsigma}} d\rho & (0 \le \varsigma < 1) \\ \frac{d^n}{dz^n} \{\mathfrak{D}_z^{\varsigma-n}[\lambda](z)\} & (n \le \varsigma < n+1) \end{cases} \qquad (19)$$

where $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the analytic function $\lambda(z)$ is constrained, and the multiplicity of the complex power $(z-s)^{-\varsigma}$ is also removed as in the definition given in the definition (18).

In particular, by considering the essential integration rules, it is easy to prove that both the linearity and scalar multiplication properties are satisfied for the transformations presented in the definitions constituted in (18) and (19). For further information, the mentioned studies in the references in relation with the detailed information about various type transformations and fractional calculus, [28], [29], [30], [31], [32], [33], [34].

As a special case, we want to consider the complex power function given by

$$\lambda(z) \coloneqq z^{Q} \quad (Q \in \mathbb{C};\ \Re e(Q) > -1). \qquad (20)$$

In order to easily solve the integrals given in (18) and (19), the well-known Beta and Gamma functions play an essential role. In this context, it is important to mention these functions. The familiar Beta function (with complex parameters), *which* is denoted by

$$\mathbf{B}(\alpha,\beta) \quad (\alpha,\beta \in \mathbb{C}),$$

and also defined by

$$\mathbf{B}(\alpha,\beta) = \int_0^1 \rho^{\alpha-1}(1-\rho)^{\beta-1} d\rho\,, \qquad (21)$$

which also has its close relationship to the Gamma function in (3) given by

$$\boldsymbol{\mathcal{B}}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (22)$$

where $\Re e(\alpha) > 0$ and $\Re e(\beta) > 0$.

In particular, *when* $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$, this fundamental relationship (22) is immediately yields that

$$\boldsymbol{\mathcal{B}}(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}\,. \qquad (23)$$

By considering the specific information between (18) and (19), and using the mentioned results provided in (3), (21), (22), and (23), along with the substitution:

$$\rho = z\Omega \;\Rightarrow\; d\rho = \mathrm{z}d\Omega\,,$$

the pending integral(s) and derivative(s) of fractional-order for the complex-type power function in (20) can be easily determined. First of them, its fractional integral of real order ς can be easily determined as the relations given by

$$\mathfrak{D}_z^{-\varsigma}[z^{Q}] = \frac{1}{\Gamma(\varsigma)}\int_0^z s^{Q}(z-s)^{\varsigma-1}ds$$
$$= \frac{\Gamma(Q+1)}{\Gamma(Q+\varsigma+1)} z^{Q+\varsigma}, \qquad (24)$$

where $\varsigma > 0$.

Secondly, its fractional derivative of order ς can be also determined as the elementary result consisting of the relationships given by

$$\mathfrak{D}_z^{\varsigma}[z^{Q}] = \frac{1}{\Gamma(1-\varsigma)}\frac{d}{dz}\left(\int_0^z \frac{\tau^{Q}}{(z-s)^{\varsigma}}\, ds\right)$$
$$= \frac{\Gamma(Q+1)}{\Gamma(Q-\varsigma+1)} z^{Q-\varsigma}, \qquad (25)$$

where $0 \le \varsigma < 1$.

In addition, its fractional derivatives of order $n+\varsigma$ can be also determined as

$$\mathfrak{D}_z^{n+\varsigma}[z^{Q}\,] \equiv \frac{d^n}{dz^n}\{\mathfrak{D}_z^{\varsigma}[z^{Q}]\}$$
$$= \frac{\Gamma(Q+1)}{\Gamma(Q-n-\varsigma+1)} z^{Q-n-\varsigma}, \qquad (26)$$

where $n \le \varsigma < n+1$ and $n \in \mathbb{N}_0$.

Particularly, as an extensive example for the special family of MLFs, the following assertions:

$$\mathfrak{D}_z^{\varsigma}\left[\mathrm{E}_{\gamma,\beta}(z)\right]$$
$$= \mathfrak{D}_z^{\varsigma}\left[\sum_{\ell=0}^{\infty}\frac{1}{\Gamma(\gamma\ell+\beta)} z^{\ell}\right]$$
$$= \sum_{\ell=0}^{\infty}\left\{\frac{1}{\Gamma(\gamma\ell+\beta)}\mathfrak{D}_z^{\varsigma}\left[z^{\ell}\right]\right\} \qquad (27)$$
$$= z^{\varsigma}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1)}{\Gamma(\gamma\ell+\beta)\,\Gamma(\ell-\varsigma+1)} z^{\ell}$$
$$= z^{\varsigma}\sum_{\ell=0}^{\infty}\frac{\ell!}{\Gamma(\gamma\ell+\beta)\,\Gamma(\ell-\varsigma+1)} z^{\ell}$$

and

$$\mathfrak{D}_z^{1+\varsigma}\left[\mathrm{E}_{\gamma,\beta}(z)\right]$$
$$= \mathfrak{D}_z^{1+\varsigma}\left[\sum_{\ell=0}^{\infty}\frac{1}{\Gamma(\gamma\ell+\beta)} z^{\ell}\right]$$
$$= \sum_{\ell=0}^{\infty}\left\{\frac{1}{\Gamma(\gamma\ell+\beta)}\,\mathfrak{D}_z^{1+\varsigma}\left[z^{\ell}\right]\right\} \qquad (28)$$
$$= z^{\varsigma-1}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1)}{\Gamma(\gamma\ell+\beta)\,\Gamma(\ell-\varsigma)} z^{\ell}$$
$$= z^{\varsigma-1}\sum_{\ell=0}^{\infty}\frac{\ell!}{\Gamma(\gamma\ell+\beta)\,\Gamma(\ell-\varsigma)} z^{\ell}$$

can also be calculated by the help of the particular results specified as in (25) and (26).

As we have emphasized before, there are numerous special relationships (*or* consequences) between MLFs, SPs, and FOCs together with related topics. In this regard, particularly in the special results from (10) to (28) in this special document, the highlighted special relationships (*or* consequences) can be revealed if the mentioned parameters used are appropriately chosen.

As numerous suitable examples, by considering the essential information in the equations given by the references (19), (20) and (26), let us focus on the special results, *which* are the following combinations of the statements given in (27) and (28):

$$\begin{aligned}
&\mathfrak{D}_z^{n+\varsigma}\left[\mathrm{E}_{\gamma,\beta}(z)\right] \\
&\quad\equiv \frac{d^n}{dz^n}\left\{\mathfrak{D}_z^{\varsigma}\left[\mathrm{E}_{\gamma,\beta}(z)\right]\right\} \\
&\quad\equiv \mathfrak{D}_z^{n+\varsigma}\left[\sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\Gamma(\gamma\ell+\beta)}\right] \qquad (29)\\
&\quad= \sum_{\ell=0}^{\infty}\left\{\frac{1}{\Gamma(\gamma\ell+\beta)}\mathfrak{D}_z^{n+\varsigma}\left[z^{\ell}\right]\right\} \\
&\quad= \sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1)}{\Gamma(\gamma\ell+\beta)\Gamma(\ell-n-\varsigma+1)}z^{\ell-n-\varsigma},
\end{aligned}$$

where $n \leq \varsigma < n+1$ and for all $n \in \mathbb{N}_0$. In the same time, for all $n \in \mathbb{N}$, the result (28) also shows that

$$\begin{aligned}
&\mathfrak{D}_z^{n+0}\left[\mathrm{E}_{\gamma,\beta}(z)\right] \\
&\quad\equiv \mathfrak{D}_z^{n}\left[\mathrm{E}_{\gamma,\beta}(z)\right] \\
&\quad\equiv \frac{d^n}{dz^n}\left\{\mathfrak{D}_z^{0}\left[\mathrm{E}_{\gamma,\beta}(z)\right]\right\} \\
&\quad= \frac{d^n}{dz^n}\left\{\mathrm{E}_{\gamma,\beta}(z)\right\} \qquad (30)\\
&\quad= \frac{d^n}{dz^n}\left\{\mathrm{E}_{\gamma,\gamma+\beta-1}(z)\right\} \\
&\qquad\qquad -(\beta-1)\frac{d^{n-1}}{dz^{n-1}}\left\{\mathrm{E}_{\gamma,\gamma+\beta}(z)\right\},
\end{aligned}$$

which immediately yields that

$$\begin{aligned}
&\gamma z\mathfrak{D}_z^{1}\left[\mathrm{E}_{\gamma,\beta}(z)\right] \\
&\quad\equiv \gamma z\frac{d}{dz}\left\{\mathrm{E}_{\gamma,\beta}(z)\right\} \qquad (31)\\
&\quad= \mathrm{E}_{\gamma,\beta-1}(z)-(\beta-1)\mathrm{E}_{\gamma,\gamma+\beta}(z)
\end{aligned}$$

when $n := 1$.

3 Concluding remarks

As is well known, this scientific document initially provided detailed information on fractional-order calculus and special functions, which constitute the core focus of our investigation. Subsequently, we explored various specific definitions and functions related to these topics and derived several results, including the fractional-order integrals and derivatives of the corresponding functions. These results involve particular parameters, each carrying different implications that may be connected to special identities or special elementary functions, and are primarily associated with the relationships discussed in essential equations constituted in (29)-(31).

Naturally, a thorough examination of these key results may lead to additional (special) implications or findings, which are not presented here.

We leave the further investigation of such extensions to interested researchers, as our current focus shifts to more specific studies concerning the family of fractional differential equations (FDEs).

In light of the special information emphasized in the third section, under the necessary conditions, by considering the function Ω as a complex-type power series of the form:

$$\Omega := \Omega(z) = \sum_{\ell=0}^{\infty} c_{\ell}\, z^{\ell/m},$$

the favorable solution to the fractional-order-type differential equation:

$$\mathfrak{D}_z^{1}[\Omega] + \mathfrak{D}_z^{1/m}[\Omega] - \Omega = 0,$$

can also be determined for some $m = 2,3,4,\ldots$. This line of reasoning may lead to several notable relationships in the context of fractional-order differential equations. For more comprehensive researches and broader applications, attention can be directed toward the mentioned works cited in the references of this investigation that involve various fractional-order-type differential equations, [35].

References:

[1] N. N. Lebedev, Special Functions and Their Applications, translated by R. A. Silverman, New Jersey, Prentice-Hall Inc, 1965.

[2] R. Beals, R. Wong, Special Functions: A Graduate Text, Cambridge University Press, 2010. Tables, New York, Dover, 1972.

[3] L. C. Andrews, Special Functions for Engineers and Applied Mathematics, New York, MacMillan Comp., 1984.

[4] A. M. Mathai, A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Clarendon Press, Oxford, UK, 1993.

[5] G. E. Andrews, R. Askey and and R. R. Ranjan, Special functions, Encyclopedia of

Mathematics and its Applications, Vol. 71. Cambridge University Press, 1999.

[6] A. M. Mathai and H. J. Haubold, Special Functions for Applied Scientists, Springer, New York, NY, USA, 2008.

[7] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications: Theory and Applications, Springer Monographs in Mathematics, Berlin, Heidelberg: Springer Berlin, Heidelberg, 2014.

[8] R. Gorenflo, J. Loutchko and Y. Luchko, Computation of the Mittag-Leffler function and its derivatives, Fract. Calc. Appl. Anal., Vol. 5, 2002, pp. 491–518.

[9] K. B. Oldham, The Fractional Calculus, Mathematics in Science and Engineering, vol.111, Academic Press, New York, 1974.

[10] I. Podlubny, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering. Academic Press, 1998.

[11] H. Irmak, Some operators of fractional calculus and their applications regarding various complex functions analytic in certain domains, Studia Universitatis Babeş-Bolyai Mathematica, Vol. 69, No. 3, 2024, pp. 471-482.

[12] K. Diethelm, The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type, Springer, New York, 2010.

[13] D. Valério, J. T. Machado, V. Kiryakova, Some pioneers of the applications of fractional calculus, Fractional Calculus and Applied Analysis, Vol. 7, No. 2, 2014, pp. 552-578.

[14] H. Pratap, S. Kumar and G. Singh, Brief History of Fractional Calculus: A Survey, Migration Letters, Vol. 21, No. S7, 2024, pp. 238-243.

[15] K. S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[16] G. Samko, A. A. Kılbas, O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.

[17] E. C. de Oliveira, J. A. Tenreiro Machado, A Review of Definitions for Fractional Derivatives and Integral, Mathematical Problems in Engineering, vol. 2014, Article ID 238459.

[18] Z. Li, L. Liu, S. Dehghan, Y. Chen, D. Xue, A review and evaluation of numerical tools for fractional calculus and fractional order controls, Int. J. Control., Vol. 90, 2017, pp. 1165-1181.

[19] H. M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent development, Kyungpook Mathematical Journal, Vol. 60, No. 1, 2020, pp. 73-116.

[20] M. G. Mittag-Leffler, Sur la nouvelle fonction $E(x)$, C. R. Acad. Sci. Paris, Vol. 137, 1903, pp. 554-558.

[21] A. Wiman, Über den Fundamentalsatz in der Teorie [sic] der Funktionen, Acta Math., Vol. 29, 1905, pp. 191-201.

[22] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, Bull. Amer. Math. Soc., Vol. 54, 1948, pp. 1115-1116.

[23] M. Abramowitz, I. A. Milton, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, New York, Dover, 1972.

[24] H. Irmak, A Special Note According to Possible Applications of Fractional-Order Calculus for Various Special Functions, *WSEAS Transactions on Advances in Engineering Education,* Vol. 21, 2024, pp. 122-127. https://doi.org/10.37394/232010.2024.21.15

[25] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

[26] H. Irmak, P. Agarwal and R. P. Agarwal, The complex error functions and

various extensive results together with implications pertaining to certain special functions, Turkish Journal of Mathematics, Vol. 46, No. 2, 2022, pp. 662-667.
[27] W. Rudin, Real and Complex Analysis, McGraw Hill., Boston, 1987.
[28] I. N. Sneddon, *The use of Integral Transforms,* MacGraw-Hill, International Editions, New York, 1972.
[29] S. Owa, On the distortion theorems. I, Kyungpook Mathematical Journal., Vol. 18, 1978, pp. 53-59.
[30] A. S. Irmak and H. Irmak, A different perspective on undeniable roles of some well-defined special functions in various multidisciplinary applications, Equations, Vol. 4, 2024, pp. 76-82. https://doi.org/10.37394/232021.2024.4.9
[31] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Springer, Berlin, 2013.
[32] D. Cafagna, Past and present - fractional calculus: A mathematical tool from the past for present engineers, *IEEE Industrial Electronics Magazine*, Vol. 1, No. 2, 2007, pp. 35-40.
[33] A. Poularikas, Transforms and Applications Handbook, Boca Raton, FL, CRC Press, 2010.
[34] M.-P. Chen, H. Irmak and H. M. Srivastava, Some Families of Multivalently Analytic Functions with Negative Coefficients, Vol. 214, No. 2, 1997, pp. 674-690.
[35] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, New York, Wiley, 1993.

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Conflict of Interest

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