Problem of Determining the Density of Sources in a Multidimensional Heat Equation with the Caputo Time Fractional Derivative

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Abstract: - In this paper, we propose a new formula for representing the solution of the third initial-boundary value problem for multidimensional fractional heat equation with the Caputo derivative. This formula is obtained by the continuation method used in the theory of partial differential equations with integer derivatives. The Green's function of the problem is also constructed in terms of the Fox $H-$ function. Involving the results of solving a direct problem and the overdetermination condition, a uniqueness theorem for the definition of the spatial part of the multidimensional source function is proved.

Key-Words: - Fractional heat equation, Robin boundary condition, Green's function, Fox $H-$ function, fundamental solution, direct problem, inverse problem, uniqueness.

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1 Formulation of the Problems

We consider the following multidimensional fractional heat equation in a half space:

$$\partial_t^\alpha u - \Delta u = g(x,y,t), \quad t>0, \quad (x,y)\in \mathbb{R}_+^{n+1}, \tag{1}$$

the solution of which satisfies the initial condition

$$u(x,y,0)=\varphi(x,y),\ (x,y)\in \mathbb{R}_+^{n+1}\cup\{0\} \tag{2}$$

and the boundary condition

$$u_y(x,0,t)-hu(x,0,t)=0, \quad \{t\ge 0, \quad x\in\mathbb{R}^n\cup\{0\}\}, \tag{3}$$

where $\mathbb{R}_+^{n+1}=\{(x,y)=(x_1,x_2,\cdots,x_n,y)\in \mathbb{R}^{n+1}|y>0\}$, the Caputo fractional differential operator ∂_t^α of the order $0<\alpha<1$ is defined by [1, pp. 90-99]:

$$\partial_t^\alpha u(x,y,t) := I_{0+}^{1-\alpha}u_t(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,y,\tau)}{(t-\tau)^\alpha}d\tau,$$

$$\partial_t^1 u(x,y,t)=u_t(x,y,t),$$

$$I_{0+}^\alpha u(x,y,t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(x,y,\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$

$I_{0+}^\alpha u(x,y,t)$ is the Riemann–Liouville fractional integral of the function $u(x,y,t)$ with respect to t, Δ is the Laplace operator concerning the variables x,y and h is a given finite number.

For the given functions $g(x,y,t)$, $\varphi(x,y)$ the problem of finding the solution to the initial - boundary problem (1) - (3) will be called the *direct problem.* A *regular solution* of this problem consists of determining the function $u(x,y,t)$, such that

1) $u(x,y,t)$ is twice continuously differentiable in x,y for each $t>0$;
2) for each $(x,y)\in\mathbb{R}_+^{n+1}$ function $u(x,y,t)$ is continuous in t for $t>0$ and its fractional integral
$$(I_{0+}^{1-\alpha}u)(x,y,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{u(x,y,\tau)d\tau}{(t-\tau)^\alpha} \tag{4}$$
is continuously differentiable in t for $t>0$;
3) $u(x,y,t)$ satisfies (1)- (3) in classical sense.

In *inverse problem*, assuming $g(x,y,t)=f(x,y)\phi(t)$ on the right side of (1), where $\phi(t)$ is a known function, we are interested in finding the function $f(x,y)$ $(x,y)\in\mathbb{R}_+^{n+1}$, if the solution to problem (1)-(3) satisfies the following overdetermination condition:

$$u(x,0,t)=F(x,t),\ t>0,\ x\in\mathbb{R}^n, \tag{5}$$

$F(x, t)$ is a given function.

At the present time, fractional PDEs have been found as useful and applicable tools in applied sciences. The processes of heat transfer and diffusion phenomena in media with a fractal structure are called anomalous, [1], [2]. The mathematical modeling for describing transport processes in such media is well developed based on fractional calculus, [1], [2], [3], [4].

In the literature, inverse problems for classical second order differential equations of parabolic type have been studied quite deeply. There are the linear inverse source and nonlinear inverse coefficient problems for heat equations with different types of the initial and initial-boundary problems (direct problems) and over-determination conditions, [5]. In this direction we note that the works, [6], [7], [8], are concerned with inverse memory recovery problems in parabolic integro-differential equations of the second order with integral terms of convolution type. In, [9], [10], there were proven that if the kernel of convolution integral in a classical integro–differential diffusion equation coincides with the two parametric Mittag–Leffler function of the special argument then this equation describes the anomalously diffusive transport of solute in heterogeneous porous media, [11]. The methods for solving various initial-boundary value problems for differential equations with fractional time derivatives in the sense of Riemann-Liouville and Caputo using functions of the Mittag-Leffler type are given in the well-known monograph, [3], and article, [12].

In recent years, fractional differential equations have attracted much attention and some analytical methods for solutions of the initial and initial-boundary problems for such equations have been proposed, [13], [14], [15], [16], [17], [18], [19], [20]. In works, [21], [22], the author obtained the exact solution of the fractional diffusion equation in half-space with the Dirichlet boundary condition. In, [23], the fractional diffusion equation in half-space was subject to the homogeneous Dirichlet boundary condition and the homogeneous Neumann boundary condition. The fractional diffusion equation in half-space with the Robin boundary condition was considered in, [24]. Using the integral transform methods, including the Laplace transform and the Fourier transform it was obtained the exact solution of the problem.

Among the inverse problems for the fractional diffusion equation with Riemann-Liouville and Caputo type derivatives, the very common are inverse source problems with different over-determination conditions, [25], [26], [27], [28], [29], [30], [31], and the literature in them). In the work, [29], there also were only obtained the uniqueness theorem for the inverse problem of determining the various time-independent smooth coefficients appearing in time fractional diffusion equations, from measurements of the solution on a certain subset at fixed time.

The inverse problem discussed here can be treated as that of determining the density of heat sources, which is described by the function $f(x, y)$, acting in the semispace $y \geq 0$. This problem has a very definite physical sense in applications: if the function ϕ to take as $\phi(t) = e^{-\lambda t}$, it is associated with problem of finding the density of radioactive heat sources by the thermal radiation on the Earth's surface (condition (5)). In this case the number λ defines the half-life of a radioactive element.

In this work, we construct the exact solution of the multidimensional fractional heat (diffusion) equation in half space with Robin type boundary conditions. This formula is obtained by the continuation method used in the theory of PDEs with integer derivatives. The Green's function of the problem is constructed in terms of the Fox $H-$ function, [32]. Using these results we prove the uniqueness of the solution to the inverse problem.

2 Cauchy Problem and Auxiliary Lemma

In the beginning, we will deal with determining a solution to the following Cauchy problem:

$$\partial_t^\alpha v - \Delta v = g(x, y, t), \quad t > 0,\ (x, y) \in \mathbb{R}^{n+1}, \tag{6}$$

$$v(x, y, 0) = \varphi(x, y), \quad (x, y) \in \mathbb{R}^{n+1}. \tag{7}$$

The solution to the problem (6) and (7) is determined by the formula, [17], [33]:

$$v(x,t) = \int_{\mathbb{R}^{n+1}} G_{\alpha,1}(x-\xi, y-\eta, t)\varphi(\xi,\eta)\,d\xi d\eta + \int_0^t \int_{\mathbb{R}} G_{\alpha,\alpha}(x-\xi, y-\eta, t-\tau) \cdot g(\xi,\eta,\tau)\,d\xi d\eta d\tau, \quad t > 0,\ (x, y) \in \mathbb{R}^{n+1}, \tag{8}$$

where $d\xi = d\xi_1 \cdots d\xi_n$,

$$G_{\alpha,\beta}(x, y, t) =$$

$$\frac{t^{\beta-1}}{\left(\sqrt{\pi(|x|^2+y^2)}\right)^{n+1}} H_{1,2}^{2,0}\left[\frac{|x|^2+y^2}{4t^\alpha}\middle|_{\left(\frac{n+1}{2},1\right),(1,1)}^{(\beta,\alpha)}\right] \quad (9)$$

is the fundamental solution of the fractional diffusion operator $\partial_t^\alpha - \Delta$, $H_{p,q}^{m,n}\left[z\middle|_{(b_j,B_j)_1^q}^{(a_j,A_j)_1^p}\right]$ is the generalized hypergeometric $H-$ function (Fox $H-$ function). For the definition and properties of this function, [29].

The function $G_{\alpha,\beta}$ is infinitely differentiable at $(x,y) \neq 0$ for $t>0$. The regularity of the function $G_{\alpha,\beta}$ and some of its derivatives at $(x,y) \neq 0$ is determined by the regularity of the $H-$ function and its derivatives. The appearance of singularities of the fundamental solution and its derivatives for $(x,y)=0$ is an essential difference between the fractional diffusion equation and the classical equations of parabolic type.

From the asymptotic behavior of the $H-$ function for large values of the argument and the formulas of differentiation $H-$ functions, [17], [19], [21], [32].

if $|x|^2+y^2 \geq t^\alpha$, then

$$\left|D_{x,y}^m G_{\alpha,\beta}(x,t)\right| \leq C t^{\beta-1-\frac{(n+m+1)\alpha}{2}} \cdot$$
$$\cdot \exp\left(-\sigma t^{-\frac{\alpha}{2-\alpha}}(|x|^2+y^2)^{\frac{1}{2-\alpha}}\right) \quad |m| \leq 2. \quad (10)$$

Here the letters C, σ denote various positive constants and $D_{x,y}^m := \frac{\partial^{|m|}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n} \partial y^{m_0}}$, $|m| = m_0 + m_1 + \cdots + m_n$.

The following statement is true:

Lemma 1. *Let the function $\Phi(x,y)$ be defined and have bounded derivatives up to the 2 order inclusive, for $(x,y) \in \mathbb{R}^{n+1}$, and a linear combination $\sum_{k=0}^2 a_k \frac{\partial^k \Phi}{\partial y}(x,y)$, where $a_k = const$, is odd with respect to the point $y=0$ for a fixed $x \in \mathbb{R}^n$. Then the function*

$$u(x,y,t) = $$
$$= \int_{\mathbb{R}^{n+1}} G_{\alpha,\beta}(x-\xi, y-\eta, t)\Phi(\xi,\eta)d\xi d\eta \quad (11)$$

satisfies the condition

$$\sum_{k=0}^{2} a_k \frac{\partial^k u(x,y,t)}{\partial y^k}\Big|_{y=0} = 0. \quad (12)$$

To **prove** this, we note that the following equalities are true:

$$\frac{\partial^k G_{\alpha,\beta}}{\partial y^k}(x-\xi, y-\eta, t) =$$
$$(-1)^k \frac{\partial^k G_{\alpha,\beta}}{\partial \eta^k}(x-\xi, y-\eta, t).$$

In view of (10) and the conditions of the lemma, one can differentiate (11) 2 times under the integral sign. As a result, we get:

$$\sum_{k=0}^{n} a_k \frac{\partial^k u(x,y,t)}{\partial y^k} =$$
$$\sum_{k=0}^{n} a_k \int_{\mathbb{R}^{n+1}} (-1)^k \frac{\partial^k G_{\alpha,\beta}}{\partial y^k}(x-\xi, y-\eta, t) \cdot$$
$$\cdot \Phi(\xi,\eta) d\xi d\eta.$$

Integrating by parts on the right side of this equality, we “throw over” the derivatives $\frac{\partial^k}{\partial \eta^k}$ to the function $\Phi(\xi,\eta)$. At the same time, taking into account that the non-integral terms vanish in accordance with (10), we obtain:

$$\sum_{k=0}^{2} a_k \frac{\partial^k u(x,y,t)}{\partial y^k} = \int_{\mathbb{R}^{n+1}} G_{\alpha,\beta}(x-\xi, y-\eta, t)$$
$$\cdot \sum_{k=0}^{2} a_k \frac{\partial^k \Phi(\xi,\eta)}{\partial \eta^k} d\xi d\eta.$$

Under the conditions of lemma the integrand in the last formula is odd with respect to $\eta=0$ at $y=0$. Therefore, the relation (12) is valid.

3 Solution of the Direct Problem

Lemma 1 allows us to solve the problem for the homogeneous heat equation:

$$\partial_t^\alpha u - \Delta u = 0,\ t>0,\ (x,y) \in \mathbb{R}_+^{n+1}, \quad (13)$$

with the initial condition:

$$u(x,y,0) = \varphi(x,y), \ \ (x,y) \in \mathbb{R}_+^{n+1}, \quad (14)$$

and a homogeneous boundary condition of the form

$$\sum_{k=0}^{2} a_k \frac{\partial^k u(x,y,t)}{\partial y^k}\Big|_{y=0} = 0. \quad (15)$$

To do this, we continue the function $\varphi(x,y)$ for $y<0$, defining a new function $\Phi(x,y)$, which satisfies conditions

$$\Phi(x,y) = \begin{cases} \varphi(x,y) \ \text{ at } \ y \geq 0, \\ \sum_{k=0}^{2} a_k \dfrac{\partial^k \Phi(x,y)}{\partial y^k} = -\sum_{k=0}^{2} a_k \dfrac{\partial^k \varphi^k(x,s)}{\partial s^k}|_{s=-y} \ \text{ at } \ y \leq 0 \end{cases} \tag{16}$$

and is continuous together with derivatives up to the 2 −th order inclusive on $\mathbb{R}$ at every fixed $x \in \mathbb{R}^n$.

We note that in the case of the homogeneous boundary condition (3), in the formula (16) there will be $a_0 = -h$, $a_1 = 1$ and $a_2 = 0$.

Suppose that the function $\varphi(x,y)$ satisfies the matching condition: $\varphi_y(x,0) - h\varphi(x,0) = 0$. According to Lemma 1, it is necessary to continue the function $\varphi(x,y)$ for $y \leq 0$ in such a way that the function $\Phi_y(x,y) - h\Phi(x,y)$ (at every fixed $x \in \mathbb{R}^n$) is odd with respect to y, where $\Phi(x,y)$ is the continuation of the function $\varphi(x,y)$ on $\mathbb{R}^{n+1}$. Obviously $\Phi(x,y) = \varphi(x,y)$ for $(x,y) \in \mathbb{R}^{n+1}_+$. For determining the function $\Phi(x,y)$ for $y < 0$ we obtain the Cauchy problem for the following differential equation:

$$\begin{cases} \Phi_y(x,y) - h\Phi(x,y) = \lambda(x,y) \quad y < 0, \\ \Phi(x,0) = \varphi(x,0), \end{cases}$$

where $\lambda(x,y) := -\varphi_y(x,-y) + h\varphi(x,-y)$.
Solving this problem, we find $\Phi(x,y)$ for $y \leq 0$

$$\Phi(x,y) = \varphi(x,-y) + 2h\int_0^y e^{h(y-z)}\,\varphi(x,-z)dz.$$

Thus, the $\varphi(x,y)$ function continues as follows:

$$\Phi(x,y) = \begin{cases} \varphi(x,y) \quad y \geq 0, \\ \varphi(x,-y) + 2h\displaystyle\int_0^y e^{h(y-z)}\,\varphi(x,-z)dz, \quad y \leq 0. \end{cases} \tag{17}$$

Writing now the solution to the problem

$$\partial_t^\alpha u - \Delta u = 0,\ t > 0,\ (x,y) \in \mathbb{R}^{n+1},$$
$$u(x,y,0) = \Phi(x,y),\ \ (x,y) \in \mathbb{R}^{n+1},$$

in the form of an analog of formula (8), where the function $\Phi(x,y)$ is determined by the formula (17):

$$u(x,y,t) = \int_{\mathbb{R}^{n+1}} G_{\alpha,1}(x-\xi, y-\eta, t)\Phi(\xi,\eta)d\xi d\eta$$

$$= \int_0^\infty \int_{\mathbb{R}^n} G_{\alpha,1}(x-\xi, y-\eta, t)\varphi(\xi,\eta)d\xi d\eta$$
$$+ \int_{-\infty}^0 \int_{\mathbb{R}^n} G_{\alpha,1}(x-\xi, y-\eta, t)\varphi(\xi,-\eta)d\xi d\eta$$
$$+2h\int_{-\infty}^0 \int_{\mathbb{R}^n} G_{\alpha,1}(x-\xi, y-\eta, t)\cdot$$
$$\cdot\int_0^\eta e^{h(\eta-z)}\,\varphi(\xi,-z)dz d\xi d\eta,$$

after transformations, we get

$$u(x,y,t) = \int_0^\infty \int_{\mathbb{R}^n} [G_{\alpha,1}(x-\xi, y-\eta, t) + G_{\alpha,1}(x-\xi, y+\eta, t)$$
$$-2h\int_0^\infty G_{\alpha,1}(x-\xi, y+\eta+z, t)e^{-hz}dz]$$
$$\cdot\varphi(\xi,\eta)d\xi d\eta.$$

As a result, we obtain an expression for the Green's function of the Robin problem for the fractional heat equation on the half-line:

$$G^R_{\alpha,1}(x-\xi, y, \eta, t) =$$
$$G_{\alpha,1}(x-\xi, y-\eta, t) + G_{\alpha,1}(x-\xi, y+\eta, t)$$
$$-2h\int_0^\infty G_{\alpha,1}(x-\xi, y+\eta+z, t)e^{-hz}dz.$$

or, taking into account the formula (9)

$$G_{\alpha,1}^{R}(x-\xi,y,\eta,t) = \frac{1}{\left(\sqrt{\pi(|x-\xi|^2+(y-\eta)^2)}\right)^{n+1}} H_{1,2}^{2,0}\left[\frac{|x-\xi|^2+(y-\eta)^2}{4t^\alpha}\middle|_{\left(\frac{n+1}{2},1\right),(1,1)}^{(1,\alpha)}\right] + \frac{1}{\left(\sqrt{\pi(|x-\xi|^2+(y+\eta)^2)}\right)^{n+1}} H_{1,2}^{2,0}\left[\frac{|x-\xi|^2+(y+\eta)^2}{4t^\alpha}\middle|_{\left(\frac{n+1}{2},1\right),(1,1)}^{(1,\alpha)}\right] -2h\int_0^\infty \frac{1}{\left(\sqrt{\pi(|x-\xi|^2+(y+\eta+z)^2)}\right)^{n+1}} H_{1,2}^{2,0}\left[\frac{|x-\xi|^2+(y+\eta+z)^2}{4t^\alpha}\middle|_{\left(\frac{n+1}{2},1\right),(1,1)}^{(1,\alpha)}\right] e^{-hz}dz. \tag{18}$$

Using the Duhamel's principle the constructed above Green's function $G_{\alpha,1}^{D}(x-\xi,y,\eta,t)$, one can find a solution to the problem (1)-(3) (on the Duhamel's principle for the fractional diffusion equation see, for example, [34]):

$$u(x,y,t) = \int_0^\infty G_{\alpha,1}^{R}(x-\xi,y,\eta,t)\varphi(\xi)d\xi d\eta + \int_0^t\int_0^\infty (t-\tau)^{\alpha-1} G_{\alpha,\alpha}^{R}(x-\xi,y,\eta,t-\tau) \cdot g(\xi,\eta,\tau)d\xi d\eta d\tau. \tag{19}$$

where

$$G_{\alpha,\alpha}^{R}(x-\xi,y,\eta,t-\tau) = (t-\tau)^{\beta-1}G_{\alpha,1}^{R}(x-\xi,y,\eta,t-\tau). \tag{20}$$

4 Particular Case of the Green's Function

Representing the $H-$ function by means of a Mellin-Barnes type integral in the following form, [32]:

$$H_{1,2}^{2,0}\left[\frac{|x|^2+y^2}{4t^\alpha}\middle|_{\left(\frac{n+1}{2},1\right),(1,1)}^{(\beta,\alpha)}\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{n+1}{2}+s\right)\Gamma(1+s)}{\Gamma(\beta+\alpha s)}\left(\frac{|x|^2+y^2}{4t^\alpha}\right)^{-s}ds$$

with $\sigma\in(-\alpha,1)$, the fundamental solution (9) can be rewritten as

$$G_{\alpha,\beta}(x,t) = \frac{t^{\beta-1}}{\left(\sqrt{\pi(|x|^2+y^2)}\right)^{n+1}} \cdot \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{n+1}{2}+s\right)\Gamma(1+s)}{\Gamma(\beta+\alpha s)}\left(\frac{|x|^2+y^2}{4t^\alpha}\right)^{-s}ds. \tag{21}$$

In particular, $\alpha=\beta=1$ (classical heat equation) the representation (21) takes the form

$$G_{1,1}(x,y,t) = \frac{1}{\left(\sqrt{\pi(|x|^2+y^2)}\right)^{n+1}} \cdot \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma\left(\frac{n+1}{2}+s\right)\left(\frac{|x|^2+y^2}{4t}\right)^{-s}ds = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}} \cdot \frac{1}{2\pi i}\int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \Gamma(s)\left(\frac{|x|^2+y^2}{4t}\right)^{-s}ds,\ \sigma^*>0.$$

The contour of integration in the integral of the last formula can be transformed to the loop $L_{-\infty}$, which is started and ended at $-\infty$, encircling all poles $s_j=-j$, $j=0,1,2,\cdots$ of the function $\Gamma(s)$. In view of the Jordan lemma, the Cauchy residue theorem and the formula $res_{s=-j}\Gamma(s) = (-1)^j/(j!)$, $j=0,1,2,\cdots$, we get the following equality:

$$G_{1,1}(x,y,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}} \cdot \frac{1}{2\pi i}\int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \Gamma(s)\left(\frac{|x|^2+y^2}{4t}\right)^{-s}ds = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{|x|^2+y^2}{4t}\right)^{-j}.$$

Thus the fundamental solution $G_{1,1}$ to the heat equation takes its classical form

$$G_{1,1}(x,y,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{|x|^2+y^2}{4t}\right)^{-j} = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}}\exp\left(-\frac{|x|^2+y^2}{4t}\right).$$

Then, as follow from the formula (20) the Green function of the problem (1)-(3) in the case of $\alpha=1$ (i.e. (1) is a multidimensional classical inhomogeneous heat equation) has the following form:

$$G_{1,1}^R(x-\xi, y, \eta, t-\tau) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n+1}}[\exp\left(-\frac{|x-\xi|^2+(y-\eta)^2}{4(t-\tau)}\right) + \exp\left(-\frac{|x-\xi|^2+(y+\eta)^2}{4(t-\tau)}\right) - -2h\int_0^\infty \exp\left(-\frac{(|x-\xi|^2+(y+\eta+z)^2}{4(t-\tau)} - hz\right)dz].$$

5 Investigation of Inverse Problem

Let $g(x,y,t) = f(x,y)\phi(t)$. There holds the following statement about the uniqueness of the solution of the inverse problem.

Theorem. *Let $\phi(t) \neq 0,\ t \in [0,\infty)$, is a bounded function, $f(x,y)$ is a bounded function, having at every y a finite Fourier transform*

$$\tilde{f}(\nu, y) = \int_{\mathbb{R}^n} f(x,y)e^{i(\nu,x)}dx, \quad \nu = (\nu_1, \dots, \nu_n)$$

depending on y in a continuous way. In this case the function $f(x,y)$ is uniquely defined by the given function $F(x,y)$.

Proof. Under fulfilling the conditions of Theorem, the solution to the direct problem (1) - (3) accordance with the formula (19) on based of the estimates (10) satisfies the conditions of applicability the Fourier-Laplace transform:

$$\tilde{u}(\nu, y, s) = \int_{\mathbb{R}^n} dx \int_0^\infty u(x,y,t)e^{i(\nu,x)-st}dt, \quad Re(s) > 0.$$

After applying this transformation, the equations (1) - (3) and (5) are reduced to the form

$$\tilde{u}_{yy} - (s^\alpha + |\nu|^2)\tilde{u} + +s^{\alpha-1}\tilde{\varphi}(\nu,y) + \tilde{\phi}(s)\tilde{f}(\nu,y) = 0, \quad t>0,\ y>0,\ \left(\tilde{u}_y + h\tilde{u}\right)_{y=0} = 0, \tag{22}$$

$$\tilde{u}|_{y=0} = \tilde{F}(\nu, y), \quad t \geq 0, \tag{23}$$

where $\tilde{\phi}(s)$ is the Laplace transform of the function $\phi(t)$:

$$\tilde{\phi}(s) = \int_0^\infty \phi(t)e^{-st}dt, \quad Re(s) > 0.$$

At every fixed value of the parameters s and ν (22) is a boundary value problem for an ordinary differential equation with respect to y. A bounded solution to this problem can be easily constructed using the Green's function $G(y,\eta,\mu)$, $\mu = \sqrt{s^\alpha + |\nu|^2}$ for problem (22). With respect to the variable y this function is continuous and bounded on the segment $[0,\infty]$ and satisfies (in the generalized sense) the relations:

$$G_{yy} - \mu^2 G = \delta(y-\eta), \quad \left(G_y + hG\right)_{y=0} = 0, \quad G_y|_{y=\eta+0} - G_y|_{y=\eta-0} = 1,$$

where $\delta(\cdot)$ is Dirac's delta function.

One can easily show that the function $G(y,\eta,\mu)$ has the form

$$G(y,\eta,\mu) = \frac{1}{h-\mu}\begin{cases}\left(\cosh\mu y - h\dfrac{\sinh\mu y}{\mu}\right)e^{-\mu\eta}, & 0 \leq y \leq \eta, \\ \left(\cosh\mu\eta - h\dfrac{\sinh\mu\eta}{\mu}\right)e^{-\mu y}, & \eta \leq y.\end{cases} \tag{24}$$

In view of Green's function the solution to the inverse problem (22), (23) is written in the following form

$$\tilde{u}(\nu, y, s) = \int_0^\infty G(y,\eta,\mu)\left[s^{\alpha-1}\tilde{\varphi}(\nu,\eta) + \tilde{\phi}(s)\tilde{f}(\nu,\eta)\right]d\eta.$$

Setting here $y = 0$, we obtain the Laplas equation for $\tilde{f}(\nu, y)$

$$\tilde{F}(\nu, s) = \frac{1}{h - \sqrt{s^\alpha + |\nu|^2}} \int_0^\infty e^{-\eta\sqrt{s^\alpha+|\nu|^2}}\left[s^{\alpha-1}\tilde{\varphi}(\nu,\eta) + \tilde{\phi}(s)\tilde{f}(\nu,\eta)\right]d\eta, \quad Re(s) > 0. \tag{25}$$

In this equation ν is as a parameter. The function $\tilde{\phi}(s)$ being analytical in the domain $Re(s) > 0$, it can be zero only at isolated points. Therefore (25) can be rewritten in the following form

$$\int_0^\infty e^{-\mu y}\tilde{f}(\nu, y)dy = \Phi(\nu, \mu), \tag{26}$$

where the function

$$\Phi(\nu,\mu)=\frac{1}{\tilde{\phi}\left((\mu^2-|\nu|^2)^{\frac{1}{\alpha}}\right)}\cdot\left[(h-\mu)\tilde{F}\left(\nu,\left((\mu^2-|\nu|^2)^{1/\alpha}\right)\right)-\int_0^\infty e^{-\mu y}\,\tilde{\varphi}(\nu,y)dy\right],$$

is known in the domain $G(\nu)=\left\{\mu\colon Re\left((\mu^2-|\nu|^2)^{1/\alpha}\right)>0,\ Re(\mu)>0\right\}$.

The function $\Phi(\nu,\mu)$ as we can see from (26), at every fixed ν is a Laplace transform with respect to the variable y of the function $\tilde{f}(\nu,y)$. But the function $\tilde{f}(\nu,y)$ is uniquely determined by the Laplace transform values within the domain $G(\nu)$, for instance, it can be found by the formula, [35].

$$\tilde{f}(\nu,y)=\lim_{n\to\infty}\left\{\left[\frac{(-1)^n}{n!}s^{n+1}\frac{\partial^n}{\partial s^n}\Phi(\nu,s)\right]_{s=n/y}\right\}.$$

Since $\tilde{f}(\nu,y)$, $\nu\in\mathbb{R}^n$ uniquely defines $f(x,y)$, then Theorem is proven.

6 Conclusion

In this paper, the technique of the continuation method of the solution from the infinite axis was applied to derive an explicit solution to the third initial-boundary problems for multidimensional time-fractional heat equation with the Caputo fractional derivative of the order α $(0<\alpha<1)$. This formula for solution contains the Green's function Robin boundary condition. The Green's function of the problem is constructed in terms of the Fox $H-$ function, which is popular in the theory of fractional calculus. It is shown, the obtained formula coincides with the well-known formula for solving the corresponding problem for $\alpha\to 1-$. Based on the results of solving a direct problem and the overdetermination condition, a uniqueness theorem for the definition of the spatial part of the multidimensional source function is proved.

It is of interest both theoretically and practically to obtain exact formulas for solutions to the multidimensional time- and space-fractional diffusion-wave equations (in the case of the fractional Laplacian $(-\Delta)^{\alpha/2}$ with $0<\alpha\le 2$) in half-space with the Dirichlet, Neumann, Robin boundary conditions. So far, such problems are open.

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