A Study on Soft Compactness in Soft R_0 and Soft R_1 Spaces

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Abstract: To address challenges in solving certain problems in topology, it is often necessary to impose specific conditions on topological spaces. This has led to the development of new concepts, such as separation axioms. In this paper, we investigate new properties of soft R_0 and soft R_1 spaces using the concept of soft points. Our research examines when a soft compact set behaves similarly to a soft closed set in these spaces, helping to clarify their structural characteristics. We also explore related findings, shedding light on how soft compactness and sof closedness interact. Additionally, we present various findings related to this topic, contributing to a deeper understanding of soft topological spaces.

Key-Words: Soft topology, soft R_0 and soft R_1 space, soft compact space, soft closed space

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1 Introduction

Soft set (SS, in short) theory is a powerful set theory that emerged as a powerful mathematical tool for managing uncertainties, [1]. This innovation sparked significant interest and rapid research progress across various disciplines. As discussed in [2], [3], [4], [5], [6], and others, several researchers have investigated various classical concepts in topology, both in soft and fuzzy sets, or in fuzzy soft sets. Following this, the concept of soft topological spaces (STSs, in short) was introduced in [7], which initiated the study of soft separation axioms by many researchers. This marked the beginning of extensive research into soft topological structures. Building on this foundation, the study of soft separation axioms was extended by integrating the idea of soft points (SPs for short), [8]. Then soft R_0 (SR_0 for short) and soft R_1 (SR_1 for short) spaces are introduced, which are proposed as new soft separation axioms and their properties are studied in depth, [9]. In particular, SR_0 spaces captured the fundamental idea of openness, helping to better understand continuity and connectedness in topologies affected by uncertainty. On the other hand, SR_1 spaces offer a higher degree of separation, allowing for a more detailed analysis of topological structures when faced with uncertain conditions.

Meanwhile, compact sets, traditionally known for their closed and bounded properties in classical topology, take on a new interpretation within the framework of soft sets. This paper aims to build on the foundation laid by earlier studies, continuing the investigation of SR_0 and SR_1 spaces with a particular emphasis on exploring the characteristics of compact sets within these spaces. Section 2 will cover the fundamental definitions and key properties that underpin this study, providing a structured approach to our analysis. In section 3, we will present and discuss new insights into the behavior of compact sets in SR_0 and SR_1 spaces, focusing on their soft closeness properties and revealing further details that contribute to a deeper understanding of these topological structures. Our first result establishes an important equivalence condition for a STS to be classified as a SR_0 space. This result demonstrates the close relationship between the SR_0 property and the decomposition of soft open sets (SOSs, in short) into soft closed components, providing a deeper understanding of the structure of SR_0 spaces. Furthermore, we extend the SR_0 property to soft subsets, proving that this property is not only preserved for the entire space but also holds for any arbitrary soft subset and its complement.

We also explore the interplay between soft compactness and soft closedness in SR_1 spaces. Our findings show that the relationship between these concepts plays a significant role in understanding how set unions behave within the context of SR_1 spaces. Additionally, we prove that the closure of a soft compact set can be expressed as the soft union (S union, in short) of the closures of its SPs. These results help deepen our understanding of soft topological spaces and shed light on how soft compactness and soft closeness interact within soft R_0 and SR_1 spaces. This exploration enhances our knowledge of the underlying mechanics of soft topology, especially in the context of separation axioms, and provides a foundation for further research in this area.

2 Preliminaries

In this section, we will introduce some definitions and results from Molodtsov's soft set theory that will assist in proving our findings in the following section. Throughout this study, X will denote any universe of objects, E will denote a set of suitable parameters for the elements in X, and $P(X)$ will denote the power set of X. The set A is any subset of E.

Definition 2.1. *([1]) The pair (F, A) is called a SS over X, where F is a mapping and is defined from A to $P(X)$.*

Accordingly, a SS is a parameterized family of subsets of X. The set $F(e)$ for every $e \in A$ from this family can be thought of as the set of e-elements of the SS (F, A) or the set of approximation elements of the SS.

The SS (F, A) is also symbolized by F_A, and we denote the family of all SSs over X by $SS(X, A)$.

Definition 2.2. *([10]) Let F_A and G_B be two SSs in the common universe of X. Then:*

(i) For every $e \in A$ and $A \subseteq B$, if $F(e)$=$G(e)$, then F_A is said to be a soft subset of G_B. This situation is symbolized by $F_A \widetilde{\subseteq} G_B$.

(ii) If $F_A \widetilde{\subseteq} G_B$ and $G_B \widetilde{\subseteq} F_A$, then these SSs are named soft equals.

(iii) For every $e \in A$, $F(e) = \emptyset$, the set F_A is named null SS on X, symbolized by $\widetilde{\Phi}$.

(iv) If $F(e) = X$ for every $e \in A$, the set F_A is named the universal SS on X, symbolized by $\widetilde{X}$.

(v) $C = AUB$ and for each $e \in C$

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & A \cap B \end{cases}$$

The SS H_C is named the soft union (SU, in short) of the SSs F_A and G_B, symbolized by $H_C = F_A \cup G_B$.

(vi) $C = A \cap B$ and for every $e \in C$, $H(e) = G(e)$ or $G(e)$ (since both are the same set) the set H_C is named the soft intersection (SI, in short) of the SSs F_A and G_B, symbolized by $F_A \cap G_B = H_C$.

Definition 2.3. *[11] Let F_A be a SS and $x \in X$. If $F(a) = \{x\}$ and $F(a') = \phi$ for all $a' \in A - \{a\}$, then F_A is named a SP in $\widetilde{X}$, symbolized by ${P_a}^x$.*

The complement of a SP ${P_a}^x$ is a SS over X, symbolized by $({P_a}^x)^c$. The SP ${P_a}^x \in F_A$, if for the element $a \in A$, $x \in F(a)$. The two SPs P_a^x and P_b^y in $\widetilde{X}$ are named distinct if $x \neq y$ or $a \neq b$ or both.

The set of all SPs in $\widetilde{X}$, symbolized by $SP(X, A)$.

Definition 2.4. *([7]) A STS is the triplet (X, τ^*, E), and τ^* is the collection of SSs over X, which satisfies the following axioms:*

(i) $\widetilde{\Phi}, \widetilde{X} \in \tau^*$,

(ii) the SI of any two SSs in τ^* is in τ^*,

(iii) the SU of any number of SSs in τ^* is in τ^*.

Remark that the definition of STS is a generalization of the definition of topological spaces. In other words, the collection of topological spaces on the universal set X is a sub collection of STSs on X.

Definition 2.5. *([7]) A SS F_A is said to be a SOS in STS (X, τ^*, E) if $F_A \in \tau^*$ and a SS G_A is said to be a soft closed set (SCS, in short) if ${F_A}^c \in \tau^*$.*

The family of all SCSs in (X, τ^*, E), symbolized by τ^{*^c}.

Definition 2.6. *([7]) Let (X, τ^*, E) be a STS and $F_A \in SS(X, A)$, then the soft closure of F_A, symbolized by $\overline{F_A}$, is defined by*

$$\overline{F_A} = \widetilde{\cap}\{G_B : G_B \in \tau^{*^c} \text{ and } F_A \widetilde{\subseteq} G_B\}.$$

It is clear that $F_A \widetilde{\subseteq} \overline{F_A}$.

Theorem 2.7. *([7]) Let (X, τ^*, E) be a STS over X and F_A, G_A are two SSs over X, then*

(i) F_A is a SCS if and only if $\overline{F_A} = F_A$,

(ii) $F_A \widetilde{\subseteq} G_A$ if $\overline{F_A} \widetilde{\subseteq} \overline{G_A}$.

Definition 2.8. *([8]) A STS (X, τ^*, E) is said to be:*

(i) Soft T_0 (ST_0, in short) space , if for every pair of distinct SPs P_a^x and P_b^y such that $x, y \in X$ and $x \neq y$, there exist soft open sets F_A and G_A such that $P_a^x \in F_A$, $P_b^y \notin F_A$ or $P_b^y \in G_A$, $P_a^x \notin G_A$.

(ii) Soft T_1 (ST_1, in short) space , if for every pair of distinct SPs P_a^x and P_b^y such that $x, y \in X$ and $x \neq y$, there exists soft open sets F_A and G_A such that $P_a^x \in F_A$, $P_b^y \notin F_A$ and $P_b^y \in G_A$, $P_a^x \notin G_A$.

(ii) Soft T_2 (ST_2, in short) space, if for every two SPs P_a^x and P_b^y such that $x, y \in X$ and $x \neq y$, there exist SOSs F_A and G_A such that $P_a^x \in F_A$, $P_b^y \in G_A$ and $F_A \widetilde{\cap} G_A = \widetilde{\Phi}$.

Remark 2.9. *([8])*
(1) Every ST_1 space is a ST_0 space.
(2) Every ST_2 space is a ST_1 space.

Definition 2.10. *([8]) A STS (X, τ^*, E) is said to be soft regular space (SRS, in short) if, for every SCS H_A and SP P_a^x with $P_a^x \notin H_A$, there exist SOSs F_A and G_A such that $P_a^x \in F_A$, $H_A \widetilde{\subseteq} G_A$ and $F_A \widetilde{\cap} G_A = \widetilde{\Phi}$.*

Definition 2.11. *([8]) A STS (X, τ^*, E) is said to be soft compact if every soft open cover of $\widetilde{X}$ has a finite subcover of $\widetilde{X}$.*

Theorem 2.12. *([8]) Every soft closed subset of a soft compact space is a soft compact set.*

Definition 2.13. *([9]) A STS (X, τ^*, E) is said to be:*
(i) SR_0 space, if for every pair of distinct SPs P_a^x and P_b^y with $P_a^x \in \overline{P_b^y}$ implies $P_b^y \in \overline{P_a^x}$.
(ii) SR_1 space, if for every pair of distinct SPs P_a^x and P_b^y with $P_a^x \neq P_b^y$, there exist SSs $F_A, G_A \in \tau^*$, such that $P_a^x \in F_A$, $P_b^y \in G_A$ and $F_A \widetilde{\cap} G_A = \widetilde{\Phi}$.

We will use the following theorems in our main results.

Theorem 2.14. *([9]) Every SR_1 space is SR_0 space.*

Theorem 2.15. *([9]) Every SRS is SR_i space, $i = 0, 1$.*

3 Results

Our first result establishes an important equivalence condition for identifying a STS as an SR_0 space. This result emphasizes the essential relationship between the soft R_0 property and the way SOSs can be broken down into soft closed components, helping to better understand the structure of these spaces.

Theorem 3.1. *A STS (X, τ^*, E) is SR_0 if and only if every SOS is a SU of SCSs.*

Proof. Let (X, τ^*, E) be a SR_0 space. Let F_A be any SOS in $\widetilde{X}$ and P_a^x be any SP in F_A. To prove that F_A is a soft union of SCSs, it is enough to prove that $\{\overline{P_a^x}\} \subset F_A$. Suppose there is a SP $P_b^y \in \overline{P_a^x}$ such that ${P_b}^y \notin F_A$ which implies P_a^x has a SOS G_A which does not contains P_b^y. Since $\widetilde{X}$ is a SR_0 topological space, P_b^y has a soft open set H_A not containing P_a^x. Also $P_b^y \in \overline{P_a^x}$ implies every SOS containing P_b^y also contains P_a^x, which is a contradiction. Thus, $P_b^y \in F_A$ which implies $\{\overline{P_a^x}\} \subset F_A$. Hence, $F_A = \widetilde{\bigcup}_{P_a^x \in F_A} \overline{P_a^x}$.

Conversely, let F_A is a SOS containing P_a^x but not P_b^y where $P_a^x, P_b^y \in \widetilde{X}$. By assumption, every SOS is a SU of SCSs. i.e. $F_A = \widetilde{\bigcup}_{\alpha} F_{A_\alpha}$, where each F_{A_α} is a SCS. Therefore, there exists α such that $P_a^x \in F_{A_\alpha}$ and $P_b^y \notin F_{A_\alpha}$. This implies that $P_b^y \in {F_{A_\alpha}}^c$ and $P_a^x \notin {F_{A_\alpha}}^c$. So, (X, τ^*, E) is a SR_0 space. This completes the proof □

The following corollary extends the understanding of SR_0 spaces to the level of soft subsets which asserts that the soft SR_0 property is preserved not only for the entire space but also for any arbitrary soft subset and its complement.

Corollary 3.2. *A STS (X, τ^*, E) is SR_0 if and only if any soft subset F_A of $\widetilde{X}$ is a SU of SCSs, whenever ${F_A}^c$ is a SU of SCSs.*

Proof. The proof is clear from the Theorem 3.1. □

The next theorem provides valuable findings about the relationship between soft compactness, soft closedness, and the nature of set unions in the context of SR_1 spaces.

Theorem 3.3. *Let (X, τ^*, E) be an SR_1 topological space. Then any soft compact set F_A in the space (X, τ^*, E) is a SCS if and only if both F_A and ${F_A}^c$ are SUs of SCSs.*

Proof. Assume F_A is any SCS in the space (X, τ^*, E). Then by Theorem 2.14 and Theorem 3.1, we can easily say that F_A^c is a SU of SCSs. Hence, by Corollary 3.2 F_A is also a SU of SCSs.

Conversely, let F_A be any SS such that $F_A = \widetilde{\bigcup}_\alpha F_{A_\alpha}$, where each F_{A_α} is a SCS in (X, τ^*, E). There is enough to prove that F_A is a SCS. Assume that F_A is not SCS. Then, there exists a soft net $\{P_{a_\lambda}^{x_\lambda}\}_\lambda$ in F_A s.t. $P_{a_\lambda}^{x_\lambda} \to P_a^x$ and $P_a^x \notin F_A$. As F_A is a soft compact so $\{P_{a_\lambda}^{x_\lambda}\}_\lambda$ has a soft limit point ${P_b}^y$ in F_A. Therefore, there exists a α such that ${P_b}^y \in F_{A_\alpha}$ and ${P_a}^x \notin F_{A_\alpha}$. Then ${F_{A_\alpha}}^c$ is a soft neighbourhood of SP P_a^x not containing P_b^y.

As (X, τ^*, E) is a SR_1 space so that P_a^x and P_b^y have disjoint neighbourhoods. This contradicts that $P_{a_\lambda}^{x_\lambda} \to P_a^x$ and P_b^y is limit point of $\{P_{a_\lambda}^{x_\lambda}\}_\lambda$. Then F_A is SCS closed. This completes the proof. □

From Theorem 2.15 and Theorem 3.3, we immediately have the following corollary:

Corollary 3.4. *Let (X, τ^*, E) be an SRS, then any soft compact set F_A in the space (X, τ^*, E) is a SCS if and only if both F_A and ${F_A}^c$ are unions of SCSs.*

Corollary 3.5. *For any compact set F_A of SR_1 space (X, τ^*, E), the following are equivalent:*
(i) F_A is a SCS.
(ii) F_A and ${F_A}^c$ are of the form $P_A \widetilde{\cup} Q_A$. Where P_A and Q_A are arbitrary SOSs and SCSs in the space (X, τ^*, E), respectively.

Proof. From Theorem 2.15, any compact set F_A of SR_1 space is a SCS if and only if F_A and ${F_A}^c$ are SUs of SCSs. So, there is enough to prove that F_A and ${F_A}^c$ are of the form $P_A \tilde{\cup} Q_A$. We have F_A is a SU of SCSs i.e. $F_A = \underset{i}{\tilde{\cup}} P_{A_i}$, where P_{A_i} are SCS and $i = 1, 2, ...$ Thus, for any α $F_A = \{\underset{i}{\tilde{\cup}} P_{A_i}\} \tilde{\cup} P_{A_\alpha}$ where α any number from $i = 1, 2, ...$ From Theorem 3.1, we have $\underset{i}{\tilde{\cup}} P_{A_i} = P_A$ and also take $P_{A_\alpha} = Q_A$, where P_A and Q_A are arbitrary SOSs and SCSs in the space (X, τ^*, E). Then, we get $F_A = P_A \tilde{\cup} Q_A$, and the proof is completed. □

From results, we get the following corollaries.

Corollary 3.6. *In a space with the SR_1 property, any set of the form $P_A \tilde{\cup} Q_A$ or $P_A \tilde{\cap} Q_A$ is a SCS if it is a soft compact, where P_A and Q_A are arbitrary SOSs and SCSs in the space (X, τ^*, E), respectively.*

Corollary 3.7. *In SR_1 space, every soft open compact set is a SCS.*

Proof. Its proof is obvious from Theorems 3.1 and 3.3. □

In the following result, we prove that the closure of a soft compact set can be expressed as the SU of the soft closures of its SPs.

Corollary 3.8. *Let F_A be a soft compact subset of an SR_1 space (X, τ^*, E). Then:*

(i) $\overline{F_A} = \tilde{\cup}\{\overline{{P_a}^x} : {P_a}^x \in F_A\}$ and $\overline{F_A}$ is a soft compact.

(ii) If G_A is any SS such that $F_A \tilde{\subset} G_A \tilde{\subset} \overline{F_A}$, then G_A is a soft compact set.

Proof. We know that

$$\tilde{\cup}\{\overline{{P_a}^x} : {P_a}^x \in F_A\} \subset \overline{F_A}.$$

Assume $P_A = \tilde{\cup}\{\overline{{P_a}^x} : {P_a}^x \in F_A\}$. Firstly, we will prove P_A is a soft compact. By assumption, F_A is a soft compact. Consider G_A be any arbitrary family of SOSs covering P_A i.e. G_A is a soft open cover of F_A such that $F_A \subset \bigcup_{i=1}^{n} G_{A_\alpha}$, where $G_{A_\alpha} \in G_A$ are SOSs. As we know in SR_i space $(i = 1, 0)$, the soft closure of any SP is contained in every soft neighborhood of that SP. So, we can say that the finite soft cover F_A will also cover P_A i.e. $P \subset \bigcup_{i=1}^{n} G_{A_\alpha}$. Hence, P_A is a soft compact set i.e. $F_A \tilde{\subset} P_A \tilde{\subset} \overline{F_A}$, then P_A is a soft compact set. This proves (ii). Also, we have $P_A = \tilde{\cup}\{\overline{{P_a}^x} : {P_a}^x \in F_A\}$, i.e. P_A, a SS, is SU of SCSs.

Now from Theorem 3.3, we have P_A is a SCS, where $F_A \tilde{\subset} P_A \tilde{\subset} \overline{F_A}$. So $P_A = \overline{F_A}$. This proves (i). This completes the proof. □

Corollary 3.9. *In a SR_1 space (X, τ^*, E),*

(i) A SU of SCSs or SI of SOSs is a SCS if it is a soft compact.

(ii) If $\mathcal{G}_A$ is a family of soft subsets of the space (X, τ^*, E) such that $\widetilde{\bigcup}\{\overline{F_A} : F_A \in \mathcal{G}_A\}$ for particular $\overline{\widetilde{\bigcup}\{F_A : F_A \in \mathcal{G}_A\}}$ is a soft compact, then $\overline{\widetilde{\bigcup}\{F_A : F_A \in \mathcal{G}_A\}} = \widetilde{\bigcup}\{\overline{F_A} : F_A \in \mathcal{G}_A\}$.

Proof. (i) This is directly followed by Theorem 3.3. □

4 Conclusion

The paper examines SR_0 and SR_1 spaces, focusing on the behavior of soft compact sets within these spaces. By using the concept of SPs, we identify the conditions under which a soft compact set behaves like a closed set in SR_0 and SR_1 spaces. This study builds on the foundational ideas of SS theory and soft topology, investigating how soft separation axioms and compactness interact. The theorems and corollaries presented establish important relationships between soft compactness, SCSs, and SOSs, particularly in SR_1 spaces.

Through this research, we offer valuable findings that enhance our understanding of how soft topological structures behave in uncertain conditions. The results provide a clearer picture of how soft compactness and soft closeness operate in these spaces, contributing to the growing field of topology in environments characterized by uncertainty. These insights have broader implications for fields where managing uncertainty and complex structures is essential, opening new directions for further research and applications in SS theory and beyond.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed, and edited the content as needed and They have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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