# On the Nonlocal Cahn-hilliard Inpainting Model with Singular Nonlinearity

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*Abstract:* In this paper, we consider the nonlocal Cahn-Hilliard equation with a fidelity term and singular nonlinearities, which has numerous applications in biology and image inpainting. First, we prove the existence of a unique solution. We approximate the singular nonlinearities by regular nonlinearities, from which we derive important priori-estimates for the solution of the approximating problem. We then establish the existence of a solution to the approximating problem using the Faedo-Galerkin method. Additionally, we demonstrate that the solution of the Cahn-Hilliard equation in question is of higher regular by applying Sobolev embedding theorems. Finally, we discuss the important separation property of the solution that allows us to obtain a global (in time) unique solution of the problem.

*Key-Words:* Nonlocal Cahn-Hilliard, Well-possednes, Singular potential, Image Inpainting, Separation property, Galerkin algorithm

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## 1 Introduction

The Cahn-Hilliard equation, proposed in 1958, describes phase separation between two materials . This model is the natural choice from a mathematical point of view. However, from a physical (microscopic) perspective, when the domain is too large, the $\varepsilon$-neighborhood in the classical model becomes insufficient. To address this, Giacomin and Lebowitz proposed a nonlocal model in 1992 to describe the same phenomenon from a microscopic standpoint. Subsequently, Bates and Han proposed a modification of the Giacomin-Lebowitz model. We refer the readers for more details [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [19], [20], [21].

Giacomin and Lebowitz pointed out that this equation cannot be rigorously derived from microscopic principles. Using statistical mechanics and the hydrodynamic limit, they formulated a continuum model-a nonlocal Cahn-Hilliard equation-governed by the following free energy functional:

$$\mathcal{E}_{nCH1} = -\frac{1}{2}\int_\Omega\int_\Omega M(x-y)\varphi(x)\varphi(y)dxdy + \int_\Omega S(\varphi(x))dx.$$

This equation is known as Helmholtz free energy, and $M : \mathbb{R}^l \longrightarrow \mathbb{R}^l$ represents a smooth convolution kernel, fulfilling $M(x) = M(-x)$. Additionally, $S$ is a convex potential, typically approximated by a convex polynomial and defined as:

$$S(m) = (m+1)\ln(m+1)+(1-m)\ln(1-m),\ m \in ]-1,1[. \quad (1)$$

Then, the nonlocal Cahn-Hilliard equation given by

$$\frac{\partial\varphi}{\partial t} = \Delta v, \quad v := \frac{\partial \mathcal{E}_{nCH1}}{\partial \varphi} = f(\varphi) - M *_{\Omega} \varphi, \tag{2}$$

where

$$(M*_{\Omega}\varphi)(x,t) = \int_{\omega} M(x-y)\varphi(y,t)dy, \quad x \in \Omega, t \geq 0.$$

The nonlocal Cahn-Hilliard energy can also be expressed in the form

$$\mathcal{E}_{nCH2} = \frac{1}{4}\int_{\Omega}\int_{\Omega} M(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_{\Omega} F(\varphi(x))dx,$$

was proposed by the authors Bates and Han in [3,4]. Here, $F$ denotes the double-well potential, as in the classical Cahn-Hilliard model, and is a non-convex function. It is typically approximated by a smooth potential of the following form:

$$F(m) = \frac{m^4}{4} + \frac{\lambda_1 - \lambda_2}{2} m^2, \quad \forall m \in \mathbb{R}, \tag{3}$$

where $\lambda_j, j = 1,2$ are positive constants. Otherwise, if $\lambda_1 - \lambda_2 > 0$, then $F$ can be seen as a smooth approximation of $S$.

### 1.1 Notations

Let $\Omega \subset \mathbb{R}^l$ be bounded, and $Y$ a real Banach space with dual $Y^*$. We also introduce the following two spaces:

$$H^1(0,T,Y,Y^*) = \{\rho \in L^2(0,T,Y); \rho_t \in L^2(0,T,Y^*)\}$$

and

$$H_0^1(\Omega) = \{\rho \in H^1(\Omega); \overline{\rho} = 0\},$$

where $\overline{\rho} = \langle \rho, \frac{1}{|\Omega|}\rangle \quad \forall \rho \in (H^1(\Omega))^*$.

The inner product $\langle , \rangle$ represents the pairing of $H^1(\Omega)$ and $(H^1(\Omega))^*$ and the variables $\rho', \partial_t\rho, \frac{\partial\rho}{\partial t}$, and $\rho_t$ represent the partial derivative of $\rho$ with respect to time $t$.

Lastly, for any $\varrho \in H^1([0,T]\times\Omega)$, we denote its partial derivative with respect to time by $\partial_t\varrho$ or $\frac{\partial\varrho}{\partial t}$ and its partial derivative with respect to the $x_j$-variable by $\partial_j\varrho$.

Suppose $\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $\gamma : \mathbb{R}^N \longrightarrow \mathbb{R}$ are measurable functions. Then, the convolution of $\lambda$ and $\gamma$ defined as :

$$\lambda * \gamma(x) = \int_{\Omega} \lambda(x-y)\gamma(y)dy \text{ for } x \in \mathbb{R}^N$$

The values of $C, R$ and $E$ may vary, even within the same line.

### 1.2 Assumptions

Consider the following assumptions:

**(A)** The convolution kernel $M : \mathbb{R}^l \longrightarrow \mathbb{R}$ satisfies the following conditions:

$$M(x) = M(-x), \text{ a.a. } x \in \mathbb{R}^l, \tag{A1}$$

$$\sup_{x\in\Omega}\int_{\Omega} |M(x-y)| < +\infty, \tag{A2}$$

$$\begin{gathered}\forall p \in [1,+\infty], \ \exists r_p > 0 \text{ such that} \\ \|M * \vartheta\|_{W^{1,p}(\Omega)} \leq r_p \|\vartheta\|_{L^p(\Omega)},\end{gathered} \tag{A3}$$

$$\exists R > 0 \text{ such that } \|M * \vartheta\|_{W^{2,2}(\Omega)} \leq R\|\vartheta\|_{W^{1,2}(\Omega)}. \tag{A4}$$

**(B)** The initial condition $\phi_0$ is assumed to satisfy

$$\varphi_0 \text{ is measurable}, \tag{B1}$$

$$0 \leq \varphi_0(x) \leq 1 \text{ a.e. } x \in \Omega, \tag{B2}$$

$$0 < \overline{\varphi}_0 < 1. \tag{B3}$$

**(C)** The damaged image $h \in L^2(\Omega)$ takes values between 0 and 1.

**(D)** The constant $\lambda_0$ is positive and assumed to be large.

**(E)** The indicator function $\chi_{\Omega\backslash D}(x)$ is defined by:

$$\chi_{\Omega\backslash D}(x) = \begin{cases} 0 & , x \in D \\ 1 & , x \in \Omega\backslash D. \end{cases}$$

## 2 Problem Formulation

We study the initial-boundary value problem given by:

$$\partial_t\varphi - \nabla\cdot(\mu\nabla v) = \lambda_0\chi_{\Omega\backslash D}(x)(h-\varphi), \text{ in } \Omega\times(0,T), \tag{4}$$

$$v = f'(\varphi) + \theta, \text{ in } \Omega\times(0,T), \tag{5}$$

$$\theta(x,t) = \int_{\Omega} M(|x-y|)(1-2\varphi(y,t))dy \text{ in } \Omega\times(0,T), \tag{6}$$

$$n\cdot\mu\nabla v = 0, \text{ on } \partial\Omega\times(0,T), \tag{7}$$

$$\varphi(0) = \varphi_0 \text{ in } \Omega, \tag{8}$$

where $\lambda_0\chi_{\Omega\backslash D}(x)(h-\varphi)$ is a fidelity term used in image inpainting.

The function $h(x)$ represents the damaged image, $D$ denotes the inpainting domain, and $\lambda_0$ is a constant. Equations (1.7)–(1.11) correspond to the nonlocal Cahn-Hilliard equation with a fidelity term and singular nonlinearities. Moreover, $n$ is the outward unit normal to $\partial\Omega$, and $M$ is the convolution kernel. The functions $f$ and $\mu$ are defined as:

$$f(\varphi) = \varphi\log(\varphi) + (1-\varphi)\log(1-\varphi),$$

$$\mu = \frac{1}{f''(\varphi)} = \varphi(1-\varphi),$$

where $\mu$ is the degenerate mobility.
The existence, uniqueness, regularity, and separation properties of the nonlocal Cahn-Hilliard equation with a fidelity term and singular nonlinearities have not been established in previous works. However, the authors in [12] proved the well-posedness of the convective nonlocal Cahn-Hilliard equation with a reaction term. Furthermore, in [?], the existence and uniqueness of the nonlocal Cahn-Hilliard equation with a reaction term are established.

# 3 Problem Solution

We consider the nonlocal Cahn-Hilliard equation with a fidelity term, singular (logarithmic) nonlinearities, and degenerate mobility (i.e., non-constant) in the form:

$$\partial_t\varphi - \nabla\cdot(\mu(\nabla v)) = \lambda_0\chi_{\Omega\setminus D}(x)(h-\varphi), \text{ in } \psi, \quad (9)$$

$$v = f'(\varphi) + \theta, \text{ in } \psi, \quad (10)$$

$$\theta(x,t) = \int_\Omega M(|x-y|)(1-2\varphi(y,t))dy, \text{ in } \psi, \quad (11)$$

$$n\cdot\mu\nabla v = 0, \text{ on } \Psi, \quad (12)$$

$$\varphi(0) = \varphi_0, \text{ in } \Omega, \quad (13)$$

where $\psi = \Omega\times(0,T)$, $\Omega \subseteq \mathbb{R}^l$, and $\Psi = \partial\Omega\times(0,T)$. Furthermore, the functions $f$, and $\mu$ are defined as:

$$f(\varphi) = \varphi\ln\varphi + (1-\varphi)\ln(1-\varphi), \quad (14)$$

$$\mu = \frac{1}{f''(\varphi)} = \varphi(1-\varphi). \quad (15)$$

**Definition 3.1** *Suppose that conditions (A1)-(A4) and (B1)-(B3) hold. Let $T\in(0,\infty)$. If*

$$\varphi \in H^1(0,T,H^1(\Omega),(H^1(\Omega))^*), \quad (16)$$

$$0\le\varphi\le 1 \ a.e \ in \ \psi, \quad (17)$$

$$\theta = M*(1-2\varphi) \ a.e. \ in \ \psi,$$

$$\theta\in C([0,T],W^{1,\infty}(\Omega)),$$

$$\varphi(0)=\varphi_0 \ in \ L^2(\Omega),$$

*then, $\varphi$ is a weak solution of (9)-(13) on $[0,T]$,*

**Theorem 3.2** *Under the above assumptions, there exists a unique weak solution of (**??**)–(**??**) (in the sense of the preceding definition) satisfying*

$$\varphi\in H^1(0,T,H^1(\Omega),(H^1(\Omega))^*)(\hookrightarrow C([0,T],L^2(\Omega)))$$

### 3.0.1 Uniqueness

We proceed to establish the uniqueness of the solution.

**Proof:** Let $\varphi_i$, $i\in\{1,2\}$, be two solutions of the system (9)-(13) with corresponding initial data $\varphi_{o,i}, i\in\{1,2\}$. Our goal is to prove that the solution is unique, i.e., $\varphi_1=\varphi_2$.
The variational formulation of equations (9)-(10) reads as follows:
We multiply (9)-(10) by a test function $\phi\in H^1(\psi)$ respectively and integrate over $\Omega$:

$$((\partial_t\varphi,\phi)) = -((\nabla\cdot\mu(\nabla v),\nabla\phi)) + ((\chi_{\Omega\setminus D}(x)(h-\varphi,\phi)), \quad (18)$$

$$((v,\nabla\phi)) = ((f'(\varphi)),\nabla\phi)) + ((\theta,\nabla\phi)). \quad (19)$$

Using (15), (18), and (19), we get for $i\in\{1,2\}$:

$$\begin{aligned}&((\partial_t\varphi_i,\phi)) = -((\nabla\varphi_i+\mu_i\nabla\theta_i,\nabla\phi))\\&+((\chi_{\Omega\setminus D}(x)(h-\varphi_i,\phi)) \ \forall\phi\in H^1(\psi), \text{ a.e in } (0,T),\end{aligned} \quad (20)$$

$$\mu_i = \mu(\varphi_i) = \varphi_i(1-\varphi_i), \quad \text{for i} \in \{1,2\}, \quad (21)$$

$$\theta_i = M*(1-2\varphi_i), \quad \text{for i}\in\{1,2\}. \quad (22)$$

Let $\phi$ be the difference between the two solutions, i.e. set $\phi=\varphi=\varphi_1-\varphi_2$.
For each $i\in\{1,2\}$, we have:

$$\begin{aligned}((\partial_t\varphi_1,\phi)) = &-((\nabla\varphi_1+\mu_1\nabla\theta_1,\nabla\phi))\\&+((\chi_{\Omega\setminus D}(x)(h-\varphi_1,\phi)),\end{aligned} \quad (23)$$

$$\begin{aligned}((\partial_t\varphi_2,\phi)) = &-(\nabla\varphi_2+\mu_2\nabla\theta_2,\nabla\phi))\\&+((\chi_{\Omega\setminus D}(x)(h-\varphi_2,\phi)).\end{aligned} \quad (24)$$

By subtracting now equations (23) and (24), we obtain:

$$\begin{aligned}((\partial_t\varphi,\varphi)) &= -((\nabla\varphi,\nabla\varphi)) - ((\mu_1\nabla\theta_1-\mu_2\nabla\theta_2,\nabla\varphi))\\&+((\chi_{\Omega\setminus D}(x)(h-\varphi_1)-\chi_{\Omega\setminus D}(x)(h-\varphi_2),\varphi)).\end{aligned} \quad (25)$$

Furthermore, equation (25) is equivalent to:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\varphi|^2 &= -\int_\Omega|\nabla\varphi|^2 - \int_\Omega(\mu_1\nabla\theta_1-\mu_2\nabla\theta_2)\nabla\varphi\\&+\int_{\Omega\setminus D}\left(\chi_{\Omega\setminus D}(x)(h-\varphi_1)-\chi_{\Omega\setminus D}(x)(h-\varphi_2)\right)\varphi.\end{aligned} \quad (26)$$

Then, integrating equation (26) over $(0,t)$, $t\in(0,T]$ and using the fact that

$$\int_0^t\frac{d}{dt}\varphi = \varphi(t)-\varphi_0,$$

we get

$$\begin{aligned}
&\frac{1}{2}\|\varphi(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi_{01}-\varphi_{02}\|^2_{L^2(\Omega)}\\
&= -\int_0^t\int_\Omega |\nabla\varphi|^2 - \int_0^t\int_\Omega (\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2)\nabla\varphi\\
&+ \int_0^t\int_{\Omega\setminus D}\left(\chi_{\Omega\setminus D}(x)(h-\varphi_1) - \chi_{\Omega\setminus D}(x)(h-\varphi_2)\right)\varphi\\
&\le -\int_0^t\int_\Omega |\nabla\varphi|^2 + \int_0^t\int_\Omega |\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2|\,|\nabla\varphi|\\
&+ \int_0^t\int_{\Omega\setminus D}\left|\chi_{\Omega\setminus D}(x)(h-\varphi_1) - \chi_{\Omega\setminus D}(x)(h-\varphi_2)\right||\varphi|.
\end{aligned} \tag{27}$$

Finally, we obtain the following estimates by applying Young's inequality:

$$\begin{aligned}
\int_\Omega |\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2|\,|\nabla\varphi| &\le \frac{1}{2}\int_\Omega |\nabla\varphi|^2\\
&+ \frac{1}{2}\int_\Omega |\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2|^2,
\end{aligned} \tag{$*$}$$

$$\begin{aligned}
&\int_\Omega |\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2|^2\\
&= \int_\Omega |\mu_1\nabla\theta_1 - \mu_2\nabla\theta_2 - \mu_1\nabla\theta_2 + \mu_1\nabla\theta_2|^2\\
&\le \int_\Omega |\mu_1|^2|\nabla\theta_1 - \nabla\theta_2|^2 + \int_\Omega |\nabla\theta_2|^2|\mu_1-\mu_2|^2\\
&\le C\|\nabla\theta_1 - \nabla\theta_2\|^2_{L^2(\Omega)} + \int_\Omega |\varphi(1-\varphi_1-\varphi_2)|^2|\nabla\theta_2|^2\\
&\le C\|\nabla\theta_1 - \nabla\theta_2\|^2_{L^2(\Omega)} + C\int_\Omega |\varphi|^2|\nabla\theta_2|^2\\
&\le C\|\nabla\theta_1 - \nabla\theta_2\|^2_{L^2(\Omega)} + C\|\nabla\theta_2\|^2_{L^\infty(\Omega)}\|\varphi\|^2_{L^2(\Omega)}\\
&\le C4r_2^2\|\varphi\|^2_{L^2(\Omega)} + C4r_\infty^2\|\varphi\|^2_{L^2(\Omega)}\\
&\le C\|\varphi\|^2_{L^2(\Omega)},
\end{aligned} \tag{$**$}$$

$$\begin{aligned}
&\int_{\Omega\setminus D}\left|\chi_{\Omega\setminus D}(h-\varphi_1) - \chi_{\Omega\setminus D}(h-\varphi_2)\right|\\
&\le \int_\Omega |h-\varphi_1-(h-\varphi_2)|\,|\varphi|\\
&\le \int_\Omega |\varphi_1-\varphi_2|\,|\varphi|\\
&\le \int_\Omega |\varphi|\,|\varphi|\\
&\le \|\varphi\|^2_{L^2(\Omega)}.
\end{aligned} \tag{$***$}$$

Next, by substituting $(*), (**)$ and $(***)$ into (3.19), we obtain:

$$\begin{aligned}
&\frac{1}{2}\|\varphi(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi_{01}-\varphi_{02}\|^2_{L^2(\Omega)}\\
&= -\int_0^t\int_\Omega |\nabla\varphi|^2 + \frac{1}{2}\int_0^t\int_\Omega |\nabla\varphi|^2\\
&\quad + \frac{1}{2}C\int_0^t \|\varphi\|^2_{L^2(\Omega)} + \int_0^t \|\varphi\|^2_{L^2(\Omega)}\\
&\le -\frac{1}{2}\int_0^t \|\nabla\varphi\|^2_{L^2(\Omega)} + C\int_0^t \|\varphi\|^2_{L^2(\Omega)} + \int_0^t \|\varphi\|^2_{L^2(\Omega)}\\
&\le C\int_0^t \|\varphi\|^2_{L^2(\Omega)}.
\end{aligned} \tag{28}$$

Then, we obtain:

$$\|\varphi(t)\|^2_{L^2(\Omega)} \le \|\varphi_{01}-\varphi_{02}\|^2_{L^2(\Omega)} + C\int_0^t \|\varphi\|^2_{L^2(\Omega)}.$$

Applying Gronwall's Lemma, we reach:

$$\|\varphi(t)\|^2_{L^2(\Omega)} \le e^{Ct}\|\varphi_{01}-\varphi_{02}\|^2_{L^2(\Omega)}.$$

Furthermore, since $\varphi_{01} = \varphi_{02}$,

$$\|\varphi(t)\|^2_{L^2(\Omega)} = 0 \Longleftrightarrow \varphi = 0 \Longleftrightarrow \varphi_1 = \varphi_2.$$

Hence, the problem (3.1)-(3.5) have a unique solution. □

### 3.0.2 Existence

We consider the approximate problem $P_\varepsilon$ defined as follows ($\forall \in H^1(\Omega)$, a.e in $(0,T)$):

$$((\partial_t\varphi,\phi)) + ((\mu_\varepsilon\nabla v,\nabla\phi)) = ((\chi_{\Omega\setminus D}(h-\varphi),\phi)) \tag{29}$$

$$v = f'_\varepsilon(\varphi) + \theta \text{ a.e in } \psi \tag{30}$$

$$\theta = M * (1-2\varphi) \text{ a.e in } \psi \tag{31}$$

$$n\cdot\mu_\varepsilon\nabla v = 0, \text{ on } \Psi, \tag{32}$$

$$\varphi(0) = \varphi_0, \text{ a.e. } x\in\psi, \tag{33}$$

where $f'_\varepsilon$ and $\mu_\varepsilon$ are defined as follows:

$$\mu_\varepsilon = max\{\mu+\varepsilon,\varepsilon\} \tag{34}$$

The function $f'_\varepsilon$ is a solution to the following Cauchy problem:

$$f''_\varepsilon = (1+2b_\varepsilon)\frac{1}{\mu_\varepsilon}, \text{ where } b_\varepsilon = \frac{(1+4\varepsilon)^{\frac{1}{2}}-1}{2}, \tag{35}$$

$$f'_\varepsilon\left(\frac{1}{2}\right) = f'\left(\frac{1}{2}\right), f_\varepsilon\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right). \tag{36}$$

On the other hand, $\mu_\varepsilon$ can be written as:

$$\mu_\varepsilon(m) = \begin{cases} \varepsilon & \text{for } m<0,\\ (m+b_\varepsilon)(1+b_\varepsilon-m) & \text{for } m\in[0,1],\\ \varepsilon & \text{for } m>1. \end{cases} \tag{37}$$

From this, we deduce that $\mu_\varepsilon$ is continuous and

$$\varepsilon \leq \mu_\varepsilon \leq \mu_\varepsilon\left(\frac{1}{2}\right) = \frac{1+4\varepsilon}{4}. \qquad (38)$$

Moreover, $f''_\varepsilon$ has the following form:

$$f''_\varepsilon(m) = \begin{cases} \dfrac{1+2b_\varepsilon}{\varepsilon} & \text{for } m<0, \\ \dfrac{1+2b_\varepsilon}{(m+b_\varepsilon)(1+b_\varepsilon-m)} & \text{for } m\in[0,1], \\ \dfrac{1+2b_\varepsilon}{\varepsilon} & \text{for } m>1, \end{cases} \qquad (39)$$

and

$$0 \leq f''_\varepsilon \leq \frac{1+2w_\varepsilon}{\varepsilon}. \qquad (40)$$

**Remark 3.3** *The function $f''_\varepsilon$ possesses the following symmetry property:*

$$f''_\varepsilon\left(\frac{1}{2}+m\right) = f''_\varepsilon\left(\frac{1}{2}-m\right) \ \forall m\in\mathbb{R}.$$

Using the fact that $f'_\varepsilon\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = 0$, it follows that $f'_\varepsilon(m) < 0$ for $m < \frac{1}{2}$ and $f'_\varepsilon(m) > 0$ for $m > \frac{1}{2}$. Hence,

$$\begin{cases} f'_\varepsilon(m) < 0 & \text{for } m<0, \\ f'_\varepsilon(m) = ln\left(\frac{b_\varepsilon+m}{1+b_\varepsilon-m}\right) & \text{for } m\in[0,1], \\ f'_\varepsilon(m) > 0 & \text{for } m>1 \ , \end{cases} \qquad (41)$$

and

$$f'_\varepsilon\left(\frac{1}{2}+m\right) = -f'_\varepsilon\left(\frac{1}{2}-m\right) \ \forall m\in\mathbb{R}. \qquad (42)$$

Additionally, knowing that $f''_\varepsilon \leq \dfrac{1+2b_\varepsilon}{\varepsilon}$ and $f'_\varepsilon\left(\frac{1}{2}\right) = 0$, we obtain:

$$\left|f'_\varepsilon(m)\right| \leq \frac{1+2b_\varepsilon}{\varepsilon}\left|m-\frac{1}{2}\right| \ \forall s\in\mathbb{R}. \qquad (43)$$

we thus infer that

$$f_\varepsilon\left(\frac{1}{2}+m\right) = f_\varepsilon\left(\frac{1}{2}-m\right) \ \forall m\in\mathbb{R}. \qquad (44)$$

Likewise, for $m > 1$, we have that

$$f'_\varepsilon(m) = \frac{1+2b_\varepsilon}{\varepsilon}(m-1) + f'_\varepsilon(1).$$

However, $f'_\varepsilon(m) \geq 0$ for $m \geq 0$, so that

$$f'_\varepsilon(m) \geq \frac{1+2b_\varepsilon}{\varepsilon}m - \frac{1+2b_\varepsilon}{\varepsilon} \ \ \forall\, m > \frac{1}{2}. \qquad (45)$$

Thanks Young's inequality, we get

$$\begin{aligned} & f_\varepsilon(m) - f_\varepsilon\left(\frac{1}{2}\right) \\ & \geq \frac{1+2b_\varepsilon}{2\varepsilon}m^2 - \frac{1+2b_\varepsilon}{\varepsilon}m - \frac{1+2b_\varepsilon}{2\varepsilon}\frac{1}{4} + \frac{1+2b_\varepsilon}{\varepsilon}\frac{1}{2} \\ & \geq \frac{1+2b_\varepsilon}{2\varepsilon}m^2 - \delta m^2 - \frac{1+2b_\varepsilon}{2\varepsilon}\frac{1}{4\delta} - \frac{1+2b_\varepsilon}{2\varepsilon}\frac{1}{4} + \frac{1+2b_\varepsilon}{\varepsilon}\frac{1}{2}, \end{aligned}$$

$\forall\delta > 0$. Since $\dfrac{1+2b_\varepsilon}{2\varepsilon} > \dfrac{1+b_\varepsilon}{2\varepsilon}$ and by choosing $\delta$ to be sufficiently small, we get:

$$f_\varepsilon(m) \geq \frac{1+b_\varepsilon}{2\varepsilon}\left(m-\frac{1}{2}\right)^2 - q'_\varepsilon \ \forall m\in\mathbb{R}, \qquad (46)$$

such that $q'_\varepsilon$ is a positive constant that depends on $\varepsilon$.
Besides, we have that

$$\begin{aligned} f_\varepsilon(m) & \geq \frac{1+b_\varepsilon}{2\varepsilon}\left(m^2 - m - \frac{1}{4}\right) - q'_\varepsilon \\ & \geq \frac{1+b_\varepsilon}{2\varepsilon}\left((1-\delta)m^2 - \frac{1}{4} - \frac{1}{8\delta}\right) - q'_\varepsilon, \end{aligned}$$

for all $\delta > 0$. By choosing $\delta$ sufficiently small and noting that $\dfrac{1+b_\varepsilon}{2\varepsilon} > \dfrac{1}{2\varepsilon}$, it follows that the below inequality holds:

$$f_\varepsilon(m) \geq \frac{1}{2\varepsilon}m^2 - q_\varepsilon \quad \forall m\in\mathbb{R}, \qquad (47)$$

where $q_\varepsilon > 0$ depends on $\varepsilon$.

**Lemma 3.4** *Let assumptions (A) be satisfied for $\varepsilon < \frac{1}{2r_2}$. Then, there exists a function $\varphi$ such that*

$$\varphi \in H^1(0,T,H^1(\Omega),(H^1(\Omega))^*) \cap L^\infty(0,T,L^2(\Omega))$$

*is solution of (29)-(33) with*

$$\left\|\mu_\varepsilon^{\frac{1}{2}}(\varphi)|\nabla v|\right\|_{L^2(0,T,L^2(\Omega))} \leq E,$$

*where $E$ is a constant that depends of $\varepsilon$.*

**Proof:** The construction of the proof is formed by using Faedo-Galerkin's method.
Let $Y = \{\varphi \in H^1(\Omega); \int_\Omega \varphi dx = 0\}$ be a closed subspace of $H^1$.
Then, there exist eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\gamma_1, \gamma_2, \ldots, \gamma_N$ with corresponding eigenfunctions $y_1, \cdots, y_N$, which form a Galerkin basis for the operator $-\Delta + Id : Y \longrightarrow Y^*$, where $Y \subseteq H^1(\Omega)$.
Consider the space $Y_N = span(\{y_j\}_{j=1}^N)$.
Define the orthogonal projection

$$P_N : H = L^2(\Omega) \longrightarrow Y_N$$

with functions:

$$\varphi_N = \sum_{j=1}^{N} \lambda_j(t) y_j \text{ and } v_N = \sum_{j=1}^{N} \gamma_j(t) y_j,$$

which solves the following approximate problem:

$$((\varphi_N', \phi)) + ((\mu_\varepsilon(\varphi_N)\nabla v_N, \nabla\phi)) \tag{48}$$
$$= ((\chi_{\Omega\setminus D}(x)(h-\varphi_N), \phi)) \ \forall \phi \in Y_N,$$
$$v_N = P_N(M * (1-2\varphi_N) + f_\varepsilon'(\varphi_N)),$$
$$\chi_{\Omega\setminus D}(x)(h-\varphi_N) = P_N(\chi_{\Omega\setminus D}(x)(h-\varphi_N)),$$
$$\varphi_N(0) = \varphi_{0N}. \tag{49}$$

From (34), (35), and (36), we deduce that the function $(\mu_\varepsilon(\varphi_N)\nabla v_N, \nabla\phi)$, for all $\phi \in Y_N$, is locally Lipschitz with respect to the variables $\lambda_j$, uniformly in $t$. This implies that there exists $T_N \in \mathbb{R}_+$ such that problem (48) admits a unique solution $\lambda_1, \ldots, \lambda_N, \gamma_1, \ldots, \gamma_N \in C^1([0, T_N); \mathbb{R})$.

Next, we derive a priori estimates to confirm that $\varphi_N$ is uniformly bounded in $N$ as follows: multiply equation (48) by the test function $\phi = v_N$. Then, obtain:

$$((\varphi_N', v_N)) + ((\mu_\varepsilon(\varphi_N)\nabla v_N, \nabla v_N))$$
$$= ((\chi_{\Omega\setminus D}(x)(h-\varphi_N), v_N)).$$

We now compute the function $((\varphi_N', v_N))$ as follows:

$$\begin{aligned}
&((\varphi_N', v_N)) = ((\varphi_N', f_\varepsilon'(\varphi_N)) + ((\varphi_N', M * (1-2\varphi_N))) \\
&= \int_\Omega f_\varepsilon'(\varphi_N') + \int_\Omega M * (1-2\varphi_N)\varphi_N' \\
&= \int_\Omega \frac{d}{dt} f_\varepsilon(\varphi_N) + \int_\Omega \frac{d}{dt}(M * (1-\varphi_N))\varphi_N \\
&= \frac{d}{dt}\left(\int_\Omega f_\varepsilon(\varphi_N) + \int_\Omega\int_\Omega M(x-y)\varphi_N(x)(1-\varphi_N(y))\right).
\end{aligned} \tag{50}$$

Therefore

$$\left(\int_\Omega f_\varepsilon(\varphi_N) + \int_\Omega\int_\Omega M(x-y)\varphi_N(x)(1-\varphi_N(y))\right)(t)$$
$$+ \int_0^t \int_\Omega \mu_\varepsilon(\varphi_N)|v_N|^2$$
$$= \int_0^t (\chi_{\Omega\setminus D}(x)(h-\varphi_N), v_N)$$
$$+ \left(\int_\Omega f_\varepsilon(\varphi_N) + \int_\Omega\int_\Omega M(x-y)\varphi_N(x)(1-\varphi_N(y))\right)(0).$$

Besides, we have $|f_\varepsilon'(m)| \leq C|m| + C$. Thus, by using this estimate and applying assumption (A) along with Young's inequality, we obtain:

$$\begin{aligned}
&\int_0^t (\chi_{\Omega\setminus D}(x)(h-\varphi_N), v_N) \\
&= \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h-\varphi_N) f_\varepsilon'(\varphi_N) \\
&+ \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h-\varphi_N), M * (1-2\varphi_N) \\
&\leq \int_\omega (h-\varphi_N)(C|\varphi_N| + C) + \int_\Omega [M * (1-2\varphi_N)](h-\varphi_N) \\
&\leq C\|\varphi_N\|_H^2 + C\|\varphi_N\|_H + C + r_2\|1-2\varphi_N\|_H\|h-\varphi_N\|_H \\
&\leq C\|\varphi_N\|_H^2 + C\|\varphi_N\|_H + C\|\varphi_N\|_H\|h\|_H + C \\
&\leq C\|\varphi_N\|_H^2 + \frac{C}{2}\|\varphi_N\|_H^2 + \frac{C}{2} + \frac{C}{2}\|\varphi_N\|_H^2 + \frac{C}{2}\|h\|_H^2 + C \\
&\leq C\|\varphi_N\|_H^2 + E.
\end{aligned}$$

Using (47), assumption (A), and Young's inequality, we get:

$$\begin{aligned}
&\int_\Omega f_\varepsilon(\varphi_N) + \int_\Omega\int_\Omega M(x-y)\varphi_N(x)(1-\varphi_N(y)) \\
&\geq \frac{1}{2\varepsilon}\int_\omega \varphi_N^2 - q_\varepsilon + \int_\Omega [M * (1-\varphi_N)]\varphi_N \\
&\geq \frac{1}{2\varepsilon}\|\varphi_N\|_H^2 - q_\varepsilon - r_2\|\varphi_N\|_H\|1-\varphi_N\|_H \\
&\geq \frac{1}{2\varepsilon}\|\varphi_N\|_H^2 - q_\varepsilon - r_2\|\varphi_N\|_H^2 - r_2|\Omega|\|\varphi_N\|_H \\
&\geq \left(\frac{1}{2\varepsilon} - r_2 - \delta\right)\|\varphi_N\|_H^2 - q_\varepsilon - \frac{r_2|\Omega|}{4\delta} \\
&\geq \left(\frac{1}{2\varepsilon} - r_2 - \delta\right)\|\varphi_N\|_H^2 - q_{\delta,\varepsilon} \quad \forall \delta > 0,
\end{aligned} \tag{51}$$

where $q_{\delta,\varepsilon}$ denotes a constant that depends on both $\delta$ and $\varepsilon$.

Since $\dfrac{1}{2\varepsilon} - r_2 > 0$, we choose $\delta$ such that $\left(\frac{1}{2\varepsilon} - r_2 - \delta\right) = C > 0$. Hence, (3.45) implies:

$$\int_\Omega f_\varepsilon(\varphi_N) + \int_\Omega\int_\Omega M(x-y)\varphi_N(x)(1-\varphi_N(y))$$
$$\geq C\|\varphi_N\|_H^2 + C. \tag{52}$$

Combining the previous estimations, we find

$$\|\varphi_N(t)\|_H^2 + \int_0^t\int_\Omega \mu_\varepsilon(\varphi_N)|\nabla v_N|^2 \leq C + E\int_0^t \|\varphi_N\|_H^2 \tag{53}$$

Applying Gronwall's Lemma to (**??**) yields:

$$\|\varphi_N\|_{L^\infty(0,T,H)} \leq C, \tag{54}$$

and

$$\left\|\mu_\varepsilon^{\frac{1}{2}}(\varphi_N)|\nabla v_N|\right\|_{L^2(0,T,H)} \leq C. \tag{55}$$

Moreover, we have that

$$\begin{aligned}
&\int_\Omega \mu_\varepsilon(\varphi_N)|\nabla v_N|^2 \geq \varepsilon \int_\Omega \left|\nabla f'_\varepsilon(\varphi_N) + \nabla M * (1-2\varphi_N)\right|^2 \\
&\geq \varepsilon \int_\Omega \left|\frac{(1+2b_\varepsilon)\nabla\varphi_N}{\mu_\varepsilon(\varphi_N)} + \nabla M * (1-2\varphi_N)\right|^2 \\
&\geq p\int_\Omega |\nabla\varphi_N|^2 + C\int_\Omega |\nabla M * (1-2\varphi_N)|^2 \\
&\quad + C\int_\Omega |\nabla\varphi_N||\nabla M * (1-2\varphi_N)| \\
&\geq p\int_\Omega |\nabla\varphi_N|^2 - Cr_2^2\|1-2\varphi_N\|_H^2 \\
&\quad - \delta\int_\Omega |\nabla\varphi_N|^2 - \frac{r_2^2}{4\delta}\|1-2\varphi_N\|_H^2 \\
&\geq (p-\delta)\|\nabla\varphi_N\|_H^2 - (C+R_\delta)r_2^2(\|\varphi_N\|_H^2+1).
\end{aligned}$$

Here, $\pi$ denotes a positive constant that depends solely on $\varepsilon$. Taking $\delta$ sufficiently small, we obtain:

$$\int_\Omega \mu_\varepsilon(\varphi_N)|\nabla v_N|^2 \geq p\|\varphi_N\|_H^2 - C\|\varphi_N\|_H^2 - C \tag{♣}$$

Substituting (♣) into (3.47), we obtain:

$$\|\varphi_N\|_{L^2(0,T,Y)} \leq E, \tag{56}$$

and

$$\|\nabla v_N\|_{L^2(0,T,H)} \leq E. \tag{57}$$

Furthermore,

$$\begin{aligned}
|\overline{v_N}| &= \left|\frac{1}{|\Omega|}\int_\Omega v_N\right| \\
&\leq E\left|\int_\Omega \frac{(1+2b_\varepsilon)\nabla\varphi_N}{\mu_\varepsilon(\varphi_N)}\right| + E\int_\Omega |\nabla M * (1-2\varphi_N)| \\
&\leq E\left(\frac{R}{2}\|\nabla\varphi_N\|_H^2 + \frac{R}{2}\right) + \frac{Er_2^2}{2}\|1-2\varphi_N\|_H^2 + \frac{Er_2^2}{2} \\
&\leq E\|\varphi_N\|_H^2 + E + E\|\varphi_N\|_H^2 + E \\
&\leq E\|\varphi_N\|_H^2 + E \leq E
\end{aligned} \tag{58}$$

which yields from Poincare-Wirtinger inequality and (58) that

$$\|v_N\|_{L^2(0,T,Y)} \simeq \|v_N\|_{L^2(0,T,V)} + |\overline{v_N}| \leq E+E \leq E. \tag{59}$$

Next, to obtain an a priori estimate for $\phi'_N$, we take $\varphi = \phi'_N$ as a test function in (114), and employ (38), (57) and (59), we obtain the equivalent formulation:

$$\begin{aligned}
((\varphi'_N,\varphi'_N)) &= -((\mu_\varepsilon(\varphi_N)\nabla v_N, \nabla\varphi'_N)) \\
&\quad + ((\chi_{\Omega\setminus D}(x)(h-\varphi_N), \varphi'_N)) \\
&\leq E\|v_N\|_H\|\nabla\varphi_N\|_H + \|h-\varphi_N\|_H\|\varphi'_N\|_H \\
&\leq (E\|v_N\|_H + R\|\varphi_N\|_H + R)\|\varphi'_N\|_H \\
&\leq (C+C+R)\|\varphi'_N\|_H \leq C\|\varphi'_N\|_H.
\end{aligned}$$

Thus,

$$\|\varphi'_N\|_{L^2(0,T,Y^*)} \leq C \tag{60}$$

As a result, from (57) and (57), we observe that $\varphi_N$, $\varphi'_N$,$v_N$ and $f'_\varepsilon(\varphi_N)$ are bounded sequences in $L^2(\Omega)$ and $L^\infty(\Omega)$. Then there exists (not relabeled) subsequences of each such that:

$$\varphi_N \rightharpoonup \varphi \text{ weakly in } L^2(0,T,Y), \tag{61}$$

$$\varphi_N \rightharpoonup \varphi \text{ weakly in } L^\infty(0,T,H), \tag{62}$$

$$\varphi'_N \rightharpoonup \varphi' \text{ weakly in } L^2(0,T,Y^*), \tag{63}$$

$$f'_\varepsilon(\varphi_N) \rightharpoonup f^*_\varepsilon \text{ weakly in } L^2(0,T,Y), \tag{64}$$

and

$$v_N \rightharpoonup v \text{ weakly in } L^2(0,T,Y). \tag{65}$$

**Remark 3.5** *Let* $(1 < p < \infty)$. *Consider a Hilbert space* $Y \subseteq H \subseteq Y^*$ *and a sequence* $\{\varphi_N\}$ *where* $\varphi_N : [0,T] \to Y$. *Suppose that*

$$\|\varphi_N\|_{L^2(0,T,Y)} \leq E,\ \|\varphi'_N\|_{L^p(0,T,Y^*)} \leq E,$$

*where* $E > 0$ *is independent of* $N$. *Then there* $\exists$ *a subsequence* $\{\varphi_{N_l}\}$ *such that*

$$\varphi_{N_l} \rightharpoonup \varphi \textit{ in } L^2(0,T,H).$$

Moreover, using Remark 3.5 and owing to (56) and (60), we get:

$$\varphi_N \to \varphi \text{ strongly in } L^2(0,T,H) \text{ and a.e. in } \psi. \tag{66}$$

Now (3.44) and (3.48) imply the following:

$$\|\chi_{\Omega\setminus D}(x)(h-\varphi_N)\|_{L^2(0,T,H)} \leq E. \tag{67}$$

Additionally, $\mu_\varepsilon$ and $\chi_{\Omega\setminus D}(x)(h-\varphi_N)$ are continuous functions. Then, by applying the Dominated Convergence Theorem, we get

$$\mu_\varepsilon(\varphi_N) \to \mu_\varepsilon(\varphi) \text{ a.e. in } \psi, \tag{68}$$

and

$$\chi_{\Omega\setminus D}(x)(h-\varphi_N) \to \chi_{\Omega\setminus D}(x)(h-\varphi) \text{ in } L^2(0,T,H). \tag{69}$$

Hence, using (65) and (69), we reach:

$$\mu_\varepsilon(\varphi_N)v_N \to \mu_\varepsilon(\varphi)v \text{ weakly in } L^2(\psi). \tag{70}$$

In fact, suppose $\phi \in L^2(0,T,H)$. Then, for $\{j \in 1,\ldots,l\}$ we have:

$$\int_0^T (\mu_\varepsilon(\varphi_N)\partial_j v_N, \phi) = \int_0^T (\partial_j v_N, \mu_\varepsilon(\varphi_N)\phi). \tag{71}$$

Also, (71) is equivalent to:

$$\int_0^T (\mu_\varepsilon(\varphi_N)\partial_j v_N, \phi) = \int_0^T (\partial_j v_N, \mu_\varepsilon(\varphi)\phi) + \int_0^T (\partial_j v_N - \partial_j v, \mu_\varepsilon(\varphi)\phi) + \int_0^T (\partial_j v_N, (\mu_\varepsilon(\varphi_N) - \mu_\varepsilon(\varphi))\phi). \tag{72}$$

Again applying the Dominated Convergence Theorem, we find

$$\left|\int_0^T (\partial_j v_N, (\mu_\varepsilon(\varphi_N) - \mu_\varepsilon(\varphi)\phi)\right| \leq \|\partial_j v_N\|_{L^2(0,T,H)} \|(\mu_\varepsilon(\varphi_N) - \mu_\varepsilon(\varphi))\phi\|_{L^2(0,T,H)} \leq E\|(\mu_\varepsilon(\varphi_N) - \mu_\varepsilon(\varphi))\phi\|_{L^2(0,T,H)}. \tag{$\bigstar$}$$

Hence

$$\left|\int_0^T (\partial_j v_N - \partial_j v, \mu_\varepsilon(\varphi)\phi)\right| \tag{$\bigstar\bigstar$}$$

$$\leq \|\partial_j v_N - \partial_j v\|_{L^2(0,T,H)} \|\mu_\varepsilon(\varphi)\phi\|_{L^2(0,T,H)}.$$

Now, as $N \to \infty$, we have $(\bigstar) \to 0$ and $(\bigstar\bigstar) \to 0$. Thus, we conclude that:

$$\int_0^T (\mu_\varepsilon(\varphi_N)\partial_j v_N, \phi) \longrightarrow 0 \text{ as } N \to \infty.$$

Lastly, using the fact that $f_\varepsilon'$ is continuous, we find that $f_\varepsilon'(\varphi) = f_\varepsilon^*$. Taking the limit $N$ to $\infty$, and applying the previous convergence results, we conclude that $\varphi$ is a solution of (9).

In addition, using Fatou Lemma, we reach:

$$\left\|\mu_\varepsilon^{\frac{1}{2}}(\varphi)|\nabla v|\right\|_{L^2(0,T,L^2(\Omega)}$$

$$\leq \liminf_{N\to\infty} \left\|\mu_\varepsilon^{\frac{1}{2}}(\varphi)|\nabla v|\right\|_{L^2(0,T,L^2(\Omega)} \leq C$$

Thus, the proof of Lemma 3.4 is complete. □

Now, we multiply (29) by the test function $\phi = \varphi_\varepsilon$ and, applying (A), (38) and Young's inequality, we obtain:

$$((\varphi_\varepsilon', \varphi_\varepsilon)) = \frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon\|^2_{L^2(\Omega)}$$

$$= -\int_\Omega \mu_\varepsilon \nabla\varphi_\varepsilon \nabla v_\varepsilon + \int_{\Omega\setminus D} \varphi_\varepsilon \chi_{\Omega\setminus D}(h - \varphi_\varepsilon)$$

$$= -\int_\Omega \mu_\varepsilon \nabla\varphi_\varepsilon f_\varepsilon'(\varphi_\varepsilon) - \int_\Omega \mu_\varepsilon \nabla\varphi_\varepsilon \nabla\theta_\varepsilon + \int_{\Omega\setminus D} \varphi_\varepsilon \chi_{\Omega\setminus D}(h - \varphi_\varepsilon)$$

$$\leq -\int_\Omega |\nabla\varphi_\varepsilon|^2 + Cr_2\|\nabla\varphi_\varepsilon\|_{L^2(\Omega)}\|1 - 2\varphi_\varepsilon\|_{H^1(\Omega)} + \frac{1}{2}\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|h\|^2_{L^2(\Omega)} - \|\varphi_\varepsilon\|^2_{L^2(\Omega)}$$

$$\leq -\int_\Omega |\nabla\varphi_\varepsilon|^2 + Cr_2(\delta\|\nabla\varphi_\varepsilon\|^2_{L^2(\Omega)} + R_\delta\|1 - 2\varphi_\varepsilon\|^2_{H^1(\Omega)} + E\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + E$$

$$\leq -\int_\Omega |\nabla\varphi_\varepsilon|^2 + Cr_2(\delta\|\nabla\varphi_\varepsilon\|^2_{L^2(\Omega)} + R_\delta\|1 - 2\varphi_\varepsilon\|^2_{H^1(\Omega)} + E\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + E$$

$$\leq (\delta - 1)\int_\Omega |\nabla\varphi_\varepsilon|^2 + R_\delta\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + R_\delta \; \forall\delta > 0,$$

where $R_\delta$ is a constant depending on $\delta$. This inequality is equivalent to:

$$\frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon\|^2_{L(\Omega)} + (1 - \delta)\|\nabla\varphi_\varepsilon\|^2_{L^2(\Omega)} \leq R_\delta\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + R_\delta.$$

Let $\delta < 1$ be fixed. Then:

$$\frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon\|_{L^2(\Omega)} + \|\nabla\varphi_\varepsilon\|^2_{L^2(\Omega)} \leq R\|\varphi_\varepsilon\|^2_{L^2(\Omega)} + R.$$

Applying Grownall's Lemma, we get:

$$\|\varphi_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla\varphi_\varepsilon\|_{L^2(0,T,L^2(\Omega))} \leq R, \tag{73}$$

from which we can deduce that:

$$\|\varphi_\varepsilon\|_{L^2(0,T,H^1(\Omega))} < R. \tag{74}$$

Likewise, choosing the test function $\phi = v_\varepsilon$ in (29), we obtain:

$$((\varphi_\varepsilon', v_\varepsilon)) + ((m_\varepsilon \nabla v_\varepsilon, \nabla v_\varepsilon)) = \frac{d}{dt}\left\{\int_\Omega f_\varepsilon(\varphi_\varepsilon) + \int_\Omega [M * (1 - \varphi_\varepsilon)]\,\varphi_\varepsilon\right\} + \int_\Omega \mu_\varepsilon|\nabla v_\varepsilon|^2 = \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h - \varphi_\varepsilon)v_\varepsilon \tag{75}$$

Owing to (A) and (74), we find

$$\left|\int_\Omega [M * (1 - \varphi_\varepsilon)]\,\varphi_\varepsilon\right| \leq \|\varphi_\varepsilon\|_{L^2(\Omega)}\|M * (1 - \varphi_\varepsilon)\|_{L^2(\Omega)} \leq R\|1 - 2\varphi_\varepsilon\|_{L^2(\Omega)} \leq R(|\Omega| - \|\varphi_\varepsilon\|_{L^2(\Omega)}) \leq R, \tag{76}$$

and

$$\begin{aligned}
&\int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)v_\varepsilon \\
&= \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)f'_\varepsilon + \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)\theta_\varepsilon \\
&\leq \int_{\varphi_\varepsilon\leq 0} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)f'_\varepsilon + \int_{\varphi_\varepsilon\geq 0} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)f'_\varepsilon \\
&+ \int_{\varphi_\varepsilon\in(0,1)} \chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon)f'_\varepsilon \\
&+ \int_{\Omega}(h-\varphi_\varepsilon)[M*(1-2\varphi_\varepsilon)] \\
&\leq \int_{\varphi_\varepsilon\in(0,1)} (h-\varphi_\varepsilon)\ln\left(\frac{\varphi_\varepsilon+b_\varepsilon}{1-\varphi_\varepsilon+b_\varepsilon}\right) + Cr_2\|1-2\varphi_\varepsilon\|_{L^2(\Omega)} \\
&\leq \int_{\varphi_\varepsilon\in(0,1)} (h-\varphi_\varepsilon)\ln(\varphi_\varepsilon+b_\varepsilon) \\
&- \int_{\varphi_\varepsilon\in(0,1)} (h-\varphi_\varepsilon)\ln(1-\varphi_\varepsilon+b_\varepsilon) + R \\
&\leq (R-\|h\|_{L^2(\Omega)})E - (R-\|h\|_{L^2(\Omega)})C + R \\
&\leq R.
\end{aligned} \tag{77}$$

Thanks to (76) and (77), we have that

$$\left\|\mu_\varepsilon^{\frac{1}{2}}|\nabla v_\varepsilon|\right\|_{L^2(\psi)} \leq R, \tag{78}$$

and

$$\left|\int_\Omega f_\varepsilon(\varphi_\varepsilon)\right| \leq R. \tag{79}$$

Similarly, for $\varphi_\varepsilon$, let $\phi = \varphi'_\varepsilon$ be the test function in (29). Then, using (2.30), and (2.67), we obtain:

$$\|\varphi'_\varepsilon\|_{L^2(0,T,(H^1(\Omega))^*)} \leq R. \tag{80}$$

Hence, owing to Remark 2 and(77), there exists a (not relabeled) subsequence $\varphi \in H^1(0,T,H^1(\Omega),(H^1(\Omega))^*)\cap L^\infty(0,T,L^2(\Omega))$ such that:

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly in } L^2(0,T,H^1(\Omega)), \tag{81}$$

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly in } L^\infty(0,T,L^2(\Omega)), \tag{82}$$

$$\varphi'_\varepsilon \rightharpoonup \varphi' \text{ weakly in } L^2(0,T,(H^1(\Omega))^*), \tag{83}$$

$$\chi_{\Omega\setminus D}(x)(h-\varphi_\varepsilon) \rightharpoonup \chi_{\Omega\setminus D}(x)(h-\varphi) \tag{84}$$

pointwise a.e in $\psi$, and

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ strongly in } L^2(0,T,L^2(\Omega)) \text{ a.e in } \psi. \tag{85}$$

In addition, we have

$$\theta_\varepsilon \to \theta = M*(1-2\varphi) \text{ in } L^2(0,T,H^1(\Omega)) \tag{86}$$

and

$$\mu_\varepsilon(\varphi_\varepsilon) \to \mu(\varphi) \text{ a.e. in } \psi. \tag{87}$$

Thus, by combining (86) and (87), we reach:

$$\mu_\varepsilon(\varphi_\varepsilon)\nabla\theta_\varepsilon \to \mu(\varphi)\nabla\theta \text{ in } L^2(\psi). \tag{88}$$

On the other hand, applying the Dominated Convergence Theorem, we find

$$\begin{aligned}
&\|\mu_\varepsilon(\varphi_\varepsilon)\nabla\theta_\varepsilon - \mu(\varphi)\nabla\theta\|_{L^2(\Omega)} \\
&\leq \|(\mu_\varepsilon(\varphi_\varepsilon)-\mu(\varphi))\nabla\theta_\varepsilon\|_{L^2(\Omega)} + \|\mu(\varphi)(\nabla\theta_\varepsilon-\nabla\theta)\|_{L^2(\Omega)} \\
&\leq \|\mu_\varepsilon(\varphi_\varepsilon)-\mu(\varphi)\|_{L^2(\Omega)}\|\nabla\theta_\varepsilon\|_{L^2(\Omega)} \\
&+ \|\mu(\varphi)\|_{L^2(\Omega)}\|\nabla\theta_\varepsilon-\nabla\theta\|_{L^2(\Omega)} \longrightarrow 0 \text{ as } \varepsilon\to 0.
\end{aligned}$$

Therefore, taking the limit $\varepsilon \to 0$, we deduce that $\varphi$ is a solution of (9)-(13).

Besides, we have that

$$\mu_\varepsilon(m) = \varepsilon \ \forall m<0 \text{ and } \forall m>1$$

and

$$f''_\varepsilon(m) = \frac{1+2b_\varepsilon}{\varepsilon} \ \forall m<0 \text{ and } \forall m>1.$$

Hence, for $m>1$, we reach:

$$f'_\varepsilon(m) \geq \frac{1+2b_\varepsilon}{\varepsilon}(m-1)+f'_\varepsilon(m) \geq \frac{1+2b_\varepsilon}{\varepsilon}(m-1)$$

and

$$f_\varepsilon(m) \geq \frac{1+2b_\varepsilon}{2\varepsilon}(m-1)^2 + f_\varepsilon(1).$$

Similarly for $m<0$, we have:

$$f_\varepsilon(m) \geq \frac{1+2b_\varepsilon}{2\varepsilon}m^2 + f_\varepsilon(0).$$

Now, for $m=\varphi_\varepsilon$, we have

$$\begin{aligned}
\int_{\varphi_\varepsilon>1}(\varphi_\varepsilon-1)^2 &\leq \frac{2\mu_\varepsilon(1)}{1+2b_\varepsilon}\int_{\varphi_\varepsilon>1}|f_\varepsilon(\varphi_\varepsilon)| \\
&- \frac{2\mu_\varepsilon(1)}{1+2b_\varepsilon}\int_{\varphi_\varepsilon>1}|f_\varepsilon(1)| \\
&\leq \frac{\mu_\varepsilon(1)}{1+2b_\varepsilon}\left(R-2\int_{\varphi_\varepsilon>1}|f_\varepsilon(1)|\right)
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
&\int_{\varphi_\varepsilon<0}\varphi_\varepsilon^2 \\
&\leq \frac{2\mu_\varepsilon(1)}{1+2b_\varepsilon}\int_{\varphi_\varepsilon<0}|f_\varepsilon(\varphi_\varepsilon)| - \frac{2\mu_\varepsilon(1)}{1+2b_\varepsilon}\int_{\varphi_\varepsilon<0}|f_\varepsilon(1)| \\
&\leq \frac{\mu_\varepsilon(1)}{1+2b_\varepsilon}\left(R-2\int_{\varphi_\varepsilon<0}|f_\varepsilon(1)|\right).
\end{aligned} \tag{90}$$

Choosing $\dfrac{\mu_\varepsilon(1)}{1+2b_\varepsilon} = o(1)$ and $f_\varepsilon = o(1)$ as $\varepsilon \to 0$ in both (89) and (90), we obtain:

$$\int_{\varphi>1} (\varphi-1)^2 = 0$$

and

$$\int_{\varphi<0} \varphi^2 = 0.$$

Therefore,

$$0 \le \varphi \le 1 \text{ a.e in } \psi.$$

In conclusion, Theorem 3.2 is proved.

## 3.1 Further regularity

**Theorem 3.6** *Suppose the assumptions of Theorem 3.2 are satisfied. Then, there exists a weak solution to (4)-(8) as in Definition 3.1. Additionally, $\varphi_0$ satisfies:*

$$\varphi_0 \in H^2(\Omega) \tag{91}$$

*and*

$$n\cdot(\nabla\varphi_0+\mu(\varphi_0)\nabla M*(1-2\varphi_0)) = 0 \text{ on } \partial\Omega. \tag{92}$$

*Then, $\varphi \in L^\infty(0,T,H^2(\Omega))$.*

The four lemmas, along with there proofs, are as follows:

**Lemma 3.7** *Assume the conditions of Theorem 3.1 are satisfied. Then, there exists $\varphi$ a solution of (4)-(8) as defined in Definition 3.1 such that*

$$\varphi_t \in L^\infty(0,T,(H^1(\Omega))^*) \cap L^2(0,T,L^2(\Omega)). \tag{93}$$

**Proof:** Firstly, we have $\varphi_t(0) \in (H^1(\Omega))^*$. Consequently, we obtain:

$$\overline{\varphi}_t = \int_\Omega \chi_{\Omega\setminus D}(x)(\varphi-h)dx \text{ for a.e. } t\in[0,T]. \tag{94}$$

However, since $\Omega$ is bounded, we have:

$$\overline{\varphi}_t \in L^\infty(0,T) \text{ and } \|\overline{\varphi}_t\|_{L^\infty(0,T)} \le E. \tag{95}$$

Denote $(H_0^1(\Omega))^*$ by $H_0^{-1}(\Omega)$. Then, it remains to show that:

$$\Phi = \varphi_t - \overline{\varphi}_t \in L^\infty(0,T,H_0^{-1}(\Omega)) \cap L^2(0,T,L^2(\Omega)).$$

Consider the realization of the Laplacian with the boundary conditions $\Delta_N : H_0^1(\Omega) \to H_0^{-1}(\Omega)$. In other words, the proof will proceed by approximating the time derivative with $\Delta_N$. Now, choosing the test function $\phi = \Delta_N^{-1}\Phi_t$ and noting that $\overline{\varphi}_t$ has a zero mean, the variational formulation of (4) becomes:

$$((\Phi_t,\Delta_N^{-1}\Phi_t))$$
$$= -((\mu(\Phi)\nabla v,\nabla\Delta_N^{-1}\Phi_t)) + ((\chi_{\Omega\setminus D}(x)(\varphi-h),\Delta_N^{-1}\Phi_t)).$$

Differentiating each term in the variational formula above with respect to t, we reach the following computations:

$$\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} = 2((\Phi_{tt},\Phi_t))_{H_0^{-1}(\Omega)}$$
$$= 2((\nabla\Delta_N^{-1}\Phi_{tt},\nabla\Delta_N^{-1}\Phi_t)) = -2((\Phi_{tt},\Delta_N^{-1}\Phi_t))_{L^2(\Omega)},$$

which then implies:

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + ((\Phi_{tt},\Delta_N^{-1}\Phi_t))_{L^2(\Omega)} = 0. \tag{96}$$

Moreover, we have

$$-((\mu(\Phi)(\nabla v)_t,\nabla\Delta_N^{-1}\Phi_t))$$
$$= -(((\mu\nabla\theta)_t,\nabla\Delta_N^{-1}\Phi_t))_{L^2(\Omega)} - ((\nabla\Phi_t,\nabla\Delta_N^{-1}\Phi_t))_{L^2(\Omega)}$$
$$= -(((\mu\nabla\theta)_t,\nabla\Delta_N^{-1}\Phi_t))_{L^2(\Omega)} + \|\Phi_t\|^2_{L^2(\Omega)}.$$

From this, and using the identity

$$-((\mu(\nabla v)_t,\nabla\Delta_N^{-1}\Phi_t)) = ((\nabla(\mu\nabla v)_t,\Delta_N^{-1}\Phi_t)),$$

we obtain:

$$((\nabla(\mu\nabla v)_t,\Delta_N^{-1}\Phi_t)) + (((\mu\nabla\theta)_t,\nabla\Delta_N^{-1}\Phi_t))_{L^2(\Omega)}$$
$$- \|\Phi_t\|^2_{L^2(\Omega)} = 0. \tag{97}$$

Thus, combining (96) and (97), we get:

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + ((\Phi_{tt},\Delta_N^{-1}\Phi_t))_{L^2(\Omega)}$$
$$= -\|\Phi_t\|^2_{L^2(\Omega)} + ((\nabla(\mu\nabla v)_t,\Delta_N^{-1}\Phi_t))$$
$$+ (((\mu\nabla\theta)_t,\nabla\Delta_N^{-1}\Phi_t))_{L^2(\Omega)}.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + \|\Phi_t\|^2_{L^2(\Omega)}$$
$$= -((\Phi_{tt},\Delta_N^{-1}\Phi_t))_{L^2(\Omega)} + (((\mu\nabla\theta)_t,\nabla\Delta_N^{-1}\Phi_t))$$
$$+ ((\nabla(\nabla\varphi+\mu\nabla\theta)_t,\Delta_N^{-1}\Phi_t))_{L^2(\Omega)}. \tag{98}$$

Furthermore, differentiating (4) with respect to $t$, we get:

$$\varphi_{tt} = \nabla\varphi_t + (\mu\nabla\theta)_t + \chi_{\Omega\setminus D}(x)(\varphi_t - h_t) \tag{99}$$

and

$$\overline{\varphi}_{tt} = \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t)dx. \qquad (100)$$

Then, Subtracting (99) and (100), we deduce the following equation:

$$\begin{aligned}\Phi_{tt} &= \varphi_{tt} - \overline{\varphi}_{tt}\\ &= \nabla\varphi_t + (\mu\nabla\theta)_t + \chi_{\Omega\setminus D}(x)(\varphi_t - h_t)\\ &+ \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t),\end{aligned}$$

that is equivalent to:

$$\begin{aligned}&-((\Phi_{tt}, \Delta_N^{-1}\phi_t))_{L^2(\Omega)} + ((\nabla(\nabla\theta + \mu\nabla\theta)_t, \Delta_N^{-1}\Phi_t))\\ &= -((\chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t))\\ &+ \left(\left(\int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t\right)\right).\end{aligned}$$

As a result, we deduce:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + \|\Phi_t\|^2_{L^2(\Omega)} = (((\mu\nabla\theta)_t, \nabla\Delta_N^{-1}\Phi_t))\\ &- ((\chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t))\\ &+ \left(\left(\int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t\right)\right).\end{aligned} \qquad (101)$$

It then follows from Young's inequality that

$$\begin{aligned}&(((\mu\nabla\theta)_t, \nabla\Delta_N^{-1}\Phi_t))\\ &\le \|\mu_t\nabla\theta + \mu\nabla\theta_t\|_{L^2(\Omega)}\|\nabla\Delta_N^{-1}\Phi_t\|_{L^2(\Omega)}\\ &\le r_\infty\|\varphi_t\|_{L^2(\Omega)}\|\Phi_t\|_{H_0^{-1}(\Omega)} + r_2\|\varphi_t\|_{L^2(\Omega)}\|\Phi_t\|_{H_0^{-1}(\Omega)}\\ &\le R\|\varphi_t\|_{L^2(\Omega)}\|\Phi_t\|_{H_0^{-1}(\Omega)}\\ &\le R(1 + \|\Phi_t\|_{L^2(\Omega)})\|\Phi_t\|_{H_0^{-1}(\Omega)}\\ &\le R\|\Phi_t\|_{H_0^{-1}(\Omega)} + R\left(\frac{1}{2}\|\Phi_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Phi_t\|^2_{H_0^{-1}(\Omega)}\right)\\ &\le \frac{1}{2}\|\Phi_t\|^2_{L^2(\Omega)} + E\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + R\|\Phi_t\|_{H_0^{-1}(\Omega)}.\end{aligned} \qquad (102)$$

Using the fact that $\Delta_N^{-1}\Phi_t \in H_0^1(\Omega)$ and Poincare's inequality, we find

$$\begin{aligned}&((\chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t))\\ &= \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t)\Delta_N^{-1}\Phi_t\\ &\le \|\varphi_t - h_t\|_{L^2(\Omega)}\|\Delta_N^{-1}\Phi_t\|_{L^2(\Omega)} \qquad (103)\\ &\le (1 + \|\Phi_t\|_{L^2(\Omega)} - \|h_t\|_{L^2(\Omega)})\|\Delta_N^{-1}\Phi_t\|_{L^2(\Omega)}\\ &\le \frac{1}{4}\|\Phi_t\|^2_{L^2(\Omega)} + R\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E\|\Phi_t\|_{H_0^{-1}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}&\left(\left(\int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Delta_N^{-1}\Phi_t\right)\right)\\ &\le \|\varphi_t - h_t\|_{L^2(\Omega)}\|\Delta_N^{-1}\Phi_t\|_{L^2(\Omega)}\\ &\le (|\Omega| + \|\Phi_t\|_{L^2(\Omega)} - \|h_t\|_{L^2(\Omega)})\|\Delta_N^{-1}\Phi_t\|_{L^2(\Omega)}\\ &\le R|\Omega|^2\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E|\Omega|\|\Phi_t\|_{H_0^{-1}(\Omega)} + \frac{1}{8}\|\Phi_t\|^2_{L^2(\Omega)}.\end{aligned} \qquad (104)$$

Thus, by combining (102), (103), (104), and applying Young's inequality in (101), we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + \frac{1}{8}\|\Phi_t\|^2_{L^2(\Omega)}\\ &\le R\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E\|\Phi_t\|_{H_0^{-1}(\Omega)}\\ &\le R\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E\left(\frac{1}{2}\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + \frac{1}{2}\right)\\ &\le E\|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E\end{aligned}$$

Integrating over $(0, t)$ gives:

$$\frac{1}{2}\|\Phi_t(t)\|^2_{H_0^{-1}(\Omega)} + \frac{1}{8}\int_0^t \|\Phi_t\|^2_{L^2(\Omega)}$$

$$\le E\int_0^t \|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E + \frac{1}{2}\|\Phi_0\|_{H_0^{-1}(\Omega)}$$

that also implies:

$$\|\Phi_t(t)\|^2_{H_0^{-1}(\Omega)} + \frac{1}{4}\int_0^t \|\Phi_t\|^2_{L^2(\Omega)} \le E\int_0^t \|\Phi_t\|^2_{H_0^{-1}(\Omega)} + E.$$

By now applying Gronwall's Lemma, we have:

$$\|\Phi_t\|_{L^\infty(0,T,(H^1(\Omega))^*)} + \|\Phi_t\|_{L^2(0,T,L^2(\Omega))} \le E.$$

Finally, by using (3.5), we obtain the estimate:

$$\|\varphi_t\|_{L^\infty(0,T,(H^1(\Omega))^*)} + \|\varphi_t\|_{L^2(0,T,L^2(\Omega))} \le E.$$

□

**Lemma 3.8** *Given that the conditions of Theorem* 3.1 *are satisfied, there exists* $\varphi$ *a solution to the system (4)-(8) as defined in Definition* 2.1*, such that*

$$\varphi_t \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)). \qquad (105)$$

**Proof:** We select the test function $\phi = \Phi_t$ and differentiate (4) with respect to time. This gives:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2_{L^2(\Omega)} = -((\mu_t\nabla\theta + \mu\nabla\theta_t + \nabla\Phi_t, \nabla\Phi_t))_{L^2(\Omega)}\\
&+ ((\chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Phi_t))_{L^2(\Omega)}\\
&- \left(\left(\left(\int_{\Omega\setminus D}\chi_{\Omega\setminus D}(x)(\varphi_t - h_t), \Phi_t\right)\right)\right)_{L^2(\Omega)}\\
&\leq -R\|\nabla\Phi_t\|^2_{L^2(\Omega)} - \|\Phi_t\|_{L^2(\Omega)}\|\nabla\Phi_t\|_{L^2(\Omega)}\\
&+ R\|\Phi_t\|_{L^2(\Omega)} + \|\Phi_t\|^2_{L^2(\Omega)} - E\|\Phi_t\|_{L^2(\Omega)} + \|\Phi_t\|^2_{L^2(\Omega)}\\
&\leq -R\left(\frac{1}{2}\|\Phi_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\Phi_t\|^2_{L^2(\Omega)}\right) - R\|\nabla\Phi_t\|^2_{L^2(\Omega)}\\
&+ R\|\Phi_t\|^2_{L^2(\Omega)} + E\|\Phi_t\|_{L^2(\Omega)}\\
&\leq R\|\Phi_t\|_{L^2(\Omega)} + E\|\Phi_t\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla\Phi_t\|^2_{L^2(\Omega)}.
\end{aligned} \tag{106}$$

Then, by integrating (106) over $(0,t)$, we infer:

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t(t)\|^2_{L(\Omega)} + \frac{1}{2}\int_0^t \|\nabla\Phi_t\|^2_{L^2(\Omega)}$$

$$\leq R\int_0^t \|\Phi_t\|_{L^2(\Omega)} + E\int_0^t \|\Phi_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Phi_t(0)\|^2_{L^2(\Omega)}.$$

Additionally, we observe that $\|\Phi_t(0)\|_{L^2(\Omega)}$ is bounded. Then, using Young's inequality, we reach:

$$\frac{d}{dt}\|\Phi_t(t)\|^2_{L(\Omega)} + \int_0^t \|\nabla\Phi_t\|^2_{L^2(\Omega)} \leq R\int_0^t \|\Phi_t\|^2_{L^2(\Omega)} + E.$$

Moreover, by applying Gronwall's Lemma, we obtain:

$$\|\Phi_t\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla\Phi_t\|_{L^2(0,T,H^1(\Omega)} \leq E.$$

Then, using the fact that $\nabla\overline{\varphi}_t = 0$, we get

$$\|\varphi_t\|_{L^\infty(0,T,L^2(\Omega))} + \|\varphi_t\|_{L^2(0,T,H^1(\Omega)} \leq E.$$

Thus, (105) follows. □

**Lemma 3.9** *Suppose the conditions of Theorem* 3.1 *are satisfied. Then, the solution $\varphi$ to the system (4)-(8) as defined in Definition* 2.1*, satisfies:*

$$\nabla\varphi \in L^\infty(0,T,L^2(\Omega)). \tag{107}$$

**Proof:** We deduce that $\nabla\varphi \in H^1(0,T,L^2(\Omega))$, owing to (105). Therefore, we obtain (111). □

**Lemma 3.10** *Assume that the conditions of Theorem* 3.1 *are satisfied. Then, there exists $\varphi$ a solution to the system (4)-(8), as defined in Definition* 2.1*, such that:*

$$\varphi \in L^\infty(0,T,H^2(\Omega)). \tag{108}$$

**Proof:** Owing to Lemma 3.2 and Lemma 3.3, we have $\varphi_t \in L^\infty(0,T,L^2(\Omega))$, $(1-2\varphi)\nabla\varphi\nabla\theta \in L^\infty(0,T,L^2(\Omega))$ and $\mu\Delta\theta \in L^\infty(0,T,L^2(\Omega))$. From this, we can also conclude that $\chi_{\Omega\setminus D}(x)(\varphi - h) \in L^\infty(0,T,L^2(\Omega))$. Hence,

$$((\Delta\varphi,\phi)) = ((\zeta,\phi))\ \forall\phi\in H^1(\Omega)\text{ a.e. } t\in(0,T), \tag{109}$$

such that

$$\zeta = \varphi_t + (1-2\varphi)\nabla\varphi\theta + \mu\Delta\theta + \chi_{\Omega\setminus D}(x)(\varphi - h) \tag{110}$$

in $L^\infty(0,T,L^2(\Omega))$. We thus infer that

$$\varphi \in L^\infty(0,T,H^1(\Omega)) \cap L^\infty(\psi)$$

$$\theta \in L^\infty(0,T,H^2(\Omega)) \cap L^\infty(0,T,W^{1,\infty}(\Omega))$$

$$\nabla\theta \in L^\infty(0,T,H^1(\Omega)) \cap L^\infty(\psi).$$

Moreover, since $\partial\Omega$ is locally Lipschitz, there exists a unit normal vector $n$ on $\partial\Omega$ such that $n \in L^\infty(\partial\Omega)$. Thus,

$$\frac{\partial\theta}{\partial n} \in L^\infty\left(0,T,H^{\frac{1}{2}}(\partial\Omega)\right) \cap L^\infty(0,T,L^\infty(\partial\Omega)). \tag{111}$$

By Lemma 3.3, we have $\nabla\mu(\varphi) = (1-2\varphi)\nabla\varphi \in L^\infty(0,T,L^2(\Omega))$, from which we deduce:

$$\mu(\varphi) \in L^\infty\left(0,T,H^{\frac{1}{2}}(\Omega)\right). \tag{112}$$

Given that $0 \leq \varphi \leq 1$ and $0 \leq \mu(m) \leq 1$ for all $m \in [0,1]$, and noting that this can be extended for all $m \in \mathbb{R}$ such that $0 \leq \mu(m) \leq 1$, we conclude:

$$\mu(\varphi) \in L^\infty(0,T,L^\infty(\Omega)). \tag{113}$$

Now, from (111), (112) and (113), we infer:

$$\mu(\varphi)\frac{\partial\theta}{\partial n} \in L^\infty\left(0,T,H^{\frac{1}{2}}(\partial\Omega)\right). \tag{114}$$

It then follows that:

$$\frac{\partial\varphi}{\partial n} = n\cdot\mu\nabla\theta = \mu(\varphi)\frac{\partial\theta}{\partial n}\text{ a.e on }\partial\Omega.$$

Then, by using (114), we deduce:

$$\frac{\partial\varphi}{\partial n} \in L^\infty\left(0,T,H^{\frac{1}{2}}(\partial\Omega)\right). \tag{115}$$

**Remark 3.11** ***Elliptic Regularity Theorem:*** *Let $\Omega \subset \mathbb{R}^l$, where $l \in \mathbb{N}$. Denote by $n$ the outer unit normal on $\partial\Omega$. Suppose $\zeta \in L^2(\Omega)$ and $\kappa \in$*

$H^{\frac{1}{2}}(\partial\Omega)$. *Then,* $\rho \in H^2(\Omega)$, *if there exists* $\rho \in H^1(\Omega)$, *such that* $\rho$ *is a weak solution satisfying:*

$$\begin{cases} \Delta\rho = \zeta \ in \ \Omega, \\ \frac{\partial\Omega}{\partial n} = \kappa \ on \ \partial\Omega. \end{cases}$$

*Furthermore, there exists* $E > 0$, *independent of* $\zeta$ *and* $\kappa$, *such that*

$$\|\rho\|_{H^2(\Omega)} \leq E\left(\|\rho\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} + \|\kappa\|_{H^{\frac{1}{2}}(\partial\Omega)}\right),$$

Lastly, by using Remark 3, we obtain:

$$\|\varphi\|_{H^2(\Omega)} \leq E\left(\|\varphi\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} + \|\frac{\partial\varphi}{\partial n}\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$

for almost every $t \in [0.T]$. Considering (120), (115), and the result $\varphi \in L^\infty(0, T, L^2(\Omega))$, we arrive at (108). □

## 3.2 The Separation Property

This part focuses on the separation of the solution $\varphi$ of the system (4)-(8) from singularities. First, we show that $\overline{\varphi}(t)$ lies between 0 and 1. After obtaining some a priori estimates, we prove the existence of a positive constant $q$ such that $q \leq \varphi(x,t) \leq 1-q$ a.e. $x \in \Omega$, $t \in [T_0, T]$. Thus, we have following theorem:

**Theorem 3.12** *Given* $l \leq 3$, *and assuming the conditions of Theorem* 3.1 *are verified, then for all* $T_0 \in (0, T)$, *there exists a constant* $q > 0$ *such that:*

$$q \leq \varphi(x,t) \leq 1-q \text{ for almost every } x \in \Omega, t \in [T_0, T]. \tag{116}$$

*Note that if*

$$\exists\ \widetilde{q} > 0 \text{ such that } \widetilde{q} \leq \varphi_0 \leq 1-\widetilde{q}, \tag{117}$$

*then* $T_0 = 0$.

We begin by proving that $0 < \overline{\varphi}(t) < 1$ for all $t \in [0, T]$ in Lemmas 4.1 and 4.2.

**Remark 3.13** *If* $0 \leq \varphi \leq 1$, *then* $\varphi$ *separates* 0 *and* 1 *in the following manner:*

$$0 < \overline{\varphi}(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x.t)dx < 1 \ \forall t \in [0, T].$$

**Lemma 3.14** *Assume that* $\varphi$ *is a weak solution of the system (2.1)-(2.5) as defined in Definition* 2.1 *and that (B3) is satisfied. Then, there exist constants* $d_0 > 0$ *and* $e_0 > 0$, *independent of* $t$, *such that:*

$$|\Omega_1^t| \geq e_0 > 0, \tag{118}$$

*where* $\Omega_1^t = \{x' \in \Omega; \varphi(x', t) \geq d_0\}$.

**Proof:** For simplicity, we assume $|\Omega| = 1$. We have

$$\overline{\varphi}(t) - \overline{\varphi}_0 = \int_0^t \frac{d}{dm}\overline{\varphi}(m)dm = \int_0^t ((\varphi', 1))$$

$$= \int_0^t \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi - h) = \int_0^t ((\chi_{\Omega\setminus D}(x)(\varphi-h), 1)).$$

Consequently, we deduce:

$$\overline{\varphi}(t) = \overline{\varphi}_0 + \int_0^t \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi - h). \tag{119}$$

Now, owing to the previous results $\varphi \in C([0, T], L^2(\Omega))$, the function $\overline{\varphi} : t \in [0, T] \mapsto \overline{\varphi}(t) \in [0, 1]$ is continuous. First, we show that there exists a constant $p > 0$, independent of $t$, that satisfies:

$$\overline{\varphi}(t) \geq p \ \forall t \ \in [0, T]. \tag{120}$$

By contradiction, assume that there exists $t_1 \in (0, T]$ such that $t_1$ is the point is the point where $\overline{\varphi}(t) = 0, t \in [0, T]$. From this, we deduce that $\varphi(t') = 0$ almost everywhere for $x \in \Omega$. We thus have that

$$\begin{aligned}
\overline{\varphi}(t) &= \overline{\varphi}_0 + \int_0^t \int_{\Omega\setminus D} \chi_{\Omega\setminus D}(x)(\varphi - h) \\
&\geq \overline{\varphi}_0 + \int_0^t \int_\Omega (\varphi - h) \\
&\geq \overline{\varphi}_0 + \int_0^t \int_\Omega \left[(\varphi(m) - h) - (\varphi(t') - h)\right] dm \\
&\geq \overline{\varphi}_0 + \int_0^t \int_\Omega \varphi(m)dm \\
&\geq \overline{\varphi}_0 + \int_0^t \overline{\varphi}(m)dm.
\end{aligned}$$

Applying Gronwall's Lemma, we get $\overline{\varphi}(t) \geq \overline{\varphi}_0 e^t > 0$ for all $t \in [0, T]$ (i.e., it is bounded below). However, we also have $\overline{\varphi}(t') = 0$, leading to a contradiction. Therefore, (120) is proved.

Additionally, let $d_0 = \frac{1}{2}p$. Then, we have:

$$|\Omega_1^t| \geq \frac{1}{2}p = e_0 > 0.$$

Now, suppose by contradiction that $|\Omega_1^t| < \frac{1}{2}p$. In this case, we get:

$$\begin{aligned} p \le \overline{\varphi}(t) &= \int_{\Omega_1^t} \varphi + \int_{\Omega\setminus\Omega_1^t} \varphi \\ &\le \int_{\Omega_1^t} 1 + \int_{\Omega\setminus\Omega_1^t} \frac{p}{2} \\ &\le |\Omega_1^t| + \frac{p}{2}\int_{\Omega\setminus\Omega_1^t} 1 \\ &\le \frac{p}{2} + \frac{p}{2}|\Omega\setminus\Omega_1^t|. \end{aligned}$$

Thus, we arrive that $|\Omega\setminus\Omega_1^t| > 1$, which contradicts the assumption $|\Omega| = 1$. ☐

**Lemma 3.15** *Consider that $\varphi$ is a weak solution of the system (4)-(8) as defined in Definition 2.1, and assume that assumption (B) holds. Then, there exist constants $d_0 > 0$ and $e_0 > 0$, independent of $t$, such that:*

$$|\Omega_1^t| \ge e_0 > 0, \tag{121}$$

*to which $\Omega_1^t = \{x' \in \Omega; \varphi(x', t) \le 1 - d_0\}$.*

**Proof:** The verification of this lemma is identical to the verification of Lemma 4.1,. ☐
There are two main propositions that are essential for proving Theorem 5.1.

**Proposition 3.16** *Suppose the conditions of theorem 4.0.1 are satisfied. Then, for every $T_0 \in (0, T)$ there exists a constant $q > 0$, depending on both $T_0$, and $\overline{\varphi}_0$, such that:*

$$q \le \varphi(x, t) \text{ for a.e. } x \in \Omega, t \in [T_0, T]. \tag{122}$$

*Moreover, $T_0 = 0$ provided that there exists $\widetilde{q} > 0$ such that:*

$$\widetilde{q} \le \varphi_0 \text{ almost every where in } \Omega. \tag{123}$$

**Proof:** The verification of Proposition 5.1 involves proving that:

$$\|\ln(\varphi(., t))\|_{L^\infty(\Omega)} \le R \;\forall t \in [T_0, T].$$

**Remark 3.17** *Given that $\varphi \in L^\infty(0, T, H^2(\Omega)) \cap C([0, T], L^2(\Omega))$, it follows that $\varphi \in C([0, T], H^m(\Omega))$ for all $m < 2$, and $\varphi \in C([0, T], L^\infty(\Omega))$ if $l \le 3$.*

First, by Remark 5, consider $0 < \varphi(t)$ almost everywhere in $\Omega$ for all $t \in [0, T]$. Define

$$M_j(t) = \|ln(\varphi(., t))\|_{L^\infty(\Omega)} \; j = 1, 2, 3, \cdots$$

Now, by choosing $j = 1$, and deriving a differential inequality of $M_j(t)$, we use the test function $\phi = \frac{1}{\varphi}$. This leads to the following:

$$\begin{aligned} \frac{d}{dt}M_1(t) &= \frac{d}{dt}\int_\Omega (-\ln(\varphi)) = -\left(\left(\varphi', \frac{1}{\varphi}\right)\right) \\ &= \int_\Omega \nabla\left(\frac{1}{\varphi}\right)(\nabla\varphi + \mu\nabla\theta) - \int_\Omega \frac{1}{\varphi}\chi_{\Omega\setminus D}(x)(\varphi - h). \end{aligned}$$

We compute

$$\frac{\chi_{\Omega\setminus D}(x)(\varphi - h)}{\varphi} \le -\frac{\varphi - h}{\varphi} + \frac{-h}{\varphi} \le -\frac{\varphi}{\varphi} = 1.$$

Then, we obtain

$$-\int_\Omega \frac{1}{\varphi}\chi_{\Omega\setminus D}(x)(\varphi - h) \le E. \tag{124}$$

This can be verified by showing that:

$$|\nabla\ln(\varphi)|^2 = -\nabla\varphi\nabla\left(\frac{1}{\varphi}\right)$$

and
$$\frac{\nabla\ln(\varphi)}{\varphi} = -\nabla\left(\frac{1}{\varphi}\right).$$

Afterwards, by applying Young's inequality, we compute as follows:

$$\begin{aligned} \frac{d}{dt}M_1(t) &\le -\int_\Omega |\nabla\ln(\varphi)|^2| - \int_\Omega \frac{\nabla\ln(\varphi)}{\varphi}\mu\nabla\theta + E \\ &\le -\int_\Omega |\nabla\ln(\varphi)|^2 - \int_\Omega (1-\varphi)\nabla\ln(\varphi)\nabla\theta + E \\ &\le -\int_\Omega |\nabla\ln(\varphi)|^2 + \frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + R\int_\Omega |\nabla\theta|^2 + E \\ &\le -\frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + R\int_\Omega |\nabla\theta|^2 + E \\ &\le -\frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + R\int_\Omega |M*(1-2\varphi)|^2 + E \\ &\le -\frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + R\; r_2^2\int_\Omega |1-2\varphi|^2 + E \\ &\le -\frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + E\int_\Omega |\varphi|^2 + R + E \\ &\le -\frac{1}{2}\int_\Omega |\nabla\ln(\varphi)|^2 + E. \end{aligned} \tag{125}$$

**Remark 3.18** *Let $\Omega \subset \mathbb{R}^l$, where $l \in \mathbb{N}$ be a bounded domain. Suppose $\rho \in H^1(\Omega)$ and $\Omega' \subseteq \Omega$ with $|\Omega'| > 0$. Then, there exists a constant $E \ge 0$, dependent on $\Omega$ and $\Omega'$, such that:*

$$\left\|\rho - \frac{1}{|\Omega'|}\int_{\Omega'}\rho\right\|_{L^2(\Omega)} \le E\frac{1}{|\Omega'|}\|\nabla\rho\|_{L^2(\Omega)}.$$

Consider $d_0$ , $\Omega_1^t$ and $e_0$ as defined in Lemma 4.1.

By remark 6, (2.11) and (4.3), we obtain:

$$\begin{aligned}
&\int_\Omega |\nabla \ln(\varphi)|^2 \geq E|\Omega_1^t|^2 \int_\Omega \left|\ln(\varphi) - \frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} \ln(\varphi)\right|^2 \\
&\geq E|\Omega_1^t|^2 \left[\int_\Omega |\ln(\varphi)|^2 - \frac{2}{|\Omega_1^t|}\int_\Omega \ln(\varphi)\int_{\Omega_1^t} \ln(\varphi)\right] \\
&\quad + E|\Omega_1^t|^2 \left[\int_\Omega \frac{1}{|\Omega_1^t|^2}\left(\int_{\Omega_1^t} \ln(\varphi)\right)^2\right] \\
&\geq E|\Omega_1^t|^2 \left(\int_\Omega |\ln(\varphi)|^2 - \frac{2}{|\Omega_1^t|}\int_\Omega \ln(\varphi)\int_{\Omega_1^t} \ln(\varphi)\right) \\
&\geq Ee_0^2 \left(\left(\int_\Omega |\ln(\varphi)|\right)^2 + \frac{2}{e_0}\int_\Omega |\ln(\varphi)|\int_{\Omega_1^t} \ln(d_0)\right) \\
&\geq E(M_1(t))^2 - EM_1(t) \\
&\geq E(M_1(t))^2 - E.
\end{aligned} \tag{126}$$

Now, combining (126) and (127), we deduce:

$$\frac{d}{dt}M_1(t) \leq -\frac{E}{2}(M_1(t))^2 + \frac{E}{2} + E \leq -E_1(M_1(t))^2 + E_2.$$

According to [30], we can show that for all $T_0 \in [0, T]$, there exists $A_1$ dependent on $\overline{\varphi}_0$ and $T_0$ and independent of $M_0$, such that:

$$M_1(t) \leq A_1 \; \forall t \in [T_0, T]. \tag{127}$$

Likewise, we derive a differential inequality for $M_j$.

$$\begin{aligned}
&\frac{d}{dt}M_j(t) = \frac{d}{dt}\left(\int_\Omega (-\ln(\varphi))\right)^{\frac{1}{j}} \\
&= \int_\Omega j\left(\left(\varphi', \frac{(-\ln(\varphi))^{j-1}}{\varphi}\right)\right)\left(-\frac{1}{j}\right)\left(\int_\Omega (-\ln(\varphi))^j\right)^{\frac{1}{j-1}} \\
&= M_j^{j-1}\int_\Omega \nabla\left(\frac{(-\ln(\varphi))^{j-1}}{\varphi}\right)(\nabla\varphi + \mu\nabla\theta) \\
&\quad - M_j^{j-1}\int_\Omega \chi_{\Omega\setminus D}(x)(\varphi - h)\frac{(-\ln(\varphi))^{j-1}}{\varphi}.
\end{aligned}$$

Owing to Holder's inequality with conjugates $j$ and $\frac{j}{j-1}$, we obtain the last term:

$$\begin{aligned}
&M_{j-1}^{j-1} = \int_\Omega (-\ln(\varphi))^{j-1} \\
&\leq \left(\int_\Omega 1\right)^{\frac{1}{j}}\left(\int_\Omega (-\ln(\varphi))^{(j-1)\frac{j}{j-1}}\right)^{\frac{j-1}{j}} \\
&\leq |\Omega|^{\frac{1}{j}} M_j^{j-1}.
\end{aligned}$$

Then, using (4.10), we get:

$$\begin{aligned}
&-M_j^{j-1}\int_\Omega \chi_{\Omega\setminus D}(x)(\varphi - h)\frac{(-\ln(\varphi))^{j-1}}{\varphi} \\
&\leq EM_j^{j-1}\int_\Omega (-\ln(\varphi))^{j-1} \\
&\leq EM_j^{j-1}M_{j-1}^{j-1} \\
&\leq EM_j^{j-1}|\Omega|^{\frac{1}{j}}M_j^{j-1} \leq E,
\end{aligned} \tag{128}$$

where $E$ is independent of $j$.

Next, by using (128) and [16], we can show the below differential inequality:

$$\frac{d}{dt}M_j \leq -E_3\frac{1}{j^2}(M_j)^2 + E_4A_1M_1 + E_5 j,$$

where $A_1$ is as in (127), from which we deduce that $\forall\, \bar{T} \in (0, T]$, the following holds:

$$\sup_{t\geq\bar{T}} M_j(t) \leq R_1(\bar{T})j^3 \; \forall j \in [1, +\infty),$$

where $R_1(\bar{T})$ is independent of the initial values $M_j(0)$ and $j$, and is decreasing on $(0, +\infty)$. Additionally, $\bar{T}R_1(\bar{T})$ increases as $\bar{T}$ becomes large. Lastly, by [16], we conclude that for all $T_0 > 0$, there exists $R > 0$ such that:

$$M_j(t) \leq R \; \forall t \geq T_0,$$

where $R$ depends on $T_0$ and $\overline{\varphi}_0$.
Taking the limit as $j \longrightarrow \infty$, we deduce:

$$\|\ln(\varphi(.,t))\|_{L^\infty(\Omega)} \leq R \; \forall t \in [T_0, T].$$

□

**Proposition 3.19** *Assume that the conditions of Theorem 4.1 are satisfied. Then, for every $T_0 \in (0, T)$, there exists a constant $q > 0$, dependent on both $T_0$ and $\overline{\varphi}_0$, such that:*

$$\varphi(x,t) \leq 1 - q \;\; a.e.\; x \in \Omega \text{ and } t \in [T_0, T]. \tag{129}$$

*Furthermore, if there exists $\widetilde{q}$ such that*

$$\varphi_0 \leq 1 - \widetilde{q}, \tag{130}$$

*then $T_0 = 0$.*

**Proof:** The proof of Proposition 4.2 follows similarly to that of Proposition 4.1, differing only in the substitution $\Phi = 1 - \varphi$. □
As a result, by combining Propositions 4.1 and 4.2, we deduce the proof of Theorem 4.1.

# 4 Conclusion

The objectives of this paper were to study the existence, uniqueness, regularity, and separation properties of the nonlocal Cahn-Hilliard equation with a fidelity term and singular nonlinearities. Uniqueness of the solution was established by assuming the existence of two solutions, multiplying equations (**??**)–(**??**) by a test function $\varphi \in H^1(\Psi)$, and analyzing the difference with respect to the initial state. From these computations and the equality of the initial conditions, we concluded that the solution is unique.

Existence of a weak solution was proved by approximating the singular logarithmic nonlinearities with more regular functions. Using the Faedo-Galerkin method, we derived a priori estimates for the approximate solutions, established their existence, and passed to the limit to obtain a solution to the original problem. Regularity was demonstrated through four key lemmas, which were instrumental in proving that $\varphi \in L^\infty(0,T;H^2(\Omega))$ in the main theorem.

We also proved the separation property, showing that $\overline{\varphi}(t)$ remains bounded between 0 and 1, and established the existence of a global (in time) unique solution.

An interesting challenge in the nonlocal Cahn-Hilliard model is the development of numerical simulations to evaluate its efficiency compared to the classical model. In particular, the question arises: "What advantages does the nonlocal Cahn-Hilliard model offer over the classical one in practical applications?" In future work, we aim to address this by developing a convergent numerical scheme for the proposed model and performing simulations to highlight the potential benefits of the nonlocal approach.

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