

An extended Taylor–Aris Framework for Thermal Front Propagation in Conjugate Pipe Flows: The Plug-Flow Limit

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Abstract: - A reduced one-dimensional unsteady description of thermal-front propagation in conjugate pipe flows is developed here for the plug-flow limit. The starting point is the analytical advection–diffusion solution for an adiabatic fluid domain, with its characteristic exponential–error-function form. The plug-flow assumption provides a zero-shear reference case. Wall storage, finite-rate fluid–wall exchange, external heat release, and solid axial conduction can then be examined without the additional contribution of velocity-profile shear. This provides the first step of a broader reduction procedure in which the different mechanisms affecting thermal-front propagation can be introduced and identified separately. The reduced description is written in terms of the fluid bulk temperature. The model separates front motion, local spreading, and the thermal level of the upstream heated region, rather than combining all departures from the reference solution into a single effective dispersion coefficient. The corresponding closure quantities are identified from finite-element simulations of the full axisymmetric conjugate problem. Hydronic radiator columns provide a representative benchmark. The results confirm the importance of this separation. For the fluid-only adiabatic case, the reduced model recovers the expected advection–diffusion response. External heat exchange lowers the temperature of the already heated upstream region while producing little change in the local front transition. With a thermally active wall, the larger deformation of the bulk-temperature profile is primarily associated with transient energy storage in the solid and the resulting depletion of the upstream branch. Because velocity-profile shear is absent in the present configuration, these effects should not be interpreted as classical Taylor–Aris dispersion. The plug-flow case instead provides a controlled benchmark that isolates the conjugate contributions. It also provides the basis for the cross-sectional reduction and identification procedure. The same framework can then be extended to Poiseuille flow, where shear-induced Taylor–Aris dispersion enters together with the wall effects identified here. The resulting ROM offers a compact and physically interpretable alternative to full conjugate simulations for applications requiring repeated transient calculations.

Key-Words: - transient heat transfer; axial dispersion; effective velocity; laminar pipe flow; fluid–wall coupling; thermal front propagation; conjugate heat transfer; reduced-order modeling; plug flow.

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1 Introduction

Thermal transport in internal flows is still commonly treated, for engineering purposes, under steady-state conditions using semi-empirical heat-transfer correlations together with global energy balances. This approach is generally sufficient for design calculations and nominal operating conditions, but it becomes less satisfactory during start-up, when the thermal state of the wall evolves together with that of the fluid. Such transients matter whenever the time history of the system is relevant, as in dynamic performance assessment, control, and time-resolved predictions of comfort or process variables.

Transient internal flows with thermally active walls occur in many applications, including compact heat exchangers, thermal-storage devices, hydronic distribution networks, and heat emitters [1], [2]. Following a change in inlet temperature, thermal energy does not reach the entire system at once. A thermal front travels downstream while advection and diffusion in the fluid interact with heat exchange and storage in the surrounding solid [3], [4], [5]. Its evolution therefore reflects several mechanisms acting over comparable time and length scales.

Taylor–Aris dispersion provides a natural reference for describing the axial evolution of such

fronts. The study reported in [6] showed that, in laminar tube flow, transverse molecular diffusion combined with a non-uniform velocity profile produces an enhanced apparent axial dispersion. This result was later generalized in [7] using moment methods, showing how a multidimensional transport problem can be reduced to an effective one-dimensional description. The connection with the present work is primarily methodological. In the plug-flow case considered here, velocity-profile shear is absent, and classical Taylor–Aris shear dispersion therefore does not arise. What is retained is the idea of obtaining a one-dimensional closure from a multidimensional transport problem. This zero-shear case also provides a natural starting point for extension to parabolic velocity profiles, where the classical shear-induced contribution can be introduced explicitly.

A thermally active wall introduces a different mechanism. Rather than acting only as a boundary condition, the wall absorbs, stores, and subsequently releases energy. The evolution of the bulk-fluid temperature can therefore no longer be described solely by the mean fluid velocity and fluid thermal diffusivity. Wall heat capacity, finite-rate fluid–wall exchange, and delayed heat release modify the transient response and introduce a memory of the previous thermal history [2], [8], [9], [10]. Transient conjugate heat-transfer studies have likewise shown the importance of fluid–solid coupling and wall dynamics in internal-flow and heat-exchanger problems [11], [12].

These effects are particularly relevant before an asymptotic transport regime has been established. Classical Taylor–Aris results are associated with sufficiently long times for transverse diffusion to smooth the cross-sectional scalar distribution and for a constant effective axial diffusivity to emerge. At shorter times, the thermal front still reflects the initial and boundary conditions, and the effective description may itself remain time dependent [13], [14]. A conjugate system introduces an additional time scale because the wall requires a finite time to heat. Changes in front trajectory and in the temperature level behind the front may therefore occur before an asymptotic closure becomes appropriate.

The present study therefore starts from the plug-flow limit. The purpose is not to treat plug flow as a general approximation for laminar pipe flow, but to use it as a controlled zero-shear reference. Removing velocity-profile shear allows wall storage, finite-rate fluid–wall exchange, external heat release, and solid axial conduction to be examined without mixing them with classical shear-

induced dispersion. The fluid-only adiabatic problem provides the corresponding baseline: a one-dimensional advection–diffusion problem with the familiar error-function front. The conjugate cases can then be compared directly with this reference, introducing the additional mechanisms progressively.

This comparison also shows why a single effective dispersion coefficient is not sufficient to describe every departure from the reference solution. A decrease in temperature behind the front, for example, is physically different from a widening of the local transition region. The reduced representation therefore treats the upstream heated branch separately from the front itself. The apparent front trajectory, its local width, and the upstream thermal level are described by distinct closure quantities identified from the cross-sectionally averaged FEM response. This separation prevents wall storage or external heat release from being interpreted automatically as additional axial spreading.

From a model-reduction perspective, the formulation lies between fully distributed simulations and strongly lumped thermal models. Reduced-order approaches seek this type of compromise: retaining the variables and mechanisms needed to reproduce the dynamics of interest while avoiding the cost of resolving the complete field at every evaluation [15], [16]. Here, the reduction is built around the bulk-fluid temperature and closure quantities with a direct physical interpretation. The same identification procedure can later be applied to a parabolic velocity profile, allowing the conjugate effects isolated here to be combined with classical Taylor–Aris shear dispersion.

Hydronic radiator columns provide a useful benchmark for this purpose. Their nominal thermal performance is generally characterized under steady conditions using semi-empirical procedures such as EN 442 [17]. These procedures are appropriate for sizing but do not describe the early response following start-up or intermittent operation. This limitation is increasingly relevant in dynamic building simulation, demand-side flexibility, and advanced control applications [18], [19], [20]. Model-predictive-control studies, for example, have shown that radiator dynamics can influence both control performance and heating-system operation [21], [22]. During warm-up, the thermal response time of the emitter may become comparable with the fluid residence time inside its columns. Fluid transport, axial spreading, wall heating, and heat

release to the room can then evolve on interacting time scales.

Experimental and numerical studies have shown that such transient behavior cannot always be represented satisfactorily by steady-state or single-capacitance models. The study in [23] examined the interaction between heating devices, room airflow, and thermal comfort, while a transient panel-radiator model based on several thermal storage elements was developed in [24]. The main difficulty is spatial: temperature gradients develop along the fluid path and within the emitter, while the thermal mass of the solid is activated progressively rather than instantaneously. This behavior is particularly evident in high-inertia emitters such as cast-iron radiators, but it is relevant more generally whenever short-time dynamics affect comfort, control, or predicted energy demand [5].

Whole-building simulation environments such as EnergyPlus, TRNSYS, and Modelica-based tools provide computationally efficient representations of hydronic emitters, but they generally do not resolve the propagation of a thermal front and the associated pre-asymptotic conjugate dynamics [18], [19], [25], [26], [27]. At the other extreme, fully distributed Computational Fluid Dynamics or finite-element models can resolve these processes, but their spatial and temporal resolution makes routine use in whole-building simulations impractical [28], [29]. An intermediate description is therefore useful: one that retains the relevant transient physics without requiring a full distributed solution at every simulation step.

The present work develops such a description. A one-dimensional, fluid-centered ROM is calibrated against axisymmetric conjugate FEM simulations, with separate closures for the upstream thermal branch, the apparent front trajectory, and the local front width. The plug-flow configuration is used as the zero-shear reference, allowing wall storage, fluid-wall exchange, external heat release, and solid conduction to be identified independently of velocity-profile shear. The resulting framework is then naturally extended to Poiseuille flow, where these conjugate effects can be combined with classical Taylor-Aris dispersion.

2 Transient Modeling Framework

2.1 Physical Problem and Front-Propagation Stage

The system consists of a cylindrical duct of length L and internal diameter D , surrounded by a solid wall of thickness s . The axial coordinate is by X , while R

denotes the radial coordinate, with $0 \leq R \leq R_0 = D/2$ in the fluid domain and $R_0 \leq R \leq R_0 + s$ in the solid wall. The geometry is shown in Figure 1 (Appendix). The wall has finite heat capacity and thermal conductivity and therefore participates actively in the transient heat exchange. The system at ambient temperature T_{amb} at $t = 0$.

The transient starts when the inlet fluid temperature is changed abruptly. A thermal front then travels along the duct while exchanging heat with the surrounding wall. The front-propagation time is of the same order as the fluid residence time and can also be comparable with the characteristic times of wall heating and fluid-wall exchange. Under these conditions, a spatially lumped description is no longer adequate. Appreciable axial temperature gradients develop in the fluid, while the wall heats progressively as the front moves downstream.

At early times, the conjugate response is governed mainly by the axial propagation of the thermal disturbance. We refer to this interval as the front-propagation stage. For its description, the fluid domain is divided into three axial regions. The hot upstream region has already been reached by the incoming hot fluid. It is followed by the thermal-front region, where the temperature changes rapidly, and the largest axial gradients occur. Farther downstream, the fluid remains essentially undisturbed, with its temperature still close to the initial equilibrium value. The three regions are evidenced in Figure 1 (Appendix).

The surrounding solid wall introduces an additional thermal time scale. As the front moves downstream, successive portions of the wall absorb and store part of the energy carried by the fluid. This exchange can alter the temperature level behind the front and, in principle, its apparent motion and local shape. The reduced model developed below is used to determine whether these changes appear primarily as upstream depletion, a shift in front velocity, or local spreading.

2.2 Modeling Assumptions

The transient behavior of a conjugate pipe flow depends on axial transport and thermal diffusion in the fluid, but also on heat storage in the wall and on the underlying flow regime. In many of the applications considered here, including hydronic networks, compact heat exchangers, and small thermal devices, the characteristic flow velocities are relatively low. The corresponding Reynolds numbers are therefore generally small, and laminar flow provides the relevant hydrodynamic regime.

As a first step in the development of the extended Taylor–Aris framework, we consider the plug-flow limit, with a velocity field that is uniform across the section and constant along the pipe. This provides the zero-shear reference needed to separate conjugate thermal effects from those associated with velocity-profile shear. Hydrodynamic entrance effects and axial development of the velocity profile are therefore outside the scope of this limiting case.

When the velocity profile is non-uniform, the transverse temperature field becomes directly involved in axial transport, because diffusion across the section interacts with velocity-profile shear. A one-dimensional description must then account for this unresolved transverse dynamics through an appropriate closure, as in the classical Taylor–Aris reduction. The plug-flow case considered here removes this shear contribution altogether. It therefore serves as the zero-shear member of the broader framework, allowing wall inertia and finite-rate fluid–wall exchange to be examined before the additional shear–diffusion coupling is introduced.

Although the velocity field is prescribed and steady, the temperature field remains fully transient throughout the front-propagation stage. Its evolution reflects the inlet temperature change together with axial transport, wall heat storage, and finite-rate fluid–wall exchange. For this reason, the usual notion of a thermally fully developed profile is of limited relevance here. The transient response is instead controlled by how the fluid residence time compares with the characteristic times of wall diffusion and conjugate heat exchange.

In the plug-flow limit, the fluid-only reference problem is governed simply by axial advection and molecular thermal diffusion. This provides a clean baseline against which the effects introduced by wall coupling and external heat exchange can be assessed. Rather than assigning every departure from this baseline to a change in front velocity or spreading, the reduced model distinguishes changes in the front trajectory and local width from variations of the upstream thermal level. The resulting description is intended for the pre-asymptotic stage, while the thermal disturbance is still traveling through the system. It also provides the reference case from which velocity-profile shear can later be introduced through a parabolic flow field.

3 Governing Equations

3.1 Fluid Domain

The fluid is assumed as incompressible and Newtonian. Its properties are assumed constant and

evaluated at a representative reference temperature, taken as $(T_{\text{in}} + T_{\text{amb}})/2$.

The flow is assumed laminar and axisymmetric. Consistent with the plug-flow limit introduced in Section 2.2, the axial velocity is prescribed as uniform across the section and equal to the mean fluid velocity U_m . The velocity field is therefore free of shear, so that the thermal response considered here reflects the fluid–wall interaction without an additional contribution from velocity-profile dispersion.

At $t = 0$, the imposed change in inlet temperature generates a hot thermal front that propagates along the axial direction X . In the fluid region, the temperature field T_f is governed by:

$$\rho_f c_f \left(\frac{\partial T_f}{\partial t} + U_m \frac{\partial T_f}{\partial X} \right) = k_f \left[\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T_f}{\partial R} \right) + \frac{\partial^2 T_f}{\partial X^2} \right] \quad (1)$$

The initial condition is $T_f(X, R, 0) = T_{\text{amb}}$. At the inlet, the imposed temperature step is written as: $T_f(0, R, t) = T_{\text{amb}} + (T_{\text{in}} - T_{\text{amb}})H(t)$, where $H(t)$ is the Heaviside function.

Symmetry at the pipe axis requires

$$\left. \frac{\partial T_f}{\partial R} \right|_0 = 0 \quad (2)$$

At the outlet, a zero axial conductive-gradient condition is imposed,

$$\left. \frac{\partial T_f}{\partial X} \right|_L = 0 \quad (3)$$

which provides a natural outflow condition when axial conduction is not intended to determine the downstream temperature field.

At the fluid–wall interface, the continuity of temperature and heat flux requires:

$$T_f(X, R_0^-, t) = T_s(X, R_0^+, t) ;$$

$$k_f \left. \frac{\partial T_f}{\partial R} \right|_{R=R_0^-} = k_s \left. \frac{\partial T_s}{\partial R} \right|_{R=R_0^+} \quad (4)$$

where T_s is the local temperature of the solid wall.

3.2 Solid Domain

The solid wall is contained within the annular domain $R_0 < R < R_0 + s$. The two-dimensional heat conduction equation is:

$$\rho_s c_s \frac{\partial T_s}{\partial t} = k_s \left[\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T_s}{\partial R} \right) + \frac{\partial^2 T_s}{\partial X^2} \right] \quad (5)$$

The wall is at the ambient temperature, $T_s(X, R, 0) = T_{\text{amb}}$. At its axial ends, the wall is taken as adiabatic,

$$\left. \frac{\partial T_s}{\partial X} \right|_0 = 0; \quad \left. \frac{\partial T_s}{\partial X} \right|_L = 0 \quad (6)$$

At the outer surface, $R = R_0 + s$, heat is exchanged through an equivalent external heat-transfer coefficient h_o , which accounts for combined convective and radiative effects:

$$-k_s \left. \frac{\partial T_s}{\partial R} \right|_{R_0+s} = h_o (T_s(X, R_0 + s, t) - T_{amb}) \quad (7)$$

4 Model Reduction Framework

The full conjugate problem retains both axial and radial temperature variations in the fluid and the wall. A formal reduction could be based on an expansion in radial eigenfunctions. Convection, however, couples the resulting modes and complicates the formulation considerably. Since the quantities of interest are mainly governed by cross-sectional energy balances, retaining the full radial temperature field is not necessary. The reduced description is therefore based on cross-sectionally averaged variables.

This choice preserves the integral energy balance and the macroscopic quantities of interest, while keeping the model compact and physically interpretable.

4.1 Cross-sectional Averaging and Reduced Variables for the Fluid Domain

The reduction is now developed for the plug-flow reference introduced in Section 2.2. With shear-induced dispersion absent, the fluid-only problem is governed by advection at the mean velocity and by molecular thermal diffusion. Wall effects are then introduced as additional terms in the one-dimensional balance for the fluid bulk temperature.

4.1.1 Integration of the Fluid Energy Equation

We begin by integrating the fluid energy equation over the cross-section $A_f = \pi R_0^2$. Under axisymmetric conditions, with $dA = 2\pi R dR$, the cross-sectionally averaged energy balance is given by:

$$\begin{aligned} \rho_f c_f \frac{\partial}{\partial t} \int_{A_f} T_f dA + \rho_f c_f \frac{\partial}{\partial X} \int_{A_f} U T_f dA = \\ = k_f \int_{A_f} \frac{\partial^2 T_f}{\partial X^2} dA - P_i \dot{q}_i \end{aligned} \quad (8)$$

where $P_i = 2\pi R_0$ is the inner perimeter and

$$\dot{q}_i = -k_f \left. \frac{\partial T_f}{\partial R} \right|_{R_0} \quad (9)$$

is the interfacial heat flux, taken as positive from the fluid to the solid wall, at this stage still depending on the local fluid temperature gradient at the wall.

The integration naturally introduces two distinct temperatures, the area-averaged temperature T_A and the bulk temperature T_b :

$$\begin{aligned} T_A(X, t) &= \frac{\int_{A_f} T_f dA}{A_f} = \frac{2 \int_0^{R_0} R T_f dR}{R_0^2}; \\ T_b(X, t) &= \frac{\int_{A_f} \rho_f U T_f dA}{\dot{m}} = \frac{2 \int_0^{R_0} R U T_f dR}{U_m R_0^2} \end{aligned} \quad (10)$$

In general, these functions are different, so the integrated equation does not provide a closed one-dimensional description. In the plug-flow limit, however, $U = U_m$ and the two temperatures T_A and T_b coincide exactly. This removes the kinematic closure associated with velocity weighting, but the interfacial heat flux still requires a separate thermal closure.

4.1.2 One-dimensional Closure for the Fluid Domain

The one-dimensional fluid balance can now be written directly in terms of the bulk temperature. In the plug-flow limit, $U(R) = U_m$ and $T_A = T_b$:

$$T_A(X, t) = T_b(X, t) \quad (11)$$

The axial diffusion term follows the same reduction. Because the cross-section is uniform and the fluid properties are constant, axial differentiation can be interchanged with cross-sectional integration:

$$\int_{A_f} \frac{\partial^2 T_f}{\partial X^2} dA = \frac{\partial^2}{\partial X^2} \int_{A_f} T_f dA = \frac{\partial^2 T_b}{\partial X^2} A_f \quad (12)$$

Once the radial temperature field is removed from the reduced description, the wall heat flux can no longer be obtained directly from the local gradient. It is therefore closed through

$$\dot{q}_i = h_i (T_b - T_{s,w}) \quad (13)$$

where h_i is an internal heat-transfer coefficient and $T_{s,w}$ denotes the solid temperature at the fluid-wall interface.

With these assumptions, the averaged fluid balance becomes

$$\begin{aligned} \rho_f c_f A_f \frac{\partial T_b}{\partial t} + \rho_f c_f A_f U_m \frac{\partial T_b}{\partial X} = \\ = \rho_f c_f A_f \alpha_f \frac{\partial^2 T_b}{\partial X^2} - h_i P_i (T_b - T_{s,w}) \end{aligned} \quad (14)$$

The reduced fluid model is therefore written entirely in terms of the bulk temperature.

4.2 One-dimensional unsteady Formulation for the Solid Domain

A one-dimensional description is also introduced for the solid wall. We assume that temperature variations across the wall thickness are small compared with the axial variations associated with the traveling thermal front. The wall temperature can then be represented by a single axial field, $T_s(X, R, t) \approx T_s(X, t)$.

This approximation is appropriate when the wall is thin compared with the axial length scale and radial diffusion across its thickness is fast enough to keep each cross-section nearly isothermal during front propagation. A convenient measure of this condition is the Biot number based on the wall thickness,

$$Bi_{s,ref} = \frac{h_{ref}s}{k_s} \ll 1 \quad (15)$$

where h_{ref} is taken as the largest heat-transfer coefficient acting on the wall. When $Bi_{s,ref} \ll 1$, each wall cross-section can be treated as thermally lumped in the radial direction while axial conduction and axial temperature variations are still retained. The interface temperature in the fluid balance can therefore be identified with the reduced wall temperature, $T_{s,w}(X, t) \approx T_s(X, t)$.

Integrating the solid energy equation over the wall cross-section gives the one-dimensional wall balance per unit axial length:

$$\rho_s c_s A_s \frac{\partial T_s}{\partial t} = k_s A_s \frac{\partial^2 T_s}{\partial X^2} + h_i P_i (T_b - T_s) - h_o P_o (T_s - T_{amb}) \quad (16)$$

where

$$A_s = \pi[(R_0 + s)^2 - R_0^2]; \quad P_o = 2\pi(R_0 + s) \quad (17)$$

are, respectively, the wall cross-sectional area and the external perimeter.

The first exchange term, $h_i P_i (T_b - T_s)$, accounts for heat transferred between the fluid and the wall, while $h_o P_o (T_s - T_{amb})$ describes heat released from the outer wall surface to the surroundings.

Equation (16) has the form of an unsteady fin equation, but the physical setting is different from that of a conventional fin. Here, the wall is not driven by a prescribed base temperature; its temperature evolves together with the fluid and is also affected by heat exchange with the surroundings.

Equations (14) and (16) together define the one-dimensional conjugate model in terms of the fluid bulk temperature $T_b(X, t)$ and the wall temperature $T_s(X, t)$. The model retains axial transport in the fluid, axial conduction and thermal storage in the wall, and heat exchange with the surroundings, without resolving the complete axial-radial temperature field.

4.3 Non-dimensional Formulation of the Reduced Model

The dimensionless form of the reduced model is useful for comparing the time scales that control the start-up transient. In particular, it helps distinguish the roles of axial advection, thermal smoothing, wall storage, and fluid-wall heat exchange.

The axial and radial coordinates are scaled with the duct length L and the tube radius R_0 , respectively. Time is scaled with the fluid residence time, $t_{res} = L/U_m$, and the reference temperature difference is the inlet-to-ambient temperature difference, $\Delta T_{max} = T_{in} - T_{amb}$. Accordingly, the dimensionless variables are:

$$\begin{aligned} x &= X / L; \quad r = R / R_0; \quad \tau = t / t_{res}; \\ \theta_b &= (T_b - T_{amb}) / (T_{in} - T_{amb}); \\ \theta_s &= (T_s - T_{amb}) / (T_{in} - T_{amb}) \end{aligned} \quad (18)$$

4.3.1 The Fluid Domain and the Baseline Solution

Recalling the dimensionless variables introduced above, the reduced one-dimensional fluid energy equation (14) becomes

$$\frac{\partial \theta_b}{\partial \tau} + \frac{\partial \theta_b}{\partial x} = -\frac{t_{res}}{t_{c,i}} (\theta_b - \theta_s) + \frac{1}{Pe_f} \frac{\partial^2 \theta_b}{\partial x^2} \quad (19)$$

where $t_{c,i} = \rho_f c_f R_0 / (2h_i)$ is the characteristic time for fluid-wall heat exchange, $Pe_f = U_m L / \alpha_f$ is the longitudinal Peclet number.

Thus, by choosing the residence time as the temporal scale, the unsteady and advective terms appear at the same order of magnitude. The relative importance of axial diffusion is then measured by $1/Pe_f$, while the ratio $t_{res}/t_{c,i}$ measures how rapidly the fluid exchanges heat with the wall compared with the time needed to travel along the duct.

A useful reference solution is obtained by removing the wall-exchange term and imposing the same step change at the inlet. For sufficiently large Pe_f , the resulting advection-diffusion problem gives the familiar error-function front

$$\theta_b^{(0)}(x, \tau) = \frac{1}{2} \operatorname{erfc} \left(\frac{x - \tau}{2\sqrt{\tau / Pe_f}} \right) \quad (20)$$

Equivalently, in dimensional variables,

$$\frac{T_b^{(0)}(X, t) - T_{amb}}{T_m - T_{amb}} = \frac{1}{2} \operatorname{erfc} \left(\frac{X - U_m t}{2\sqrt{\alpha_f t}} \right) \quad (21)$$

This solution defines the fluid-only adiabatic baseline used throughout the reduction. The first-order spatial term sets the propagation speed of the front, while the second-order term determines its local width. In the plug-flow reference case, the front therefore travels at U_m and spreads only through the molecular thermal diffusivity α_f , with no attenuation of the upstream temperature level.

Once wall coupling or external heat exchange is introduced, the bulk-temperature profile can depart from the adiabatic reference in different ways. The upstream heated region may be attenuated or distorted, the apparent front position may shift, and the local transition may broaden or narrow. Since these effects have different physical origins, the reduced formulation introduced in Section 4.3.3 treats the upstream branch separately from the local erfc front.

4.3.2 The Solid Domain

The same scaling applies to the wall balance, exposing the time scales that govern its thermal response. Using the variables defined in Eq. (18), Eq. (16) becomes

$$\frac{\partial \theta_s}{\partial \tau} = \frac{t_{res}}{t_{k,s}} \frac{\partial^2 \theta_s}{\partial x^2} + \frac{t_{res}}{t_{i,s}} (\theta_b - \theta_s) - \frac{t_{res}}{t_{o,s}} \theta_s \quad (22)$$

where the relevant wall time scales are

$$t_{k,s} = \frac{L^2}{\alpha_s}; \quad t_{i,s} = \frac{\rho_s c_s (A_s / P_i)}{h_i}; \quad t_{o,s} = \frac{\rho_s c_s (A_s / P_o)}{h_o} \quad (23)$$

Here, $t_{i,s}$ represents the characteristic time for heat uptake from the fluid, $t_{o,s}$ the characteristic time for heat release to the surroundings, and $t_{k,s}$ the axial conduction time scale in the solid wall.

Their comparison with t_{res} indicates which of these processes can evolve significantly while the thermal front crosses the duct.

A simple limiting case is obtained when axial conduction along the wall is slow compared with front propagation, so that $t_{res}/t_{k,s} \ll 1$. Each axial location of the wall then responds essentially as a local first-order thermal system:

$$\frac{\partial \theta_s}{\partial \tau} = -(\beta_i + \beta_o) \theta_s + \beta_i \theta_b;$$

$$\beta_i = \frac{t_{res}}{t_{i,s}}; \quad \beta_o = \frac{t_{res}}{t_{o,s}} \quad (24)$$

For an arbitrary history of the local fluid bulk temperature, the solution can be written as

$$\theta_s(x, \tau) = \beta_i \int_0^\tau \exp[-(\beta_i + \beta_o)(\tau - \sigma)] \theta_b(x, \sigma) d\sigma \quad (25)$$

with $\theta_s(x, 0) = 0$. The wall therefore follows the fluid temperature with a finite response time rather than instantaneously; the characteristic relaxation rate is $(\beta_i + \beta_o)$. The meaning of this local thermal memory becomes clearer if we consider a step-like fluid-temperature signal at a fixed axial position.

To make this behavior explicit, consider a step-like change in the fluid bulk temperature at a fixed axial position,

$$\theta_b(x, \tau) = H(\tau - \tau_f(x)) \quad (26)$$

where $\tau_f(x)$ is the local arrival time of the thermal disturbance. The corresponding wall response is then:

$$\theta_s(x, \tau) = \frac{\beta_i}{\beta_i + \beta_o} H(\tau - \tau_f(x)) \cdot \{1 - \exp[-(\beta_i + \beta_o)(\tau - \tau_f(x))]\} \quad (27)$$

A useful feature of Eq. (27) is that the Heaviside factor is retained. The wall responds only after the thermal disturbance reaches a given axial position and remains unaffected ahead of the front. In the absence of axial conduction along the wall, this local thermal memory therefore does not generate an additional axial spreading mechanism or shift the front position. Instead, it changes the temperature level behind the front through the finite time required to heat the wall. Even when the outer surface is adiabatic, $\beta_o = 0$, the wall continues to absorb energy after the front has passed, lowering the temperature of the already heated upstream fluid. External heat release adds a further dissipative contribution to this effect.

When axial conduction along the wall is retained, this behavior changes. Heat stored in upstream portions of the solid can then diffuse downstream, reaching regions that have not yet been heated by the fluid front.

This limiting solution supports the separation used in the reduced model. Wall thermal memory mainly affects the upstream heated branch, while the local erfc transition remains controlled by fluid advection and diffusion unless an additional axial-smoothing mechanism is present.

4.3.3 Effective Transport Closure and Conjugate Parameters

The fluid-only solution of Section 4.3.1 provides the reference against which the conjugate cases are compared. Once wall coupling or external heat exchange is introduced, the bulk-temperature profile can no longer be represented by a single adiabatic error-function front.

Here, the expression “extended Taylor–Aris framework” refers to the reduction of the cross-sectional transport problem to a set of effective one-dimensional closures. In the present zero-shear case, these closures account only for the conjugate wall effects. The classical shear-induced contribution enters only when a non-uniform velocity profile is introduced.

The reference solution separates two features that respond differently to conjugate effects: the upstream region already reached by the thermal disturbance and the local front leading into the still-undisturbed fluid.

For a fluid-only system with external heat exchange, the upstream temperature decays approximately exponentially along the duct, as expected for a linear problem with distributed lateral losses. Once a thermally active wall is introduced, the upstream profile becomes more complex. Its shape then reflects both the gradual heating of the wall and the temporary redistribution of energy between the moving fluid and the stationary solid.

The limiting analysis of Section 4.3.2 helps explain this behavior. When axial conduction along the wall is negligible, each wall section responds only after the thermal front has reached that position. The wall therefore does not create an independent spreading of the front; its main effect is to lower the temperature of the already heated upstream fluid as energy is stored in the solid. If axial conduction is retained, part of this stored energy can also diffuse downstream through the wall, adding a separate axial-smoothing mechanism that is physically distinct from shear-induced Taylor–Aris dispersion.

The characteristic times introduced above also indicate when local fluid–wall quasi-equilibrium may be approached. In the applications considered here, internal heat exchange is generally faster than heat release to the surroundings, so that $t_{i,s} \ll t_{o,s}$. Local quasi-equilibrium between fluid and wall, however, requires a stronger condition: the exchange times on both sides of the interface must be short compared with the fluid residence time, i.e., $t_{c,i} \ll t_{res}$ and $t_{i,s} \ll t_{res}$.

The relevant time-scale ratios are evaluated for the present benchmark in Section 6.2. If both ratios

are sufficiently small, the temperature difference between fluid and wall relaxes much faster than the front propagates through the duct. In that limit, a local quasi-equilibrium approximation becomes reasonable:

$$\theta(x, \tau) \equiv \theta_b(x, \tau) \approx \theta_s(x, \tau) \quad (28)$$

This approximation is used only to derive limiting reference values for the effective transport parameters. It is not imposed in either the FEM model or the identification procedure, both of which retain the finite temperature difference between fluid and wall. The effective closures are therefore identified from the full pre-asymptotic response, including wall storage, axial smoothing, and external heat release.

The effective transport structure becomes clearer when the fluid and wall balances are combined. Let $C_f = \rho_f c_f A_f$ and $C_s = \rho_s c_s A_s$ denote their heat capacities per unit length, with $\chi = C_s/C_f$. Multiplying the wall balance by χ and adding it to the fluid balance eliminates the fluid–wall exchange terms, since they represent internal energy transfer between the two domains. Under the local quasi-equilibrium approximation, the combined balance becomes:

$$(1 + \chi) \frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial x} = \left(\frac{1}{Pe_f} + \chi \frac{t_{res}}{t_{k,s}} \right) \frac{\partial^2 \theta}{\partial x^2} - \chi \frac{t_{res}}{t_{o,s}} \theta \quad (29)$$

This equation has the same structure as a one-dimensional advection-diffusion equation with attenuation. In the local-equilibrium limit, a direct comparison of coefficients gives the following reference values:

$$U^{ref} \square U_m \frac{1}{1 + \chi}; \quad D^{ref} \square \frac{\alpha_f + \chi \alpha_s}{1 + \chi}; \quad t^{ref} \square t_{o,s} \frac{1 + \chi}{\chi} \quad (30)$$

The values in Eq. (30) provide limiting references rather than constraints on the calibrated ROM. Equation (31) retains the advection–diffusion–attenuation structure of Eq. (29), but replaces the limiting coefficients with effective quantities identified from the full FEM response. Accordingly, the effective fluid-centered reduced model is written in terms of the ROM bulk-temperature field $\theta^{ROM}(X, t)$ as:

$$\frac{\partial \theta^{ROM}}{\partial t} + U_E \frac{\partial \theta^{ROM}}{\partial X} = +D_E \frac{\partial^2 \theta^{ROM}}{\partial X^2} - \frac{\theta^{ROM}}{t_{o,E}} \quad (31)$$

From this point on, the temperature field is written in dimensionless form, while the axial coordinate and time are kept dimensional. This choice allows the effective parameters to retain their usual physical units. Equation (31) then introduces three distinct elements of the closure: the effective velocity U_E , associated with the apparent front motion; the equivalent spreading coefficient D_E , associated with the local front thickness; and an upstream closure describing the thermal level behind the front.

The heat-sink term in Eq. (31) lowers the temperature of the upstream branch over the characteristic time $t_{0,E}$. In the fluid-only cases, this decay is due solely to heat released to the surroundings. Since the fluid advects the attenuated signal downstream, the same effect can be described in space through the attenuation length $L_E = U_E t_{0,E}$. This form is particularly useful here because L_E can be identified directly from the axial FEM temperature profiles. When a thermally active wall is present, the upstream response is no longer governed by external heat loss alone. Part of the energy is temporarily stored in the solid and may later be released, so a single attenuation time is no longer sufficient to describe the profile. We therefore represent the upstream branch more generally through the function $F_{up}(X,t)$.

The three closure quantities describe different features of the temperature profile and are identified separately. The front position $X_f(t)$, and hence $U_E(t)$, describes the front motion. Its local width is characterized by $D_E(t)$, while $F_{up}(X,t)$, or $L_E(t)$ in fluid-only cases, describes the temperature level behind the front. This separation prevents a reduction in upstream temperature caused by heat loss or temporary wall storage from being interpreted as local front broadening. The reduced temperature profile can therefore be written in piecewise form as

$$\theta^{ROM}(X,t) = \begin{cases} F_{up}(X,t) & ; X \leq X_h(t) \\ F_{up}(X_h(t),t) F^{(0)}(X,t) & ; X > X_h(t) \end{cases}$$

where:

$$F^{(0)}(X,t) = \frac{\operatorname{erfc}\left(\frac{X - X_f(t)}{2\sqrt{D_E(t)} t}\right)}{\operatorname{erfc}\left(\frac{X_h(t) - X_f(t)}{2\sqrt{D_E(t)} t}\right)} \quad (33)$$

Here, the function $F^{(0)}$ is the dimensionless shape of the adiabatic reference front, normalized so that $F^{(0)}(X_h(t),t)=1$.

The function $F_{up}(X,t)$ describes the upstream heated branch, $X_h(t)$ is the hot-side edge of the front, $X_f(t)$ is the apparent front position, and $D_E(t)$ controls the local erfc transition width. The normalization in the second branch ensures continuity of the reduced temperature profile at $X_h(t)$.

The function $F_{up}(X,t)$ on the configuration considered. For the fluid-only cases, the upstream branch is well described by an exponential decay, with $L_E = U_E t_{0,E}$ relating the attenuation length to the corresponding time scale when advection dominates axial transport. With a thermally active wall, the upstream profile is generally no longer exponential because it also reflects the transient charging of the solid. In that case, F_{up} is reconstructed directly from the FEM bulk-temperature profile using a polynomial representation.

Equation (32) provides the working form used in the calibration. It keeps the upstream branch and the local front transition separate, so that attenuation or wall-induced depletion is assigned to $F_{up}(X,t)$, while the front trajectory and width are described by $X_f(t)$, $U_E(t)$ and $D_E(t)$. These quantities are identified from the full conjugate FEM response and then used to reconstruct the bulk-temperature evolution during the pre-asymptotic stage.

5 Identification Strategy

The reduced formulation developed in Section 4 describes the transient response in terms of the fluid bulk temperature. Because the velocity is uniform across the section, shear-induced Taylor–Aris dispersion is absent. With a parabolic profile, transverse diffusion would instead interact with velocity shear and introduce the classical Taylor–Aris contribution [6], [7], [13]. Axial conduction through the wall is a different mechanism and may provide additional axial smoothing when significant. The present zero-shear case therefore provides a reference against which the effects of external heat loss, wall storage, finite-rate fluid–wall exchange, and wall conduction can be identified.

5.1 Reference Baseline and Benchmark Configurations

The identification procedure is used to determine how external heat exchange and wall coupling modify the plug-flow reference response. When neither wall storage nor external heat exchange is present, the thermal front reduces to the one-dimensional advection–diffusion baseline introduced in Section 4.3.

Departures from the reference profile can then be associated with the additional mechanisms introduced in each benchmark case. The reduced description treats the upstream heated branch separately from the local front transition: F_{up} accounts for attenuation or wall-induced depletion behind the front, while U_E and D_E describe its motion and the local width. In the absence of shear-induced dispersion, this separation is especially useful because changes in the overall bulk-temperature profile, such as those caused by wall storage, should not be interpreted automatically as increased local spreading.

The benchmark cases are introduced progressively, so that each additional configuration activates a new physical mechanism. The first case is the fluid-only adiabatic reference, where front propagation and molecular axial diffusion can be examined without wall storage or external heat loss.

The second case adds external heat exchange while keeping the wall thermally inactive. This isolates the effect of heat loss on the upstream branch without introducing wall storage. The upstream temperature is then described by an exponential attenuation through L_E , while the front trajectory and local width are expected to remain close to those of the adiabatic fluid-only reference.

The third case introduces a thermally active wall while keeping the outer surface adiabatic. This allows wall storage and finite-rate fluid-wall exchange to be examined without external heat loss. As the wall heats up, it temporarily draws energy from the fluid and can therefore lower the upstream temperature level. If no additional axial-smoothing mechanism becomes important, the local front itself may remain close to the fluid-only plug-flow reference. Because the outer boundary is adiabatic, the reduction of the upstream temperature reflects energy redistribution between fluid and wall rather than a net loss from the system.

The fourth case completes the sequence by allowing the thermally active wall to exchange heat with the surroundings. Wall storage and finite-rate fluid-wall exchange are therefore accompanied by an actual loss of energy through the outer surface. The upstream response then reflects both temporary storage in the wall and external thermal dissipation. Taken together, these four cases make it possible to separate the effects on the upstream thermal level from those affecting front motion and local width.

5.2 Identification Procedure

The same identification sequence is used for all benchmark cases. The FEM temperature field is first averaged over the fluid cross-section to obtain the

bulk-temperature profile and then normalized by the inlet-to-ambient temperature difference. At each sampled time t_n , the resulting axial profile is represented by a cubic interpolant on a dense grid. Its second spatial derivative is then used to locate the upstream region, the thermal-front region, and the undisturbed downstream region.

Moving downstream along the profile, the hot-side edge $X_{h,n}$ is associated with the dominant local minimum of the second derivative, while the following dominant local maximum identifies the cold-side edge $X_{c,n}$. Assuming a locally symmetric front transition, the apparent front position is defined as $X_{f,n} = (X_{h,n} + X_{c,n})/2$.

The upstream branch is identified over the interval $0 < X < X_{h,n}$, but its representation depends on the physical configuration. In the fluid-only cases, it is described by an exponential decay with attenuation length $L_{E,n}$, with the adiabatic baseline recovered as $L_{E,n} \rightarrow \infty$. When a thermally active wall is present, the upstream profile is reconstructed more generally through $F_{up}(X, t_n)$.

The discrete front positions $X_{f,n}$ are first fitted as a continuous trajectory $X_f(t)$, from which the effective velocity $U_E(t)$ is obtained by differentiation. Once this trajectory has been established, the data between $X_{h,n}$ and $X_{c,n}$ are used to determine the equivalent spreading coefficient $D_{E,n}$ by matching the width of the normalized erfc transition.

The detailed mathematical definitions and fitting steps are given in Section 6.3.

6 Numerical Calibration of the Reduced Model

A two-dimensional axisymmetric finite-element model was used as the numerical reference for identifying the effective quantities introduced in Section 5. The simulations were performed in COMSOL Multiphysics and resolve axial and radial heat transport in the fluid, together with heat conduction in the solid wall when present. The velocity field is prescribed rather than solved. In keeping with the plug-flow assumption, it is uniform across the section and equal to the mean velocity U_m .

The numerical model provides the transient bulk-temperature profiles needed to identify the different components of the one-dimensional reduced representation.

The numerical model was applied to the benchmark cases introduced in Section 5. The computational domain contains either the fluid alone or the coupled fluid-wall system. For the wall-

coupled cases, the outer surface is taken as either adiabatic or convective.

The transient FEM solutions are post-processed in terms of the dimensionless bulk temperature, $\theta_b^{\text{FEM}}(X, t)$. The resulting axial profiles provide the quantities required by the reduced model: the upstream heated branch, the two edges of the thermal front, its apparent position, and its local width.

6.1 Numerical Reference Framework

To prevent the outlet boundary condition from affecting front identification, the FEM computational domain was extended to $1.2 L$. At every sampled time, the complete thermal front therefore remained within the computational domain and was used in the fitting procedure. For graphical presentation only, the profiles shown in the figures were truncated at $x = L$.

The two-dimensional transient problem was discretized with an unstructured triangular mesh in the axisymmetric meridional plane. Spatial and temporal discretization effects were then assessed through a sensitivity analysis.

Spatial and temporal sensitivity were tested for the wall-coupled case with $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$, where the stronger external heat exchange produces the largest thermal gradients among the configurations considered. Table 1 (Appendix) reports the results. In each test, the maximum relative temperature difference was evaluated over the entire computational domain.

The reference mesh used for ROM identification had a maximum element size of $6.4 \cdot 10^{-4} \text{ m}$. For the mesh-refinement test, the number of elements was approximately doubled to 133824 while the time step was kept at $\Delta t = 0.05 \text{ s}$. The maximum element size was thereby reduced to $4.5 \cdot 10^{-4} \text{ m}$, with a minimum size of $2.85 \cdot 10^{-5} \text{ m}$. The minimum and average element quality values were 0.703 and 0.912, respectively.

The maximum relative temperature difference over the entire computational field was evaluated as:

$$E_T = \max \left| \frac{T_{\text{refined}} - T_{\text{ref}}}{T_{\text{refined}}} \right| \cdot 100$$

The refined mesh differed from the reference solution by at most 0.06%.

Temporal sensitivity was assessed on the reference mesh by reducing the time step from $\Delta t = 0.05 \text{ s}$ to 0.01 s . The maximum relative temperature difference was only $1.1 \cdot 10^{-4} \%$.

Together with the mesh-refinement result, this confirms that the reference discretization is sufficiently converged for ROM identification.

6.2 Geometry and Physical Parameters

Table 2 (Appendix) summarizes the geometry, thermophysical properties, and operating conditions used in the simulations. The selected parameters are representative of a domestic cast-iron radiator column, which provides a practical benchmark for conjugate pipe flow with a thermally active wall. The FEM solution is used here as the transient reference from which the reduced quantities are identified.

For the fluid properties, geometry, and flow conditions considered here, the internal heat-transfer coefficient is $h_i \approx 120 \text{ W m}^{-2} \text{ K}^{-1}$. The external coefficient h_o , instead, is varied parametrically, with $h_o = 10$ and $100 \text{ W m}^{-2} \text{ K}^{-1}$ considered in the benchmark cases.

For the properties listed in Table 2 (Appendix), the wall thermal conductivity is $k_s = 50 \text{ W m}^{-1} \text{ K}^{-1}$. Taking $h_{\text{ref}} = \max(h_i, h_o) = 120 \text{ W m}^{-2} \text{ K}^{-1}$, the corresponding wall Biot number is $Bi_{s,\text{ref}} = 7.2 \cdot 10^{-3}$. The radial diffusion time across the wall thickness is $s^2/\alpha_s = 0.65 \text{ s}$, whereas the fluid residence time is 306 s , corresponding to a ratio of about $2.1 \cdot 10^{-3}$. Both estimates support the one-dimensional wall approximation adopted in Section 4.

The local fluid-wall quasi-equilibrium condition introduced in Section 4.3.3 can also be checked against the present time scales. The characteristic exchange times are $t_{c,i} \approx 162 \text{ s}$ and $t_{i,s} \approx 104 \text{ s}$, corresponding to $t_{c,i}/t_{\text{res}} \approx 0.53$ and $t_{i,s}/t_{\text{res}} \approx 0.34$. These values indicate that the local quasi-equilibrium limit is only partially approached during front propagation. Eq. (29) is therefore used only as a limiting reference for the effective closure. The FEM simulations retain finite-rate fluid-wall exchange, so the identified quantities reflect the actual pre-asymptotic dynamics.

The Reynolds and Peclet numbers based on the inner diameter are $Re_D = U_m D/\nu_f = 106$ and $Pe_D = U_m D/\alpha_f = 280$, respectively. The flow is therefore laminar, while convection dominates thermal transport over the cross-sectional length scale. These conditions are representative of the hydronic benchmark considered here.

6.3 Parameter Identification from FEM Profiles

The mathematical definitions used to extract the closure quantities from the FEM bulk-temperature profiles are presented below.

6.3.1 FEM Data Processing

The FEM profiles are sampled at five equally spaced times over the front-propagation interval:

$$t_n/t_{res} = \{0.2, 0.4, 0.6, 0.8, 1\} \quad (34)$$

At each sampled time, the FEM bulk-temperature field is converted into the dimensionless form introduced in Section 4.3. The discrete axial profile is represented by a cubic interpolant, $\theta_b^{FEM}(X, t_n)$.

6.3.2 Identification of the Axial Regions

The axial regions are identified from the second spatial derivative of the interpolated FEM profile. Moving downstream, the dominant local minimum defines the hot-side edge, $X_{h,n}$, while the following dominant local maximum defines the cold-side edge, $X_{c,n}$.

These two locations define the thermal-front interval and separate it from the upstream and undisturbed downstream regions. Assuming local symmetry, the apparent front position is taken as its midpoint:

$$X_{f,n} = \frac{1}{2}(X_{h,n} + X_{c,n}) \quad (35)$$

6.3.3 Upstream-region Reconstruction

At each sampled time t_n , the upstream heated branch is denoted by $F_{up,n}(X) = F_{up,n}(X, t_n)$ and reconstructed over the interval $0 \leq X \leq X_{h,n}$. Its mathematical form depends on the benchmark configuration. For the fluid-only cases, the upstream branch is described by the exponential form:

$$F_{up,n}(X) = \exp(-X / L_{E,n}) \quad (36)$$

This form satisfies the inlet condition, $F_{up}(0) = 1$, and reduces to the adiabatic reference in the limit $L_{E,n} \rightarrow \infty$, so that $F_{up}(X) = 1$.

The attenuation length $L_{E,n}$ is obtained from the linear regression

$$\ln[\theta_b^{FEM}(X, t_n)] \approx -X / L_{E,n} \quad (37)$$

With a thermally active wall, the upstream profile is no longer generally exponential because it also reflects transient energy storage in the solid. It is therefore reconstructed over $0 \leq X \leq X_{h,n}$ using a constrained fourth-order polynomial.

The polynomial is constrained to satisfy the imposed inlet temperature:

$$F_{up,n}(0) = 1 \quad (38)$$

Two additional conditions are imposed:

$$\left. \frac{dF_{up,n}}{dX} \right|_{X=0} = a_n ; F_{up,n}(X_{h,n}) = b_n = \theta_b^{FEM}(X_{h,n}, t_n) \quad (39)$$

The first specifies the inlet slope, while the second matches the upstream branch to the FEM temperature at the hot-side edge of the front. The inlet slope is obtained from the one-dimensional energy balance for the upstream region:

$$\left. \frac{dF_{up,n}}{dX} \right|_{X=0} = -\frac{h_{eff} P_i}{\dot{m} c_f} = a_n \quad (40)$$

Introducing the normalized coordinate $\xi = X/X_{h,n}$, the fourth-order polynomial is written as

$$F_{up,n}(X) = 1 + a_n X_{h,n} \xi + (b_n - 1 - a_n X_{h,n}) \xi^2 + A_n \xi^2 (\xi - 1) + B_n \xi^2 (\xi - 1)^2 \quad (41)$$

The remaining coefficients A_n and B_n are then obtained by fitting the upstream FEM profile over the interval $[0, X_{h,n}]$.

6.3.4 Thermal-front Region

The sampled front positions $\{X_{f,n}\}$ are fitted with a continuous trajectory $X_f(t)$, whose time derivative gives the effective velocity $U_E(t)$.

At each sampled time t_n , the reduced profile in the thermal front region is evaluated from Eq. (33) and denoted by $\theta_n^{ROM}(X, D)$. The parameter D is the trial value of the equivalent spreading coefficient.

With the upstream branch and the front locations fixed, $D_{E,n}$ is determined by fitting the erfc transition over to the FEM data over the thermal-front zone.

The mismatch between the FEM profile and the reduced representation is quantified by the least-squares objective function

$$OF_n(D) = \sum_{X_j \in [X_{h,n}, X_{c,n}]} [\theta_n^{ROM}(X_j, D) - \theta_b^{FEM}(X_j, t_n)]^2 \quad (42)$$

The identified spreading coefficient $D_{E,n}$ is the positive value of D that minimizes $OF_n(D)$:

$$D_{E,n} = \arg \min_{D>0} [OF_n(D)] \quad (43)$$

6.4 Fitting of the Identified Sequences and Model Comparison

The discrete quantities identified in Section 6.3 are fitted with smooth functions of time to reconstruct the reduced bulk-temperature field over the entire front-propagation interval. The fitting forms are

kept general so that the same procedure can later be extended to Poiseuille flow, where the effective closures may evolve appreciably in time. In the present plug-flow case, by contrast, the kinematic, spreading, and attenuation parameters are expected to vary only weakly. In the following expressions, the subscripts “0” and “ ∞ ” denote the initial and long-time values, respectively, while t_D , t_L , and t_U are the relaxation times associated with D_E , L_E , and U_E .

The equivalent spreading coefficient is fitted as:

$$D_E(t) = D_{E,\infty} + (D_{E,0} - D_{E,\infty}) \exp(-t/t_D) \quad (44)$$

For fluid-only configurations, the attenuation length is described by the same relaxation form:

$$L_E(t) = L_{E,\infty} + (L_{E,0} - L_{E,\infty}) \exp(-t/t_L) \quad (45)$$

In wall-coupled configurations, the upstream branch cannot be described by a single attenuation length. The coefficients of the constrained polynomial are therefore fitted as functions of time to obtain a continuous reconstruction.

The sampled front positions $\{X_{f,n}\}$ are fitted through a trajectory generated by an exponentially relaxing effective velocity:

$$U_E(t) = U_\infty + (U_0 - U_\infty) \exp(-t/t_U) \quad (46)$$

Integration gives the corresponding front trajectory:

$$X_f(t) = U_\infty t + (U_0 - U_\infty) t_U [1 - \exp(-t/t_U)] \quad (47)$$

with $X_f(0) = 0$.

The ROM is compared with the FEM solution in terms of both front trajectory and bulk-temperature profile. The front-position error is measured by the dimensionless root-mean-square quantity

$$\varepsilon_f = \frac{1}{L} \sqrt{\frac{1}{N_t} \sum_{n=1}^{N_t} [X_f^{\text{FEM}}(t_n) - X_f^{\text{ROM}}(t_n)]^2} \quad (48)$$

where N_t is the number of sampled times.

The global profile error is defined as:

$$\varepsilon_\theta = \sqrt{\frac{1}{N_t N_X} \sum_{n=1}^{N_t} \sum_{m=1}^{N_X} [\theta_b^{\text{FEM}}(X_m, t_n) - \theta_n^{\text{ROM}}(X_m, D_{E,n})]^2} \quad (49)$$

where N_X is the number of axial sampling locations.

The first metric measures the front-position error relative to the pipe length, whereas the second measures the overall difference between the ROM and FEM dimensionless bulk-temperature fields

7 Results and Discussion

The identification procedure is now applied to the four benchmark cases. For each case, the FEM bulk-temperature profiles are compared with the ROM reconstruction in terms of the upstream branch, front trajectory, and local transition. The errors defined in Eqs. (48)–(49) quantify the agreement, while the identified closure quantities help distinguish the kinematic, spreading, and upstream effects.

7.1 Fluid-only Adiabatic Baseline

The fluid-only adiabatic case provides a direct consistency check for the reduced model. With no wall storage or external heat exchange, the FEM solution should reproduce the advection–diffusion reference introduced in Section 4.3.1.

The identified front trajectory and local spreading practically recover the nominal plug-flow quantities, while the upstream branch remains unattenuated,

$$\begin{aligned} U_E &= U_m = 2.35 \text{ mm s}^{-1}; \\ D_E &= 1.606 \cdot 10^{-7} \text{ m}^2 \text{ s}^{-1} \approx \alpha_f = 1.600 \cdot 10^{-7} \text{ m}^2 \text{ s}^{-1}; \\ F_{\text{up}}(X) &= 1 \end{aligned} \quad (50)$$

Figure 2 (Appendix) compares the FEM and ROM profiles throughout the front-propagation interval. The curves are nearly superimposed, with $\varepsilon_f = 2.26 \cdot 10^{-2}$, $\varepsilon_\theta = 9.03 \cdot 10^{-4}$, confirming close agreement with the adiabatic reference response.

This baseline confirms that the identification procedure does not introduce spurious corrections to front motion, local spreading, or upstream temperature when these effects are absent from the reference problem.

7.2 Fluid-only Flow with External Heat Exchange

This configuration isolates the effect of external heat exchange in the absence of wall thermal storage. Two values of the external heat-transfer coefficient are considered, $h_o = 10$ and $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$. As shown in Figure 3 (Appendix), the ROM accurately reproduces the FEM profiles in both cases.

For $h_o = 10 \text{ W m}^{-2} \text{ K}^{-1}$, U_E and D_E remain very close to U_m and α_f , respectively.

The upstream attenuation is described by:

$$L_E(t) = 4.707 - 1.38 \exp(-t/44.4) \text{ m} \quad (51)$$

with t in seconds.

The corresponding errors are $\varepsilon_0 = 7.15 \cdot 10^{-4}$ and $\varepsilon_f = 9.31 \cdot 10^{-5}$.

Increasing the external coefficient to $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$ has little effect on the front trajectory or local width, while the attenuation length becomes

$$L_E(t) = 0.619 - 0.115 \exp(-t/98.6) \text{ m} \quad (52)$$

The long-time value of L_E therefore decreases from about 4.71 to 0.62 m, a factor of approximately 7.6. The errors remain small, with $\varepsilon_0 = 4.617 \cdot 10^{-3}$, $\varepsilon_f = 2.187 \cdot 10^{-3}$. The main effect of stronger external heat exchange is thus a greater attenuation of the upstream temperature, with only minor changes in front motion and local spreading.

7.3 Thermally Active Wall with Adiabatic External Boundary

With a thermally active wall and an adiabatic outer surface, the FEM profiles depart markedly from the fluid-only adiabatic reference, as shown in Figure 4 (Appendix). The hot-side and cold-side front locations, however, remain close to those of the fluid-only case. Consequently, the effective velocity and the equivalent spreading coefficient also remain close to their reference values.

The main change occurs in the upstream region. Since no heat is released to the surroundings, the reduction in fluid bulk temperature behind the front results from transient energy transfer from the moving fluid to the stationary wall. It therefore reflects temporary energy storage rather than thermal dissipation.

The ROM reproduces this behavior with $\varepsilon_0 = 7.15 \cdot 10^{-3}$ and $\varepsilon_f = 2.24 \cdot 10^{-3}$. These results show that, under adiabatic wall conditions, the dominant conjugate effect is the deformation of the upstream branch, whereas the front trajectory and local width are only weakly affected.

7.4 Thermally Active Wall with Convective External Boundary

The final benchmark adds external heat exchange to the thermally active wall. This introduces a dissipative mechanism in addition to the transient energy storage already examined under adiabatic conditions. Two external heat-transfer coefficients are considered, $h_o = 10 \text{ W m}^{-2} \text{ K}^{-1}$ and $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$. Figure 5 (Appendix) compares the corresponding FEM and ROM profiles.

Compared with the adiabatic wall-coupled case, external convection mainly lowers the upstream temperature level, while the hot-side front location changes very little. This shows that the additional heat loss acts primarily on the already heated fluid

rather than on the local front transition. Wall storage therefore remains the dominant conjugate mechanism, with external heat release providing a secondary dissipative contribution.

For the milder external exchange, $h_o = 10 \text{ W m}^{-2} \text{ K}^{-1}$, the calibrated quantities are very close to their reference values. The FEM–ROM comparison gives $\varepsilon_f = 4.41 \cdot 10^{-3}$, $\varepsilon_0 = 3.49 \cdot 10^{-3}$.

For the stronger external exchange, $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$, the stronger external exchange, the identified values are $U_E = 2.36 \text{ mm s}^{-1}$, $D_E = 1.77 \cdot 10^{-7} \text{ m}^2 \text{ s}^{-1}$. They show only a slight and not physically significant departure from U_m and α_f . The corresponding errors result $\varepsilon_f = 1.38 \cdot 10^{-3}$, $\varepsilon_0 = 4.27 \cdot 10^{-3}$.

Increasing the external heat transfer therefore affects mainly the upstream temperature level, with only minor changes in front trajectory and local width. The ROM captures the combined effects of transient wall storage and external heat release while keeping their contributions distinguishable in the reconstructed bulk-temperature field.

8 Conclusions

This work developed a reduced one-dimensional unsteady framework for thermal-front propagation in conjugate pipe flows, using hydronic radiator columns as a representative benchmark. The formulation was obtained by cross-sectional reduction of the coupled fluid–wall energy problem and calibrated against two-dimensional axisymmetric finite-element simulations under plug-flow conditions.

A key feature of the model is the separate treatment of the upstream heated region and the local thermal-front transition. This distinction is important because changes in the bulk-temperature profile caused by heat loss or temporary wall storage do not necessarily imply changes in front velocity or local spreading.

The fluid-only adiabatic case recovered the reference advection–diffusion response, with the identified velocity and spreading coefficient close to the nominal fluid velocity and molecular thermal diffusivity. Adding external heat exchange mainly attenuated the temperature behind the front, while leaving its trajectory and local width nearly unchanged. In the absence of a thermally active wall and velocity-profile shear, external exchange therefore acts primarily on the upstream temperature level.

A stronger modification appeared when the wall was allowed to participate thermally. With an adiabatic outer surface, energy was transferred

temporarily from the moving fluid to the stationary solid, producing a marked reduction of the upstream temperature without a comparable change in the local front. When external convection was also included, heat release to the surroundings produced an additional reduction of the upstream level. Under the conditions examined, however, transient wall storage remained the dominant conjugate effect.

Because the prescribed velocity field contains no shear, these changes cannot be attributed to classical Taylor–Aris dispersion. The present study instead establishes the zero-shear reference of the broader closure framework, in which wall-induced effects can be identified before introducing shear–diffusion coupling. The next step is therefore to extend the same reduction and identification procedure to Poiseuille flow, where classical Taylor–Aris dispersion will act together with the conjugate mechanisms isolated here. Configurations with stronger axial conduction in the wall and more detailed external heat-transfer models also remain to be investigated.

The resulting ROM provides an intermediate description between fully distributed conjugate simulations and spatially lumped models. Its low dimensionality, combined with the separate representation of front motion, local spreading, and upstream thermal evolution, makes it suitable for repeated transient calculations in system simulation, control-oriented modeling, and dynamic building-energy analysis.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and they have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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Conflict of Interest

The authors have no conflicts of interest to declare.

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APPENDIX

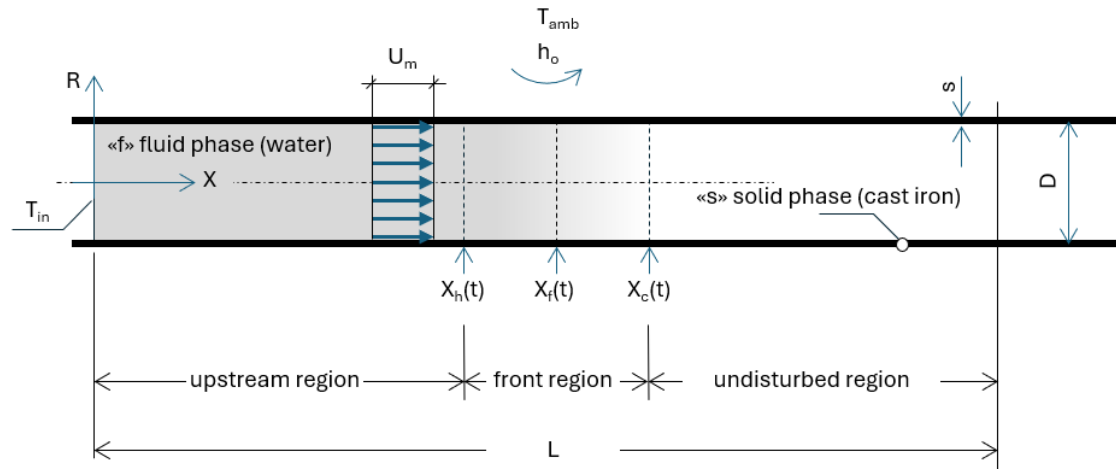


Fig. 1: Geometrical configuration and schematic representation of the thermal-front regions

Source: created by the authors

Table 1. Spatial and temporal discretization sensitivity for the wall-coupled configuration with $h_o = 100 \text{ W m}^{-2} \text{ K}^{-1}$.

Test	Number of elements	Maximum element size [m]	Time step [s]	Maximum relative temperature difference [%]
Reference discretization	67568	$6.4 \cdot 10^{-4}$	0.05	—
Mesh refinement	133824	$4.5 \cdot 10^{-4}$	0.05	$6.0 \cdot 10^{-2}$
Time-step refinement	67568	$6.4 \cdot 10^{-4}$	0.01	$1.1 \cdot 10^{-4}$

Source: created by the authors

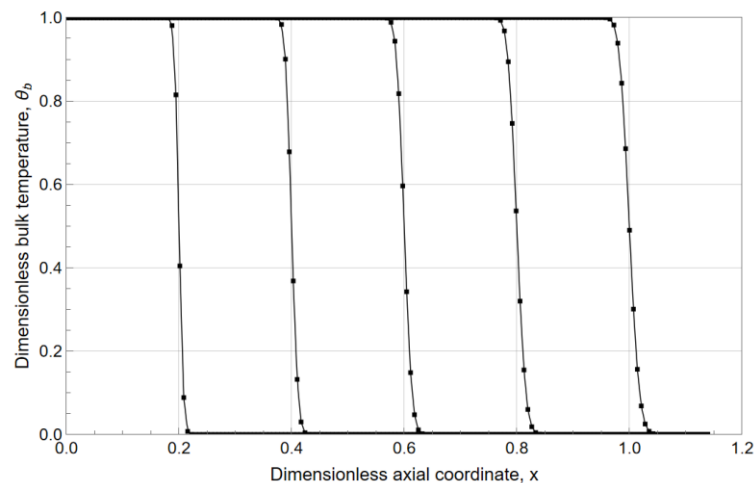


Fig. 2: Dimensionless bulk-temperature profiles for the fluid-only adiabatic baseline at selected times. Symbols denote FEM results, while solid lines denote ROM predictions.

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Table 2. Geometrical, thermophysical, and operating parameters adopted in the numerical benchmark

Category	Parameter	Symbol	Value	Unit
Geometry	Inner diameter	$D=2R_0$	$19.0 \cdot 10^{-3}$	m
	Inner radius	R_0	$9.5 \cdot 10^{-3}$	m
	Wall thickness	s	$3.0 \cdot 10^{-3}$	m
Operating conditions	Inlet fluid temperature	T_{in}	75	°C
	Initial and ambient temperature	T_{amb}	20	°C
	Mean fluid velocity	U_m	$2.35 \cdot 10^{-3}$	$m \cdot s^{-1}$
	External heat-transfer coefficient	h_o	0, 10, 100	$W \cdot m^{-2} \cdot K^{-1}$
Fluid: water	Density	ρ_f	977.8	$kg \cdot m^{-3}$
	Specific heat capacity	c_f	4186.5	$J \cdot kg^{-1} \cdot K^{-1}$
	Thermal conductivity	k_f	0.656	$W \cdot m^{-1} \cdot K^{-1}$
	Thermal diffusivity	α_f	$1.60 \cdot 10^{-7}$	$m^2 \cdot s^{-1}$
	Dynamic viscosity	μ_f	$4.12 \cdot 10^{-4}$	$kg \cdot m^{-1} \cdot s^{-1}$
	Kinematic viscosity	ν_f	$4.213 \cdot 10^{-7}$	$m^2 \cdot s^{-1}$
Solid: cast iron	Density	ρ_s	7200	$kg \cdot m^{-3}$
	Specific heat capacity	c_s	500	$J \cdot kg^{-1} \cdot K^{-1}$
	Thermal conductivity	k_s	50	$W \cdot m^{-1} \cdot K^{-1}$
	Thermal diffusivity	α_s	$1.39 \cdot 10^{-5}$	$m^2 \cdot s^{-1}$
Derived quantities	Fluid heat capacity per unit length	$C_f=\rho_f c_f A_f$	1161	$J \cdot m^{-1} \cdot K^{-1}$
	Solid heat capacity per unit length	$C_s=\rho_s c_s A_s$	746	$J \cdot m^{-1} \cdot K^{-1}$

Source: created by the authors

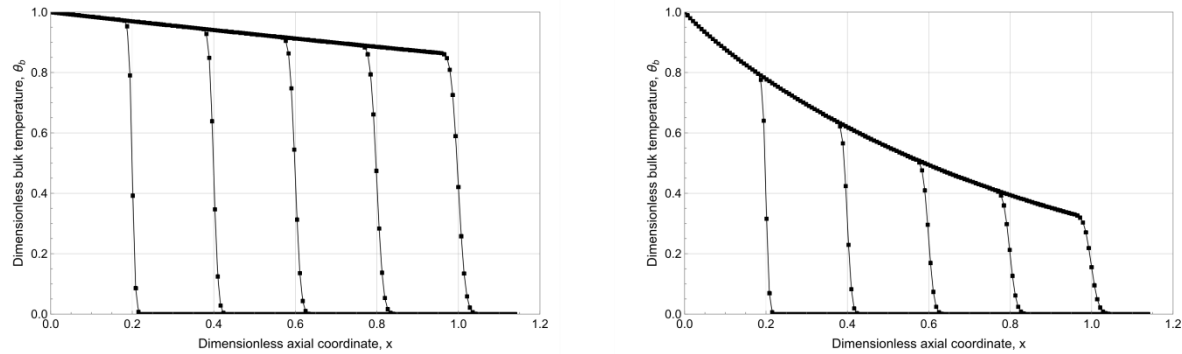


Fig. 3: Dimensionless bulk-temperature profiles for Fluid-only configurations with external heat exchange for $h_0 = 10$ and $h_0 = 100 \text{ W m}^{-2} \text{ K}^{-1}$.

Symbols denote FEM results, while solid lines denote ROM predictions.

Source: created by the authors

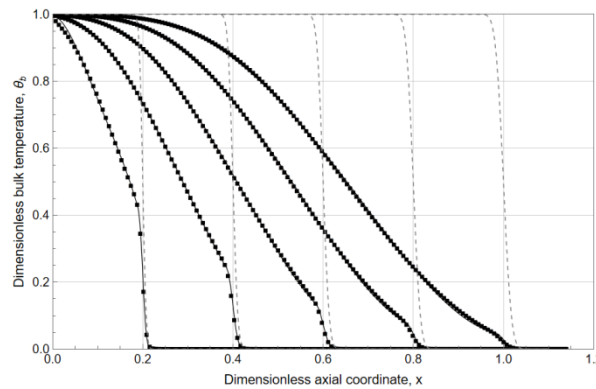


Fig. 4: Dimensionless bulk-temperature profiles for the wall-coupled adiabatic configuration. Solid lines denote ROM predictions, symbols denote FEM results, and dashed lines denote the fluid-only adiabatic reference.

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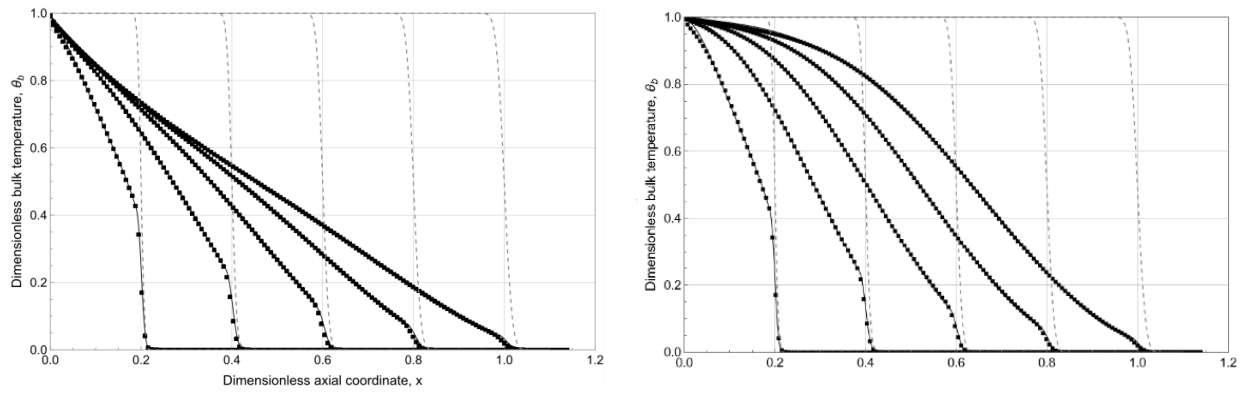


Fig. 5: Dimensionless bulk-temperature profiles for the wall-coupled configurations with external heat exchange. Solid lines denote ROM predictions, symbols denote FEM results, and dashed lines denote the fluid-only adiabatic reference.
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