MSTFrFT on Some Function Spaces

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Abstract: - The multidimensional short-time fractional Fourier transform, which involves multiplying the signal by a time-shifted window prior to performing the multidimensional fractional Fourier transform, is the subject of this work. The multidimensional short-time fractional Fourier transform on Morrey, Besov, and weighted Besov spaces is examined for boundedness.

Key-Words: - Fourier transform, fractional Fourier transform, fractional short-time Fourier transform, Morrey space, Besov space, weighted Besov space.

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1 Introduction

Of the different approaches available for the analysis and synthesis of signals, Fourier analysis is the oldest. The "basis functions" in the Fourier transform (FT) are fully spread in time and completely concentrated in frequency. One interpretation of this would be that the FT loses information about time while providing the most information about the distribution of frequencies. An averaged frequency representation across the entire time interval of the signal is given by the standard FT.

The fractional Fourier transform, or FrFT for short, is currently used in many scientific domains, such as communications, optics, wave propagation, signal and image processing, and more. In any field where the regular FT is used, the FrFT could prove to be a useful tool. In order to establish the potential of a new application, it is necessary to concentrate on an application that utilises the conventional FT and to inquire whether an enhancement or generalization can be achieved using the FrFT. The "fraction" parameter's ability to offer an extra degree of optimization freedom usually allows for improved performance or more widespread generalization. Since the FrFT is based on chirp signals, signals with linear frequency growth usually have a lot of potential for improvement. When the FrFT is unable to identify the FrF domain frequency contents, the short-time fractional Fourier transform (STFrFT) has been suggested as a solution, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14].

Let f be a function on R^d. The translation operator of f is $T_x f(t) = f(t-x)$ for $x \in R^d$ and the modulation operator of f is $M_\eta f(t) = e^{2\pi i \eta \cdot t} f(t)$ for $t, \eta \in R^d$.

As Lebesgue spaces for $p \in [1, \infty]$, we write $\left(L^p(R^d), \|\cdot\|_p\right)$. Locally p-integrable function spaces are also denoted by $L^p_{loc}(U)$, where $U \subset R^d$ is an open set.

Let $f \in L^1(R^d)$, let us define $\hat{f}$ (or $\mathrm{F}f$) by

$$\mathrm{F}f(w) = \hat{f}(w) = \int_{R^d} f(x)\, \mathrm{e}^{-2\pi i x \cdot w}\, dx, \qquad (1)$$

$w \in R^d$. The function $\hat{f}$ is called the Fourier transform of the function f. The inversion formula is

$$f(x) = \mathrm{F}^{-1}\left(\hat{f}\right) = \int_{R^d} \hat{f}(w)\, \mathrm{e}^{2\pi i x \cdot w}\, dw \qquad (2)$$

$x \in R^d$, if $f \in L^1(R^d)$ and $\hat{f} \in L^1(R^d)$.

Since the Fourier transform of f is a function of frequency and hence it lives on the vertical axis, while f lives on the horizontal axis, by the Fourier transform, the representation axis is changed from time to frequency, which corresponds to a counterclockwise rotation over an angle $\pi/2$. Because applying (1) and then (2) leads to:

$$\left(\mathrm{F}^2 f\right)(x) = \left(\mathrm{F}\left(\mathrm{F}f\right)\right)(x) = f(-x),$$

from which it follows that"
$\left(\mathrm{F}^{3} f\right)(w)=(\mathrm{F} f)(-w)$ and $\mathrm{F}^{4} f(x)=f(x)$
so the transformation is of period 4.

The fractional Fourier transform (FrFT) is a generalization of the Fourier transform with a parameter α and can be interpreted as a rotation by an angle α in the time-frequency plane. The fractional Fourier transform of a function $f \in L^{1}(R)$ is defined by:

$$F^{a} f(u)=f_{a}(u)=\int_{\square} K_{a}(u,t) f(t) dt,$$

where

$$K_{a}(u,t)=A_{\alpha}\ \mathrm{e}^{i\pi\left(\left(u^{2}+t^{2}\right)\cot\alpha-2ut\csc\alpha\right)},$$

$A_{\alpha}=\sqrt{1-i\cot\alpha}$, $\alpha=\dfrac{a\pi}{2}$, when $a \neq 2k$ for integer k. The square root is defined such that the argument of the result lies in the interval $(-\pi/2, \pi/2]$. If $a=4k$, the kernel K_{a} is defined $K_{a}(u,t)=\delta(u-t)$ and if $a=4k+2$, the kernel is defined $K_{a}(u,t)=\delta(u+t)$, where δ is the Dirac delta function. It is easy to see that, if $\alpha=\pi/2$, then the transform is the ordinary Fourier transform, and if $\alpha=-\pi/2$, then we get the ordinary inverse Fourier transform, [1], [2], [7], [8], [9]. Kamalakkannan and Roopkumar introduced the multidimensional fractional Fourier transform (MFrFT) in [4].

Although it fully loses information in relation to time, the Fourier transform provides the most information about the frequency distribution. Although there is always a tradeoff, time-frequency information is still connected in the short-time Fourier transform (STFT): the simultaneous localization in time and frequency is constrained by the uncertainty principle. For some situations, the short-time Fourier transform is superior to Fourier analysis, even though this is true for any time-frequency analysis. The short-time Fourier transform multiplies the infinitely long complex exponential with a window to localize it, adding a time dimension to the underlying function parameters. Sliding windows, another name for window functions, are functions where the amplitude tapers gently and gradually toward zero at the edges. The resulting short-time Fourier transform shows the signal's spectral composition at each corresponding time period because each block spans a separate time period.

Let g be a nonzero function (it is called a window function). The short-time Fourier transform (STFT) of a function f with respect to the window function g is given by:

$$V_{g} f(x,w)=\int_{\square^{d}} f(t)\overline{g(t-x)}\mathrm{e}^{-2\pi i t\cdot w}\, dt,$$

when the integral exists, for all $x, w \in R^{d}$. If $f, g \in L^{2}\left(R^{d}\right)$, then $V_{g} f \in L^{2}\left(R^{2d}\right)$ and $V_{g} f$ is uniformly continuous, [3].

Multidimensional short-time fractional Fourier transform (MSTFrFT) is the multiplication of the signal with a time-shifted window before the multidimensional fractional Fourier transform (MFrFT) is taken, namely:

$$\begin{aligned} V_{g}^{\alpha} f(x,w) &= \mathrm{F}_{t}^{\alpha}\left\{f(t)\overline{g(t-x)}\right\} \\ &= \int_{\square^{d}} f(t)\overline{g(t-x)} K_{\alpha}(w,t)\, dt, \end{aligned} \tag{3}$$

where $\mathrm{F}_{t}^{\alpha}\{\cdot\}$ denotes a MFrFT integral over the variable t, $\alpha \in (0,2\pi)^{d}$, $\alpha_{j} \neq n\pi$, $n \in R$, $j=1,...,d$, $K_{\alpha}(w,t)=\prod_{j=1}^{d} K_{\alpha_{j}}\left(w_{j}, t_{j}\right)$, $x, w \in R^{d}$, [12], [15]. Lemma 2.1 in [12], shows that formula (3) can be rewritten as:

$$\begin{aligned} & V_{g}^{\alpha} f(x,w) \\ & = A_{\alpha} e^{i\pi w^{2}\cdot\cot\alpha} e^{-2\pi i x\cdot w\cdot\csc\alpha}\left(\mathrm{N}_{\alpha} f * M_{diag(\csc\alpha)w} g^{\square}\right)(x) \end{aligned} \tag{4}$$

where $A_{\alpha}=\prod_{j=1}^{d} A_{\alpha_{j}}$, $\mathrm{N}_{\alpha} f(t)=f(t) e^{i\pi t^{2}\cdot\cot\alpha}$, $x\cdot w=\sum_{j=1}^{d} x_{j} w_{j}$, $g^{\square}(x)=\overline{g(-x)}$, $*$ is the ordinary convolution operator, $diag(\csc\alpha)$ is the diagonal matrix of $\csc\alpha$, $diag(\csc\alpha)w$ is the matrix product $diag(\csc\alpha)[w]$, here $[w]$ is the coordinate matrix of w in R^{d} relative to the standard basis.

2 Main Results

2.1 MSTFrFT on Morrey Spaces

Applications of Morrey spaces to regularity properties of PDE solutions, such as the study of Navier-Stokes equations, are common. These spaces are superior to Lebesgue spaces in describing local features of functions. In this section, we examine the boundedness of MSTFrFT on classical Morrey space. Firstly, we will recall the definition of classical Morrey spaces.

Let $p \in [1,\infty)$ and $v \in [0,1]$. Then the Morrey spaces $L_M^{p,v}\left(R^d\right)$ are defined by:

$$L_M^{p,v}\left(R^d\right) = \left\{ f \in L_{loc}^p\left(R^d\right) \mid \|f\|_{L_M^{p,v}\left(R^d\right)} < \infty \right\},$$

where

$$\|f\|_{L_M^{p,v}\left(R^d\right)} = \sup_{\substack{x \in R^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} |f(t)|^p \, dt \right)^{1/p}.$$

Note that the Morrey spaces are Banach spaces with this norm. Since the Schwartz class $\mathrm{S}\left(R^d\right)$, which is the space of rapidly decreasing functions, is not dense in Morrey space, we consider the space $\overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}}$, which is the closure of $\mathrm{S}\left(R^d\right)$ in $L_M^{1,v}\left(R^d\right)$, for more details, we refer to [16]. The boundedness characteristics of the multidimensional short-time fractional Fourier transform on $\overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}}$ will now be examined.

Lemma 1. Let $0 \neq g \in L^1\left(R^d\right) \cap L^2\left(R^d\right)$ and $f \in L^1\left(R^d\right)$. Then $V_g^\alpha f(\cdot,a) \in L^1\left(R^d\right)$.

Proof. $V_g^\alpha f(b,a)$ is a function of b and is such that

$$\left|V_g^\alpha f(b,a)\right| = \left| \int_{\mathbb{R}^d} f(t)\overline{g(t-b)}K_\alpha(a,t)\,dt \right| \leq |A_\alpha| \int_{\mathbb{R}^d} |f(x+b)||g(x)|\,dx .$$

Hence we have"

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|V_g^\alpha f(b,a)\right| db \\ &\leq |A_\alpha| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x+b)||g(x)|\,dx \right) db \\ &= |A_\alpha| \int_{\mathbb{R}^d} |g(x)| \left(\int_{\mathbb{R}^d} |f(x+b)|\,db \right) dx \\ &= |A_\alpha| \|g\|_1 \|f\|_1 . \end{aligned}$$

Then, it follows that $V_g^\alpha f(\cdot,a) \in L^1\left(R^d\right)$. □

Theorem 2. Let $0 \neq g \in L^1\left(R^d\right) \cap L^2\left(R^d\right)$. Then $V_g^\alpha : \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}} \to \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}}$, $f \to V_g^\alpha f(\cdot,a)$ is bounded. Moreover, for all $f \in \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}}$, we have the inequality:

$$\left\|V_g^\alpha f(\cdot,a)\right\|_{L_M^{1,v}\left(R^d\right)} \leq |A_\alpha| \|g\|_1 \|f\|_{L_M^{1,v}\left(R^d\right)}.$$

Proof. Let $f \in \mathrm{S}\left(R^d\right)$. Then, by using Lemma 1, we write:

$$V_g^\alpha f(\cdot,a) \in L^1\left(R^d\right) = \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L^1(\mathbb{R}^d)}} \subset \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(\mathbb{R}^d)}}.$$

We also write:

$$\begin{aligned} &\left\|V_g^\alpha f(\cdot,a)\right\|_{L_M^{1,v}\left(R^d\right)} \\ &= \sup_{\substack{x \in R^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} \left|V_g^\alpha f(b,a)\right| db \right). \end{aligned} \tag{5}$$

Moreover we have:

$$\begin{aligned} &\frac{1}{r^v} \int_{B(x,r)} \left|V_g^\alpha f(b,a)\right| db \\ &\leq |A_\alpha| \frac{1}{r^v} \int_{B(x,r)} \left| \int_{R^d} |f(u+b)||g(u)|\,du \right| db \\ &= |A_\alpha| \int_{R^d} |g(u)| \left(\frac{1}{r^v} \int_{B(x+u,r)} |f(z)|\,dz \right) du \\ &\leq |A_\alpha| \|f\|_{L_M^{1,v}\left(R^d\right)} \int_{R^d} |g(u)|\,du \\ &= |A_\alpha| \|f\|_{L_M^{1,v}\left(R^d\right)} \|g\|_1 \end{aligned}$$

by changing $z = u+b$. Taking the supremum of both sides of the above inequality for $x \in R^d$, $r>0$, we get:

$$\begin{aligned} &\sup_{\substack{x \in \mathbb{R}^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} \left|V_g^\alpha f(b,a)\right| db \right) \\ &\leq |A_\alpha| \|f\|_{L_M^{1,v}\left(R^d\right)} \|g\|_1 . \end{aligned} \tag{6}$$

From (5) and (6), we have

$$\left\|V_g^\alpha f(\cdot,a)\right\|_{L_M^{1,v}\left(R^d\right)} \leq |A_\alpha| \|g\|_1 \|f\|_{L_M^{1,v}\left(R^d\right)}$$

for all $f \in \mathrm{S}\left(R^d\right)$. It is known that the Schwartz space $\mathrm{S}\left(R^d\right)$ is dense in the space $\overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}\left(R^d\right)}}$. Moreover, V_g^α is a linear operator from the Schwartz space $\mathrm{S}\left(R^d\right)$ into the Banach space

$\overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$ that satisfies the above inequality. Then, by the density principle, V_g^α extends to a bounded operator

$$V_g^\alpha : \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}} \to \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}.$$

Moreover,

$$\left\|V_g^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(R^d)} \le |A_\alpha| \|g\|_1 \|f\|_{L_M^{1,v}(R^d)}$$

holds for all $f \in \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$.

This completes the proof. □

Corollary 3. If $0 \neq g \in L^1\left(R^d\right) \cap L^2\left(R^d\right)$ and $f \in \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$, then we have

$$\left\|V_g^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(R^d)} = O\left(|A_\alpha|\right).$$

Proof. By using Theorem 2, we get the result. □

Now, we will determine the $L_M^{1,v}\left(R^d\right)$- distance between two MSTFrFT.

Theorem 4. Let f and h are in $\overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$, and Φ and Ψ are nonzero functions in $L^1\left(R^d\right) \cap L^2\left(R^d\right)$. Then we have

$$\left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha h(\cdot,a)\right\|_{L_M^{1,v}(R^d)} \le |A_\alpha| \left(\|f\|_{L_M^{1,v}(R^d)} \|\Phi - \Psi\|_1 + \|f-h\|_{L_M^{1,v}(R^d)} \|\Psi\|_1 \right).$$

Proof. Let $f \in \mathrm{S}\left(R^d\right)$. Then we write,

$$\left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(R^d)}$$

$$= \sup_{\substack{x \in R^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} \left| V_\Phi^\alpha f(t,a) - V_\Psi^\alpha f(t,a) \right| dt \right)$$

$$= \sup_{\substack{x \in R^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} \left| \int_{R^d} f(y) \left(\overline{\Phi(y-t) - \Psi(y-t)} \right) K_\alpha(a,y)\, dy \right| dt \right).$$

Now,

$$\frac{|A_\alpha|}{r^v} \int_{B(x,r)} \left| \int_{R^d} f(y) \left(\overline{\Phi(y-t) - \Psi(y-t)} \right) dy \right| dt$$

$$\le \frac{|A_\alpha|}{r^v} \int_{B(x,r)} \left(\int_{R^d} |f(y)| \left| \Phi(y-t) - \Psi(y-t) \right| dy \right) dt$$

$$= \frac{|A_\alpha|}{r^v} \int_{B(x,r)} \left(\int_{R^d} |f(z+t)| \left| \Phi(z) - \Psi(z) \right| dz \right) dt$$

$$= |A_\alpha| \int_{R^d} \left| \Phi(z) - \Psi(z) \right| \left(\frac{1}{r^v} \int_{B(x,r)} |f(z+t)|\, dt \right) dz$$

$$= |A_\alpha| \int_{R^d} \left| \Phi(z) - \Psi(z) \right| \left(\frac{1}{r^v} \int_{B(z+x,r)} |f(u)|\, du \right) dz$$

$$\le |A_\alpha| \|f\|_{L_M^{1,v}(R^d)} \|\Phi - \Psi\|_1.$$

Therefore

$$\sup_{\substack{x \in R^d \\ r>0}} \left(\frac{1}{r^v} \int_{B(x,r)} \left| V_\Phi^\alpha f(t,a) - V_\Psi^\alpha f(t,a) \right| dt \right) \le |A_\alpha| \|f\|_{L_M^{1,v}(R^d)} \|\Phi - \Psi\|_1. \quad (7)$$

Then we have:

$$\left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(\square^d)} \le |A_\alpha| \|f\|_{L_M^{1,v}(R^d)} \|\Phi - \Psi\|_1$$

for all $f \in \mathrm{S}\left(R^d\right)$. Hence, by the density principle, we get:

$$\left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(R^d)} \le |A_\alpha| \|f\|_{L_M^{1,v}(\square^d)} \|\Phi - \Psi\|_1 \quad (8)$$

for all $f \in \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$. Also, using Theorem 2, we write:

$$\left\|V_\Psi^\alpha f(\cdot,a) - V_\Psi^\alpha h(\cdot,a)\right\|_{L_M^{1,v}(R^d)} \le |A_\alpha| \|f-h\|_{L_M^{1,v}(R^d)} \|\Psi\|_1 \quad (9)$$

for all $f, h \in \overline{\mathrm{S}\left(R^d\right)}^{\|\cdot\|_{L_M^{1,v}(R^d)}}$.

Then by (7), (8), and (9), we obtain:

$$\left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha h(\cdot,a)\right\|_{L_M^{1,v}(R^d)} \le \left\|V_\Phi^\alpha f(\cdot,a) - V_\Psi^\alpha f(\cdot,a)\right\|_{L_M^{1,v}(R^d)} + \left\|V_\Psi^\alpha f(\cdot,a) - V_\Psi^\alpha h(\cdot,a)\right\|_{L_M^{1,v}(R^d)}$$

$$\leq |A_\alpha| \left(\|f\|_{L_M^{1,\nu}(R^d)} \|\Phi - \Psi\|_1 + \|f - h\|_{L_M^{1,\nu}(R^d)} \|\Psi\|_1 \right).$$

□

2.2 MSTFrFT on Besov and Weighted Besov Spaces

A considerable number of classical function spaces, including Besov, Hölder, and Sobolev spaces, can be characterised by harmonic analysis methods through a variety of approaches, including Fourier or wavelet bases, frames, Littlewood-Paley decompositions, and approximation by spline functions, to name but a few. Such characterisations have historically been employed in a variety of settings, including operator theory and the theoretical and numerical analysis of partial differential equations. In this part, we characterize the MSTFrFT on Besov Spaces and weighted Besov spaces.

Definition 5. For $p \in [1,\infty]$, $q \in [1,\infty)$ and $\beta \in (0,1)$, the Besov space $B_p^{\beta,q}(R^d)$ is defined as the space of all functions $f \in L^p(R^d)$ such that:

$$\int_{R^d} \left(w_p(f,h) \right)^q \frac{dh}{|h|^{d+\beta q}} < \infty,$$

where $w_p(f,h) = \|f(\cdot + h) - f(\cdot)\|_p$ is well defined as $w_p(f,h) \leq 2\|f\|_p$. If $q = \infty$, the Besov space $B_p^{\beta,\infty}(R^d)$ is defined as:

$$B_p^{\beta,\infty}(R^d) = \left\{ f \in L^p(R^d) \,\middle|\, |h|^{-\beta} w_p(f,h) \in L^\infty(R^d \setminus \{0\}) \right\},$$

where $|h|$ stands for the Euclidean norm of $h \in R^d$. With regard to the norm denoted by:

$$\|f\|_{B_p^{\beta,q}} = \|f\|_p + \left(\int_{R^d} \left(w_p(f,h) \right)^q \frac{dh}{|h|^{d+\beta q}} \right)^{\frac{1}{q}}$$

for $q < \infty$ and the norm

$$\|f\|_{B_p^{\beta,\infty}} = \|f\|_p + \left\| |h|^{-\beta} w_p(f,h) \right\|_\infty$$

for $q = \infty$, the Besov space $B_p^{\beta,q}(R^d)$ is a Banach space.

Definition 6. A positive function k defined on R^d is called a tempered weight function if there exist positive constants C and N such that:

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta)$$

for all $\xi, \eta \in R^d$. The set of all such functions k is denoted by K.

Let $p \in [1,\infty)$. The weighted Lebesgue space $L_k^p(R^d)$ is the set of all complex-valued measurable functions on R^d such that:

$$\|f\|_{L_k^p} = \left(\int_{R^d} |f(x)|^p k(x) dx \right)^{1/p} < \infty.$$

Definition 7. For $p \in [1,\infty]$, $q \in [1,\infty)$ and $\beta \in (0,1)$, the weighted Besov space $B_{p,k}^{\beta,q}(R^d)$ is defined as the space of all functions $f \in L_k^p(R^d)$ such that

$$\int_{R^d} \left(w_{p,k}(f,h) \right)^q \frac{dh}{|h|^{d+\beta q}} < \infty,$$

where $w_{p,k}(f,h) = \|f(\cdot + h) - f(\cdot)\|_{p,k}$. If $q = \infty$, the weighted Besov space $B_{p,k}^{\beta,\infty}(R^d)$ is defined as:

$$B_{p,k}^{\beta,\infty}(R^d) = \left\{ f \in L_k^p(R^d) \,\middle|\, |h|^{-\beta} w_{p,k}(f,h) \in L^\infty(R^d \setminus \{0\}) \right\},$$

where $|h|$ stands for the Euclidean norm of $h \in R^d$. The weighted Besov space $B_{p,k}^{\beta,q}(R^d)$ is a Banach space associated with the norm defined by:

$$\|f\|_{B_{p,k}^{\beta,q}} = \|f\|_{p,k} + \left(\int_{R^d} \left(w_{p,k}(f,h) \right)^q \frac{dh}{|h|^{d+\beta q}} \right)^{\frac{1}{q}}$$

for $q < \infty$ and the norm $\|f\|_{B_{p,k}^{\beta,\infty}} = \|f\|_{p,k} + \left\| |h|^{-\beta} w_{p,k}(f,h) \right\|_\infty$ for $q = \infty$, [17], [18].

In this part, we will study on $[c,\infty)^d$, $c > 0$, instead of R^d. Also, we will take $p = 1$, $q = 1$.

Theorem 8. Let $0 \neq g \in L^1\left([c,\infty)^d\right)$. Then

$$V_g^\alpha : B_1^{\beta,1}\left([c,\infty)^d\right) \to B_1^{\beta,1}\left([c,\infty)^d\right),$$

$f \to V_g^\alpha f(\cdot, a)$ is bounded. Moreover, for all $f \in B_1^{\beta,1}\left([c,\infty)^d\right)$, we have the inequality:

$$\left\| V_g^\alpha f(\cdot, b) \right\|_{B_1^{\beta,1}} \leq |A_\alpha| \|g\|_1 \left(\|f\|_{B_1^{\beta,1}} + L\|f\|_1 \right),$$

where $L = 2 \Big/ \left((d + \beta - 1)^d c^{d(d+\beta-1)} \right)$.

Proof. Let $f \in B_1^{\beta,1}\left(\left[c,\infty\right)^d\right)$. Then we write $f \in L^1\left(\left[c,\infty\right)^d\right)$. We claim that $V_g^\alpha f\left(\cdot,a\right) \in B_1^{\beta,1}\left(\left[c,\infty\right)^d\right)$. By using the equality (4) and the Young inequality, we write

$$\left\|V_g^\alpha f\left(\cdot,b\right)\right\|_1 \le \left|A_\alpha\right| \left\|g\right\|_1 \left\|f\right\|_1 \tag{10}$$

On the other hand,

$$w_1\left(V_g^\alpha f\left(\cdot,b\right),h\right) = \left\|V_g^\alpha f\left(\cdot+h,b\right) - V_g^\alpha f\left(\cdot,b\right)\right\|_1$$

$$= \int_{[c,\infty)^d} \left|V_g^\alpha f\left(a+h,b\right) - V_g^\alpha f\left(a,b\right)\right| da$$

$$= \int_{[c,\infty)^d} \left| \int_{[c,\infty)^d} f\left(x+a+h\right)\overline{g\left(x\right)} K_\alpha\left(x+a+h,b\right) dx - \right.$$

$$\left. - \int_{[c,\infty)^d} f\left(x+a\right)\overline{g\left(x\right)} K_\alpha\left(x+a,b\right) dx \right| da$$

$$\le \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g\left(x\right)\right|$$

$$\left(\int_{[c,\infty)^d} \left| f\left(x+a+h\right) e^{-2\pi i h\cdot\left(b\cdot\csc\alpha - (x+a)\cdot\cot\alpha\right)} e^{i\pi h^2\cot\alpha} \right.\right.$$

$$\left.\left. - f\left(x+a\right)\right| da \right) dx$$

$$\le \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g\left(x\right)\right|$$

$$\left(\int_{[c,\infty)^d} \left| f\left(x+a+h\right) - f\left(x+a\right)\right| da + \right.$$

$$\left. \int_{[c,\infty)^d} \left| f\left(x+a\right)\left(e^{-2\pi i h\cdot\left(b\cdot\csc\alpha - (x+a)\cdot\cot\alpha\right)} e^{i\pi h^2\cot\alpha} - 1\right)\right| da \right) dx$$

$$\le \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g\left(x\right)\right| \left(w_1\left(f,h\right) + 2\left\|f\right\|_1\right) dx$$

$$= \left|A_\alpha\right| \left\|g\right\|_1 \left(w_1\left(f,h\right) + 2\left\|f\right\|_1\right).$$

Therefore, we have for $0 < \beta < 1$

$$\int_{[c,\infty)^d} w_1\left(V_g^\alpha f\left(\cdot,b\right),h\right) \frac{dh}{h^{d+\beta}}$$

$$\le \left|A_\alpha\right| \left\|g\right\|_1 \left(\int_{[c,\infty)^d} \left(w_1\left(f,h\right) + 2\left\|f\right\|_1\right) \frac{dh}{h^{d+\beta}} \right)$$

$$= \left|A_\alpha\right| \left\|g\right\|_1 \left(\int_{[c,\infty)^d} w_1\left(f,h\right) \frac{dh}{h^{d+\beta}} + 2\left\|f\right\|_1 \int_{[c,\infty)^d} \frac{dh}{h^{d+\beta}} \right)$$

$$= \left|A_\alpha\right| \left\|g\right\|_1 \left(\int_{[c,\infty)^d} w_1\left(f,h\right) \frac{dh}{h^{d+\beta}} + L\left\|f\right\|_1 \right) \tag{11}$$

where $L = 2\Big/\left(\left(d+\beta-1\right)^d c^{d(d+\beta-1)}\right)$. It follows from (10) and (11) that

$$\left\|V_g^\alpha f\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$= \left\|V_g^\alpha f\left(\cdot,b\right)\right\|_1 + \int_{[c,\infty)^d} w_1\left(V_g^\alpha f\left(\cdot,b\right),h\right) \frac{dh}{h^{d+\beta}}$$

$$\le \left|A_\alpha\right| \left\|g\right\|_1 \left\|f\right\|_1 + \left|A_\alpha\right| \left\|g\right\|_1 \left(\int_{[c,\infty)^d} w_1\left(f,h\right) \frac{dh}{h^{d+\beta}} + L\left\|f\right\|_1 \right)$$

$$\le \left|A_\alpha\right| \left\|g\right\|_1 \left\|f\right\|_{B_1^{\beta,1}} + L\left|A_\alpha\right| \left\|g\right\|_1 \left\|f\right\|_1.$$

This is the desired result. □

Now, we will determine the $B_1^{\beta,1}\left(\left[c,\infty\right)^d\right)$ distance between two MSTFrFT.

Theorem 9. Let f and h are in $B_1^{\beta,1}\left(\left[c,\infty\right)^d\right)$, ϕ and ψ are nonzero functions in $L^1\left(\left[c,\infty\right)^d\right)$. Then we have

$$\left\|V_\phi^\alpha f\left(\cdot,b\right) - V_\psi^\alpha h\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$\le \left|A_\alpha\right| \left(\left\|\phi-\psi\right\|_1 \left\|f\right\|_{B_1^{\beta,1}} + \left\|\psi\right\|_1 \left\|f-h\right\|_{B_1^{\beta,1}} \right)$$

$$+ L\left|A_\alpha\right| \left(\left\|\phi-\psi\right\|_1 \left\|f\right\|_1 + \left\|\psi\right\|_1 \left\|f-h\right\|_1 \right).$$

Proof. Since

$$V_\phi^\alpha f\left(a,b\right) - V_\psi^\alpha f\left(a,b\right) = V_{(\phi-\psi)}^\alpha f\left(a,b\right)$$

and

$$V_\psi^\alpha f\left(a,b\right) - V_\psi^\alpha h\left(a,b\right) = V_\psi^\alpha\left(f-h\right)\left(a,b\right),$$

we obtain by Theorem 8

$$\left\|V_\phi^\alpha f\left(\cdot,b\right) - V_\psi^\alpha h\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$\le \left\|V_\phi^\alpha f\left(\cdot,b\right) - V_\psi^\alpha f\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$+ \left\|V_\psi^\alpha f\left(\cdot,b\right) - V_\psi^\alpha h\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$= \left\|V_{(\phi-\psi)}^\alpha f\left(\cdot,b\right)\right\|_{B_1^{\beta,1}} + \left\|V_\psi^\alpha\left(f-h\right)\left(\cdot,b\right)\right\|_{B_1^{\beta,1}}$$

$$\le \left|A_\alpha\right| \left\|\phi-\psi\right\|_1 \left\|f\right\|_{B_1^{\beta,1}} + L\left|A_\alpha\right| \left\|\phi-\psi\right\|_1 \left\|f\right\|_1$$

$$+ \left|A_\alpha\right| \left\|\psi\right\|_1 \left\|f-h\right\|_{B_1^{\beta,1}} + L\left|A_\alpha\right| \left\|\psi\right\|_1 \left\|f-h\right\|_1.$$

□

Now we will investigate the $B_{1,k}^{\beta,1}\left(\left[c,\infty\right)^d\right)$-boundedness of the MSTFrFT.

Theorem 10. Suppose $0 \neq g \in L^1\left(\left[c,\infty\right)^d\right)$ is a window function such that:

$$\int_{[c,\infty)^d} \left|g(x)\right|\left(1+C|x|\right)^N dx \leq M < \infty \tag{12}$$

where M, C and N are positive constants. Then

$$V_g^\alpha : B_{1,k}^{\beta,1}\left(\left[c,\infty\right)^d\right) \to B_{1,k}^{\beta,1}\left(\left[c,\infty\right)^d\right),$$

$f \to V_g^\alpha f(\cdot,b)$ is bounded. Moreover, for all $f \in B_{1,k}^{\beta,1}\left(\left[1,\infty\right)^d\right)$, we have the inequality:

$$\left\|V_g^\alpha f(\cdot,a)\right\|_{B_{1,k}^{\beta,1}} \leq M\left|A_\alpha\right|\left(\left\|f\right\|_{B_{1,k}^{\beta,1}} + L\left\|f\right\|_{1,k}\right),$$

where $L = 2\Big/\left(\left(d+\beta-1\right)^d c^{d(d+\beta-1)}\right)$.

Proof. By the equality (4) and the Minkowski inequality, it can be readily seen that:

$$\begin{aligned}
&\left\|V_g^\alpha f(\cdot,b)\right\|_{1,k} = \int_{[c,\infty)^d} \left|V_g^\alpha f(a,b)\right| k(a)\,da \\
&= \left|A_\alpha\right| \int_{[c,\infty)^d} \left|\left(\mathrm{N}_\alpha f * M_{diag(\csc\alpha)b}\, g^{\square}\right)(a)\right| k(a)\,da \\
&= \left|A_\alpha\right| \int_{[c,\infty)^d} \left| \int_{[c,\infty)^d} \mathrm{N}_\alpha f(a-x) M_{diag(\csc\alpha)b}\, g^{\square}(x)\,dx \right| k(a)\,da \\
&\leq \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g^{\square}(x)\right| \left(\int_{[c,\infty)^d} \left|f(a-x)\right| k(a)\,da \right) dx .
\end{aligned}$$

By a property of tempered weight functions, by the change of variable $y = a - x$, we write $k(a) \leq \left(1+C|x|\right)^N k(a-x)$. Hence, we get by: (12)

$$\begin{aligned}
&\left\|V_g^\alpha f(\cdot,b)\right\|_{1,k} \\
&\leq \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g^{\square}(x)\right|\left(1+C|x|\right)^N \\
&\qquad \left(\int_{[c,\infty)^d} \left|f(y)\right| k(y)\,dy \right) dx \\
&= \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g(x)\right|\left(1+C|x|\right)^N \left\|f\right\|_{1,k}\,dx \\
&= M\left|A_\alpha\right|\left\|f\right\|_{1,k} .
\end{aligned} \tag{13}$$

On the other hand, we obtain:

$$\begin{aligned}
&w_{1,k}\left(V_g^\alpha f(\cdot,b),h\right) \\
&= \left\|V_g^\alpha f(\cdot+h,b) - V_g^\alpha f(\cdot,b)\right\|_{1,k} \\
&= \int_{[c,\infty)^d} \left|V_g^\alpha f(a+h,b) - V_g^\alpha f(a,b)\right| k(a)\,da \\
&= \int_{[c,\infty)^d} \left| \int_{[c,\infty)^d} f(x+a+h)\overline{g(x)} K_\alpha(x+a+h,b)\,dx \right. \\
&\qquad \left. - \int_{[c,\infty)^d} f(x+a)\overline{g(x)} K_\alpha(x+a,b)\,dx \right| k(a)\,da \\
&\leq \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g(x)\right| \\
&\qquad \left(\int_{[c,\infty)^d} \left| f(x+a+h) e^{-2\pi i h\cdot\left(b\cdot\csc\alpha-(x+a)\cdot\cot\alpha\right)} e^{i\pi h^2\cot\alpha} \right.\right. \\
&\qquad \left.\left. - f(x+a)\right| k(a)\,da \right) dx \\
&\leq \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g(x)\right|\left(1+C|x|\right)^N \\
&\qquad \left(\int_{[c,\infty)^d} \left| f(x+a+h) - f(x+a)\right| k(x+a)\,da + \right. \\
&\qquad \left. + \int_{[c,\infty)^d} \left| f(x+a)\left(e^{-2\pi i h\cdot\left(b\cdot\csc\alpha-(x+a)\cdot\cot\alpha\right)} e^{i\pi h^2\cot\alpha} - 1\right)\right| k(x+a)\,da \right) dx \\
&\leq \left|A_\alpha\right| \int_{[c,\infty)^d} \left|g(x)\right|\left(1+C|x|\right)^N \left(w_{1,k}(f,h) + 2\left\|f\right\|_{1,k}\right) dx \\
&\leq \left|A_\alpha\right| M\left(w_{1,k}(f,h) + 2\left\|f\right\|_{1,k}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{[c,\infty)^d} w_{1,k}\left(V_g^\alpha f(\cdot,b),h\right)\frac{dh}{h^{d+\beta}} \\
&\leq \left|A_\alpha\right| M\left(\int_{[c,\infty)^d} \left(w_{1,k}(f,h) + 2\left\|f\right\|_{1,k}\right)\frac{dh}{h^{d+\beta}} \right) \\
&= \left|A_\alpha\right| M\left(\int_{[c,\infty)^d} w_{1,k}(f,h)\frac{dh}{h^{d+\beta}} + 2\left\|f\right\|_{1,k} \int_{[c,\infty)^d} \frac{dh}{h^{d+\beta}} \right) \\
&= \left|A_\alpha\right| M\left(\int_{[c,\infty)^d} w_{1,k}(f,h)\frac{dh}{h^{d+\beta}} + L\left\|f\right\|_{1,k} \right).
\end{aligned} \tag{14}$$

It follows from (13) and (14) that

$$\left\|V_g^\alpha f(\cdot,b)\right\|_{B_{1,k}^{\beta,1}}$$

$$\le \left\| V_g^\alpha f(\cdot,b) \right\|_{1,k} + \int_{[c,\infty)^d} w_{1,k}\left(V_g^\alpha f(\cdot,b), h \right) \frac{dh}{h^{d+\beta}}$$

$$\le M|A_\alpha| \|f\|_{1,k} + |A_\alpha| M \left(\int_{[c,\infty)^d} w_{1,k}(f,h) \frac{dh}{h^{d+\beta}} + L\|f\|_{1,k} \right)$$

$$= M|A_\alpha| \left(\|f\|_{1,k} + \int_{[c,\infty)^d} w_{1,k}(f,h) \frac{dh}{h^{d+\beta}} \right) + LM|A_\alpha| \|f\|_{1,k}$$

$$= M|A_\alpha| \left(\|f\|_{B_{1,k}^{\beta,1}} + L\|f\|_{1,k} \right).$$

□

The following result's proof can be carried out similarly to how Theorem 9 is proved.

Theorem 11. Let f and h are in $B_{1,k}^{\beta,1}\left([c,\infty)^d \right)$, and $\phi, \psi \in L^1\left([c,\infty)^d \right)$ be nonzero functions that satisfy condition (12) for $\phi - \psi$ and ψ with constants M_1 and M_2, respectively. Then we have:

$$\left\| V_\phi^\alpha f(\cdot,b) - V_\psi^\alpha h(\cdot,b) \right\|_{B_{1,k}^{\beta,1}}$$

$$\le M_1 |A_\alpha| \left(\|f\|_{B_{1,k}^{\beta,1}} + L\|f\|_{1,k} \right)$$

$$+ M_2 |A_\alpha| \left(\|f-h\|_{B_{1,k}^{\beta,1}} + L\|f-h\|_{1,k} \right).$$

Example 12. Let $d=1$ and $f(x) = g(x) = e^{i2\pi kx}$, $k \in R$. Then the STFrFT of f with respect to g is

$$V_g^\alpha f(x,w) = \int e^{i2\pi kt} e^{-i2\pi k(t-x)} K_\alpha(w,t)\, dt$$

$$= A_\alpha e^{i\pi w^2 \cot\alpha} e^{i2\pi kx} \int e^{i(\pi\cot\alpha)t^2} e^{-i2\pi t w \csc\alpha}\, dt$$

$$= A_\alpha e^{i\pi w^2 \cot\alpha} e^{i2\pi kx} \mathrm{F}\left(e^{i(\pi\cot\alpha)t^2} \right)(w\csc\alpha)$$

$$= A_\alpha e^{i\pi w^2 \cot\alpha} e^{i2\pi kx} \mathrm{F}\left(e^{i(\pi\cot\alpha)t^2} \right)(w\csc\alpha)$$

$$= A_\alpha e^{i\pi w^2 \cot\alpha} e^{i2\pi kx} \sqrt{\tan\alpha}\, e^{-i\left(\left(\frac{\pi w \csc\alpha}{\pi\cot\alpha} \right)^2 - \frac{\pi}{4} \right)}$$

$$= A_\alpha \sqrt{\tan\alpha}\, e^{i\pi w^2 \cot\alpha} e^{i2\pi kx} e^{-i\left(w^2 \sec^2\alpha - \frac{\pi}{4} \right)}.$$

3 Conclusion

A powerful tool for analyzing the chirp signal is the fractional Fourier transform (FrFT). But in certain situations, it is unable to locate the fractional Fourier domain (FrFD)-frequency contents. To address this issue, the short-time fractional Fourier transform (STFrFT) is suggested. In the short-time fractional Fourier domain (STFrFD), it simultaneously shows the time and FrFD-frequency data. In this study, we investigate the continuity of STFrFT for multidimensional case, which has an important place in signal processing, on Morrey, Besov, and weighted Besov spaces, which also have important applications in harmonic analysis.

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