Fixed point results for integral type contractions in the framework of complete neutrosophic metric spaces

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Abstract: - Fixed point theory is a core mathematical concept with wide-ranging applications across various disciplines. In this paper, we present common fixed point theorems for (ϕ, ψ)-neutrosophic contractions in the context of complete neutrosophic metric spaces. We also derive several related results concerning fixed points based on our primary theorem. Additionally, we highlight how these findings extend and deepen the understanding of neutrosophic contractions.

Key-Words: - Banach contraction principle, Common fixed point, Fixed point, Neutrosophic metric, Integral contractions.

Received: February 25, 2025. Revised: April 27, 2025. Accepted: May 21, 2025. Published: July 17, 2025.

1 Introduction

Fuzzy Sets (FSs), introduced by [1], have greatly impacted various scientific fields but have not always provided satisfactory solutions to some problems. This led to the development of new approaches, such as Intuitionistic Fuzzy Sets (IFSs) by [2], to address these issues. Additionally, Smarandache introduced the Neutrosophic Set (NS), [3], as an advanced extension of traditional set theory. The applications of fuzzy sets and its extensions are wide as the reader can see in [4], [5], [6]. In fact, neutrosophic sets has a wide applications, for example the authors in [7], characterize efficient and appropriately optimal solutions concerning related scalar optimization problems. For a broader understanding of the applicability of neutrosophic sets and their various applications, we direct the reader to the works cited in [8], [9], [10], [11], and the references included therein.

The Banach fixed-point theorem, [12], also known as the Banach contraction principle, is a fundamental result in mathematics, particularly in the study of metric spaces. This theorem guarantees both the existence and uniqueness of fixed points for certain self-maps in metric spaces, and provides a systematic approach for identifying these fixed points. Following Banach's result, numerous generalizations have been introduced, with some focusing on modifications to the distance function, as shown in [13], [14], [15], [16]. Another approach to generalizing Banach's theorem involves altering the conditions imposed on the self-map, as demonstrated in [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]. Additionally, several authors have established results concerning common fixed points, as seen in [31], [32].

2 Prliminary

Triangular norms (t-norms), introduced by [33], are fundamental in mathematical analysis. Menger's approach used probability distributions to measure distances between elements, generalizing the triangle inequality in metric spaces.

Through this manuscript, $\mathbb{R}^+ = [0, \infty)$, $J_0 = [0, 1]$.

Definition 2.1. *[33]*

Consider an operation $\oplus : J_0 \times J_0 \to J_0$. This operation is called continuous TN (CTN) if it satisfies the following conditions: for any elements $\kappa, \kappa', \iota, \iota' \in J_0$.

1. $\kappa \oplus 1 = \kappa$,
2. *If $\kappa \leq \kappa'$ and $\iota \leq \iota'$, than $\kappa \oplus \iota \leq \kappa' \oplus \iota'$,*
3. *$\oplus$ is continuous,*
4. *$\oplus$ is commutative and associative.*

Definition 2.2. *[33]*

Consider an operation $\odot : J_0 \times J_0 \to J_0$. This operation is called continuous TC (CTC) if it satisfies the following conditions: for all elements $\kappa, \kappa', \iota, \iota' \in J_0$.

1. $\kappa \odot 0 = \kappa$,
2. If $\kappa \leq \kappa'$ and $\iota \leq \iota'$, than $\kappa \odot \iota \leq \kappa' \odot \iota'$,
3. $\odot$ is continuous,
4. $\odot$ is commutative and associative.

Definition 2.3. *[34] Let F represent an arbitrary set, and define*
$N = \{< \zeta, \Xi_U(\zeta), \Upsilon_U(\zeta), \Omega_U(\zeta) >: \zeta \in F\}$ *as a neutrosophic structure such that $N : F \times F \times \mathbb{R}^+ \to J_0$. The symbols $\oplus$ and $\odot$ denote the operations of the continuous triangular norm (CTN) and continuous triangular conorm (CTC), respectively. The quadruple $V = (F, N, \oplus, \odot)$ is referred to as a neutrosophic metric space (NMS) when the following conditions hold for all $\zeta, \eta, c, \in F$.*

1. $0 \leq \Xi(\zeta, \eta, \theta) \leq 1$, $0 \leq \Upsilon(\zeta, \eta, \theta) \leq 1$, $0 \leq \Omega(\zeta, \eta, \theta) \leq 1$, $\forall \theta \in \mathbb{R}^+$
2. $0 \leq \Xi(\zeta, \eta, \theta) + \Upsilon(\zeta, \eta, \theta) + \Omega(\zeta, \eta, \theta) \leq 3$, $\forall \theta \in \mathbb{R}^+$
3. $\Xi(\zeta, \eta, \theta) = 1$, for $\theta > 0$ iff $\zeta = \eta$
4. $\Xi(\zeta, \eta, \theta) = \Xi(\eta, \zeta, \theta)$, for $\theta > 0$
5. $\Xi(\zeta, \eta, \theta) \oplus \Xi(\eta, c, \nu) \leq \Xi(\zeta, c, \theta + \nu)$, for $\nu, \theta > 0$
6. $\Xi(\zeta, \eta, \cdot) : \mathbb{R}^+ \to J_0$ is continuous
7. $\lim_{\theta \to \infty} \Xi(\zeta, \eta, \theta) = 1$
8. $\Upsilon(\zeta, \eta, \theta) = 0$, for $\theta > 0$ iff $\zeta = \eta$
9. $\Upsilon(\zeta, \eta, \theta) = \Upsilon(\eta, \zeta, \theta)$, for $\theta > 0$
10. $\Upsilon(\zeta, \eta, \theta) \odot \Upsilon(\eta, c, \nu) \geq \Upsilon(\zeta, c, \theta + \nu)$, for $\nu, \theta > 0$
11. $\Upsilon(\zeta, \eta, \cdot) : \mathbb{R}^+ \to J_0$ is continuous
12. $\lim_{\theta \to \infty} \Upsilon(\zeta, \eta, \theta) = 0$
13. $\Omega(\zeta, \eta, \theta) = 0$, for $\theta > 0$ iff $\zeta = \eta$
14. $\Omega(\zeta, \eta, \theta) = \Omega(\eta, \zeta, \theta)$, for $\theta > 0$
15. $\Omega(\zeta, \eta, \theta) \odot \Omega(\eta, c, \nu) \geq \Omega(\zeta, c, \theta + \nu)$, for $\nu, \theta > 0$
16. $\Omega(\zeta, \eta, \cdot) : \mathbb{R}^+ \to J_0$ is continuous
17. $\lim_{\theta \to \infty} \Omega(\zeta, \eta, \theta) = 0$
18. If $\theta \leq 0$, then $\Xi(\zeta, \eta, \theta) = 0$, $\Upsilon(\zeta, \eta, \theta) = \Omega(\zeta, \eta, \theta) = 1$

Then, the triplet $N = (\Xi, \Upsilon, \Omega)$ is referred to as a neutrosophic metric (NM) on the set F. The functions $\Xi(\zeta, \eta, \theta)$, $\Upsilon(\zeta, \eta, \theta)$, and $\Omega(\zeta, \eta, \theta)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ζ and η in relation to the parameter θ, respectively.

Definition 2.4. *[34] Let (ζ_n) be a sequence in $V = (F, N, \oplus, \odot)$. Then*

1. (ζ_n) converges to $\zeta \in F$ iff for a given $\epsilon \in (0, 1)$, $\theta > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

   $\Xi(\zeta_n, \zeta, \theta) > 1-\epsilon$, $\Upsilon(\zeta_n, \zeta, \theta) < \epsilon$, $\Omega(\zeta_n, \zeta, \theta) < \epsilon$

   i.e.,

   $$\lim_{n \to \infty} \Xi(\zeta_n, \zeta, \theta) = 1, \lim_{n \to \infty} \Upsilon(\zeta_n, \zeta, \theta) = 0,$$
   $$\lim_{n \to \infty} \Omega(\zeta_n, \zeta, \theta) = 0.$$

2. (ζ_n) is called Cauchy iff for a given $\epsilon \in (0, 1)$, $\theta > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$

   $\Xi(\zeta_n, \zeta_m, \theta) > 1-\epsilon$, $\Upsilon(\zeta_n, \zeta_m, \theta) < \epsilon$, $\Omega(\zeta_n, \zeta_m, \theta) < \epsilon$

   i.e.,

   $$\lim_{n,m \to \infty} \Xi(\zeta_n, \zeta_m, \theta) = 1, \lim_{n,m \to \infty} \Upsilon(\zeta_n, \zeta_m, \theta) = 0,$$
   $$\lim_{n,m \to \infty} \Omega(\zeta_n, \zeta_m, \theta) = 0$$

3. V is called complete if each Cauchy sequence is convergent to an element in F.

In [35], the authors defined NC-contractions on neutrosophic metric spaces and showed that every NC-contraction has a unique fixed point under special considerations.

Definition 2.5. *[35] Let V be a NMS. A mapping $f : F \to F$ is called neutrosophic contraction if there is $k \in (0, 1)$ such that for each $\zeta, \eta \in F$ and $\theta > 0$, we have*

$$\frac{1}{\Xi(f\zeta, f\eta, \theta)} - 1 \leq k \left(\frac{1}{\Xi(\zeta, \eta, \theta)} - 1 \right),$$

$$\frac{1}{\Upsilon(f\zeta, f\eta, \theta)} - 1 \geq k \left(\frac{1}{\Upsilon(\zeta, \eta, \theta)} - 1 \right),$$

and

$$\frac{1}{\Omega(f\zeta, f\eta, \theta)} - 1 \geq k \left(\frac{1}{\Omega(\zeta, \eta, \theta)} - 1 \right).$$

Numerous studies have been conducted on neutrosophic metric spaces. In [36], the authors investigated fixed point theorems within complete neutrosophic fuzzy metric spaces specifically for integral type contractions. Similarly, in [37], the focus was on fixed point results in complete neutrosophic fuzzy metric spaces concerning quasi contractions. In our study, we initially present a common fixed point theorem for pairs of self-maps in complete neutrosophic metric spaces, where the condition is of integral type. Subsequently, we derive several corollaries related to the common fixed point based on the primary result. Finally, we obtain fixed point results that emerge as consequences of the main findings.

3 Main Result

To facilitate our main result we need the following two classes of functions

$\Phi = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ :$ φ is Lebesgue integrable with $0 < \int_0^c \varphi(\mathrm{b})d\mathrm{b} < \infty,\ \forall c > 0\}$

$\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ : \psi$ is continuous with $\lim_{n\to\infty} \psi^n(s) = 0,\ \forall s > 0\}$.

Remark 3.1. *If $\psi \in \Psi$, then*

1. $\psi(s) < s$, *for* $s > 0$,
2. $\psi(0) = 0$.

By employing the classes of Φ and Ψ functions within the framework of neutrosophic metric spaces, we present the notion of a (ϕ, ψ)-Neutrosophic contraction. As a result, we demonstrate that these contractions have a unique common fixed point. Subsequently, we apply our primary findings to explore a range of fixed point theorems in complete neutrosophic metric spaces.

We will now define a (ϕ, ψ)-Neutrosophic contraction.

Definition 3.2. *Let V be a NMS, $\phi \in \Phi$, and $\psi \in \Psi$. A pair of self maps (f,g) on F is called (ϕ, ψ)-neutrosophic contraction ((ϕ, ψ)-NC) if for each $\zeta, \eta \in F$ and each $\theta > 0$, we have*

$$\int_0^{\frac{1}{\Xi(f\zeta, g\eta, \theta)}-1} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\frac{1}{\Xi(\zeta,\eta,\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\right),$$

$$\int_0^{\Upsilon(f\zeta, g\eta, \theta)} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\Upsilon(\zeta,\eta,\theta)} \varphi(\mathrm{b})d\mathrm{b}\right),$$

and

$$\int_0^{\Omega(f\zeta, g\eta, \theta)} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\Omega(\zeta,\eta,\theta)} \varphi(\mathrm{b})d\mathrm{b}\right).$$

Theorem 3.3. *Let V be a complete NMS, Suppose that there are $\phi \in \Phi$, and $\psi \in \Psi$ such that $(f, g) : F \to F$ is (ϕ, ψ)-NC. Consequently, the functions f and g possesses a unique common fixed point.*

Proof. Let $\zeta_0 \in F$ represent an arbitrary point. We examine the sequence (ζ_n) characterized by the relation $\zeta_{2n+1} = f\zeta_{2n}$, $\zeta_{2n+2} = g\zeta_{2n+1}$, $n \ge 0$. Let $n \in \mathbb{N}$. If n is even, then $n = 2s$, $s \in \mathbb{N}$.

$$\begin{aligned}\int_0^{\frac{1}{\Xi(\zeta_n,\zeta_{n+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b} &= \int_0^{\frac{1}{\Xi(\zeta_{2s},\zeta_{2s+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &= \int_0^{\frac{1}{\Xi(g\zeta_{2s-1},f\zeta_{2s},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &\le \psi \int_0^{\frac{1}{\Xi(\zeta_{2s-1},\zeta_{2s},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &= \psi \int_0^{\frac{1}{\Xi(\zeta_{n-1},\zeta_n,\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\end{aligned}$$

If n is odd, then $n = 2t + 1$, $t \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\int_0^{\frac{1}{\Xi(\zeta_n,\zeta_{n+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b} &= \int_0^{\frac{1}{\Xi(\zeta_{2t+1},\zeta_{2t+2},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &= \int_0^{\frac{1}{\Xi(f\zeta_{2t},g\zeta_{2t+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &\le \psi \int_0^{\frac{1}{\Xi(\zeta_{2t},\zeta_{2t+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\\ &= \psi \int_0^{\frac{1}{\Xi(\zeta_{n-1},\zeta_n,\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\end{aligned}$$

Thus, in each case, we get

$$\int_0^{\frac{1}{\Xi(\zeta_n,\zeta_{n+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\frac{1}{\Xi(\zeta_{n-1},\zeta_n,\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\right).$$

By the same way, we can also show that

$$\int_0^{\Upsilon(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\Upsilon(\zeta_{n-1},\zeta_n,\theta)} \varphi(\mathrm{b})d\mathrm{b}\right),$$

and

$$\int_0^{\Omega(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathrm{b})d\mathrm{b} \le \psi\left(\int_0^{\Omega(\zeta_{n-1},\zeta_n,\theta)} \varphi(\mathrm{b})d\mathrm{b}\right).$$

By induction on n, we get that

$$\int_0^{\frac{1}{\Xi(\zeta_n,\zeta_{n+1},\theta)}-1} \varphi(\mathrm{b})d\mathrm{b} \le \psi^n\left(\int_0^{\frac{1}{\Xi(\zeta_0,\zeta_1,\theta)}-1} \varphi(\mathrm{b})d\mathrm{b}\right),$$

$$\int_0^{\Upsilon(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq \psi^n\left(\int_0^{\Upsilon(\zeta_0,\zeta_1,\theta)} \varphi(\mathfrak{b})d\mathfrak{b}\right),$$

and

$$\int_0^{\Omega(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq \psi^n\left(\int_0^{\Omega(\zeta_0,\zeta_1,\theta)} \varphi(\mathfrak{b})d\mathfrak{b}\right).$$

By taking the limit, we get

$$\lim_{n\to\infty}\int_0^{\frac{1}{\Xi(\zeta_n,\zeta_{n+1},\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b} = 0,$$

$$\lim_{n\to\infty}\int_0^{\Upsilon(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathfrak{b})d\mathfrak{b} = 0,$$

and

$$\lim_{n\to\infty}\int_0^{\Omega(\zeta_n,\zeta_{n+1},\theta)} \varphi(\mathfrak{b})d\mathfrak{b} = 0.$$

Which implies that

$$\begin{array}{l}\lim_{n\to\infty}\Xi(\zeta_n,\zeta_{n+1},\theta) = 1,\\ \lim_{n\to\infty}\Upsilon(\zeta_n,\zeta_{n+1},\theta) = 0,\\ \text{and}\\ \lim_{n\to\infty}\Omega(\zeta_n,\zeta_{n+1},\theta) = 0.\end{array} \tag{1}$$

We aim to demonstrate that (ζ_n) is Cauchy by proving that (ζ_{2n}) is Cauchy.

Assume the contrary that is

$$\lim_{n,m\to\infty}\Xi(\zeta_{2n},\zeta_{2m},\theta) \neq 1,$$

$$\text{or } \lim_{n,m\to\infty}\Upsilon(\zeta_{2n},\zeta_{2m},\theta) \neq 0,$$

$$\text{or } \lim_{n,m\to\infty}\Omega(\zeta_{2n},\zeta_{2m},\theta) \neq 0.$$

Case 1: If $\lim_{n,m\to\infty}\Xi(\zeta_{2n},\zeta_{2m},\theta) \neq 1$, then exist an $\epsilon > 0$ and $\theta > 0$ along with two subsequences (ζ_{2n_k}) and (ζ_{2m_k}) derived from (ζ_{2n}), where $(2m_k)$ is selected as the smallest index satisfying the condition.

$$\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta) \leq 1-\epsilon, \quad m_k > n_k > k. \tag{2}$$

This implies that

$$\Xi(\zeta_{2n_k},\zeta_{2m_k-1},\theta) > 1-\epsilon. \tag{3}$$

chose $\delta > 0$. Then

$$\begin{array}{l}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta+\delta)\\ \geq \Xi(\zeta_{2n_k},\zeta_{2m_k-1},\theta)\oplus\Xi(\zeta_{2m_k-1},\zeta_{2m_k},\delta)\\ > (1-\epsilon)\oplus\Xi(\zeta_{2m_k-1},\zeta_{2m_k},\delta)\end{array}$$

Using Equation 1, we get

$$\liminf_{k\to\infty}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta+\delta) \geq (1-\epsilon).$$

Also,

$$\begin{array}{rl}(1-\epsilon) & \leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta+\delta)\\ & = \liminf_{k\to\infty}\lim_{\delta\to 0^+}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta+\delta)\\ & = \liminf_{k\to\infty}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta).\end{array}$$

Also, from 2, it follows

$$\liminf_{k\to\infty}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta) \leq (1-\epsilon).$$

So, we get

$$\lim_{k\to\infty}\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta) = (1-\epsilon).$$

Again, we have

$$\begin{array}{l}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta+\delta)\\ \geq \Xi(\zeta_{2n_k-1},\zeta_{2n_k},\delta)\oplus\Xi(\zeta_{2n_k},\zeta_{2m_k-1},\theta)\\ > \Xi(\zeta_{2n_k-1},\zeta_{2n_k},\delta)\oplus(1-\epsilon).\end{array}$$

Using Equation 1, we get

$$\liminf_{k\to\infty}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta+\delta) \geq (1-\epsilon).$$

Also,

$$\begin{array}{rl}(1-\epsilon) & \leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta+\delta)\\ & = \liminf_{k\to\infty}\lim_{\delta\to 0^+}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta+\delta)\\ & = \liminf_{k\to\infty}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta).\end{array}$$

From Definition 3.2 and Eq 2, we get

$$\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta) \leq \Xi(\zeta_{2n_k},\zeta_{2m_k},\theta) \leq (1-\epsilon).$$

So,

$$\liminf_{k\to\infty}\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta) = (1-\epsilon).$$

Using Definition 3.2, we get

$$\int_0^{\frac{1}{\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b} \le \psi\left(\int_0^{\frac{1}{\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b}\right).$$

By taking the limit inferior, we get

$$\liminf_{k\to\infty}\int_0^{\frac{1}{\Xi(\zeta_{2n_k},\zeta_{2m_k},\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b}$$
$$\le \liminf_{k\to\infty}\psi\left(\int_0^{\frac{1}{\Xi(\zeta_{2n_k-1},\zeta_{2m_k-1},\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b}\right).$$

Hence,

$$\int_0^{\frac{1}{1-\epsilon}-1}\varphi(\mathfrak{b})d\mathfrak{b} \le \psi\left(\int_0^{\frac{1}{1-\epsilon}-1}\varphi(\mathfrak{b})d\mathfrak{b}\right) < \int_0^{\frac{1}{1-\epsilon}-1}\varphi(\mathfrak{b})d\mathfrak{b},$$

which is a contradiction, and hence, $\lim_{n,m\to\infty}\Xi(\zeta_{2n},\zeta_{2m},\theta) = 1$.

The other two cases are similar.
Hence (ζ_n) is a Cauchy sequence, thus, there is $u \in F$ such that $\zeta_n \to u$.

Definition 3.2 gives that

$$\int_0^{\frac{1}{\Xi(fu,g\zeta_n,\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b} \le \psi\left(\int_0^{\frac{1}{\Xi(u,\zeta_n,\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b}\right)$$
$$\to 0 \text{ as } n\to\infty,$$
$$\int_0^{\Upsilon(fu,g\zeta_n,\theta)}\varphi(\mathfrak{b})d\mathfrak{b} \le \psi\left(\int_0^{\Upsilon(u,\zeta_n,\theta)}\varphi(\mathfrak{b})d\mathfrak{b}\right)$$
$$\to 0 \text{ as } n\to\infty,$$

and

$$\int_0^{\Omega(fu,g\zeta_n,\theta)}\varphi(\mathfrak{b})d\mathfrak{b} \le \psi\left(\int_0^{\Omega(u,\zeta_n,\theta)}\varphi(\mathfrak{b})d\mathfrak{b}\right)$$
$$\to 0 \text{ as } n\to\infty.$$

Which implies that ζ_{n+1} converges to fu, hence $u = fu$.

By the same argument, we can show that $gu = u$.

Now, let $v \in F$ with $v = fv$, and $v = gv$. If $u \ne v$, then from Definition 3.2, it follows that

$$\int_0^{\Omega(u,v,\theta)}\varphi(\mathfrak{b})d\mathfrak{b} = \int_0^{\Omega(fu,gv,\theta)}\varphi(\mathfrak{b})d\mathfrak{b}$$
$$\le \psi\left(\int_0^{\Omega(u,v,\theta)}\varphi(\mathfrak{b})d\mathfrak{b}\right) < \int_0^{\Omega(u,v,\theta)}\varphi(\mathfrak{b})d\mathfrak{b},$$

which is a contradiction. So $u = v$.

□

If we let $\psi(s) = qs$, where q is in the interval $[0,1)$, we arrive at the subsequent conclusion:

Corollary 3.4. *Let V be a complete NMS, Suppose that $f,g : F \to F$ satisfies the following for each $\zeta,\eta \in F$ and each $\theta > 0$, we have:*

$$\int_0^{\frac{1}{\Xi(f\zeta,g\eta,\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b} \le q\int_0^{\frac{1}{\Xi(\zeta,\eta,\theta)}-1}\varphi(\mathfrak{b})d\mathfrak{b},$$
$$\int_0^{\Upsilon(f\zeta,g\eta,\theta)}\varphi(\mathfrak{b})d\mathfrak{b} \le q\int_0^{\Upsilon(\zeta,\eta,\theta)}\varphi(\mathfrak{b})d\mathfrak{b},$$

and

$$\int_0^{\Omega(f\zeta,g\eta,\theta)}\varphi(\mathfrak{b})d\mathfrak{b} \le q\int_0^{\Omega(\zeta,\eta,\theta)}\varphi(\mathfrak{b})d\mathfrak{b}.$$

Consequently, the functions f, and g possesses a unique common fixed point.

If we establish the function $\psi(s) = qs$, where q is constrained within the interval $[0,1)$, and define $\varphi(\mathfrak{b}) = 1$, we arrive at the subsequent conclusion:

Corollary 3.5. *Let V be a complete NMS, Suppose that $f,g : F \to F$ satisfies the following for each $\zeta,\eta \in F$ and each $\theta > 0$, we have:*

$$\frac{1}{\Xi(f\zeta,g\eta,\theta)} - 1 \le q\left(\frac{1}{\Xi(\zeta,\eta,\theta)} - 1\right),$$
$$\Upsilon(f\zeta,g\eta,\theta) \le q\Upsilon(\zeta,\eta,\theta),$$

and

$$\Omega(f\zeta,g\eta,\theta) \le q\Omega(\zeta,\eta,\theta).$$

Consequently, the functions f, and g possesses a unique common fixed point.

Now, we derive further fixed point results based on our main result.

By taking $f = g$ in Theorem 3.3, we get the following result:

Theorem 3.6. *Let V be a complete NMS, Suppose that there are $\phi \in \Phi$, and $\psi \in \Psi$ such that $f : F \to F$ satisfy the following conditions for each $\zeta,\eta \in F$ and each $\theta > 0$, we have*

$$\int_0^{\frac{1}{\Xi(f\zeta,f\eta,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b} \leq \psi\left(\int_0^{\frac{1}{\Xi(\zeta,\eta,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b}\right),$$

$$\int_0^{\Upsilon(f\zeta,f\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq \psi\left(\int_0^{\Upsilon(\zeta,\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b}\right),$$

and

$$\int_0^{\Omega(f\zeta,f\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq \psi\left(\int_0^{\Omega(\zeta,\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b}\right).$$

Hence, the functions f, possesses a unique fixed point.

We will now present an example to demonstrate our primary theorem.

Example 3.7. *Let $F = [0,1]$ with the standard metric by $d(x,y) = |x-y|$. Also, Let the t-norm and t-conorm be defined as follows $x \oplus y = \min\{x,y\}$, $x \odot y = \max\{x,y\}$. Additionally, let the fuzzy sets be defined as follows:*

$$\Xi(x,y,\theta) = \frac{\theta}{\theta + d(x,y)},\ \Upsilon(x,y,\theta) = \frac{d(x,y)}{\theta + d(x,y)}$$

$$\Omega(x,y,\theta) = \frac{d(x,y)}{\theta + d(x,y)}.$$

Also, define $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ via $\varphi(a) = a$, $\psi(a) = \frac{1}{2}a$.

Define $f : F \to F$ by $f(x) = \frac{1}{2}x$.

Then, f has a unique fixed point.

Proof. From [34], $V = (F, \Xi, \Upsilon, \Omega)$ represents a complete NMS.

Now, for any elements x and y in F and for any $\theta > 0$, we have

$$\begin{aligned}
\int_0^{\frac{1}{\Xi(fx,fy,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b} &= \int_0^{\frac{1}{\frac{\theta}{\theta+d(fx,fy)}}-1} \mathfrak{b}d\mathfrak{b} \\
&= \int_0^{\frac{d(fx,fy)}{\theta}} \mathfrak{b}d\mathfrak{b} \\
&= \int_0^{\frac{|\frac{x}{2}-\frac{y}{2}|}{\theta}} \mathfrak{b}d\mathfrak{b} \\
&\leq \frac{1}{2}\int_0^{\frac{|x-y|}{\theta}} \mathfrak{b}d\mathfrak{b} \\
&= \psi \int_0^{\frac{1}{\Xi(x,y,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b}
\end{aligned}$$

The remaining cases can be illustrated in a similar fashion. □

If we let $\psi(s) = qs$, in Theorem 3.6 where q is in the interval $[0,1)$, we arrive at the subsequent conclusion:

Corollary 3.8. *Let V be a complete NMS, Suppose that $f : F \to F$ satisfies the following for each $\zeta, \eta \in F$ and each $\theta > 0$, we have:*

$$\int_0^{\frac{1}{\Xi(f\zeta,f\eta,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b} \leq q\int_0^{\frac{1}{\Xi(\zeta,\eta,\theta)}-1} \varphi(\mathfrak{b})d\mathfrak{b},$$

$$\int_0^{\Upsilon(f\zeta,f\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq q\int_0^{\Upsilon(\zeta,\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b},$$

and

$$\int_0^{\Omega(f\zeta,f\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b} \leq q\int_0^{\Omega(\zeta,\eta,\theta)} \varphi(\mathfrak{b})d\mathfrak{b}.$$

Consequently, the function f possesses a unique fixed point.

If we establish the function $\psi(s) = qs$, in Theorem 3.6, where q is constrained within the interval $[0,1)$, and define $\varphi(\mathfrak{b}) = 1$, we arrive at the subsequent conclusion:

Corollary 3.9. *Let V be a complete NMS, Suppose that $f : F \to F$ satisfies the following for each $\zeta, \eta \in F$ and each $\theta > 0$, we have:*

$$\frac{1}{\Xi(f\zeta,f\eta,\theta)} - 1 \leq q\left(\frac{1}{\Xi(\zeta,\eta,\theta)} - 1\right),$$

$$\Upsilon(f\zeta,f\eta,\theta) \leq q\Upsilon(\zeta,\eta,\theta),$$

and

$$\Omega(f\zeta,f\eta,\theta) \leq q\Omega(\zeta,\eta,\theta).$$

Consequently, the function f possesses a unique fixed point.

4 Conclusion

Fixed-point theory encompasses a range of theorems that analyze the impact of transformations on points within a designated set, guaranteeing the existence of at least one invariant point. These theorems play a vital role in demonstrating the existence of solutions to various equations and systems across multiple mathematical disciplines. In this study, we introduced integral-type contractions in complete neutrosophic metric spaces for a pair of self-maps, demonstrating that these contractions yield a unique common fixed point. Subsequently, we derived fixed-point results based on this primary finding. This significant result paves the way for numerous additional outcomes. Future research may aim to extend our results to other types of distance spaces or enhance them by incorporating appropriate conditions. Another avenue for future investigation could involve exploring applications to fractional differential equations through the lens of neutrosophic sets, as referenced in [38], [39], [40].

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.