A Hankel Determinant Evaluation: An Asymmetric and Nonlinear Variant of Filbert-like Matrices

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Abstract: We define a new variant of Filbert-like matrices with entries defined by ratios of terms from general Fibonacci and Lucas sequences as well as they are defined differently based on the parity of the row indices. For each matrix, we find L and U matrices related to LU-decomposition, their inverses, inverse of the each matrix, and its determinant. All results are valid for general q. The results for general Fibonacci and Lucas numbers are obtained as natural consequences for the particular selection of q.

Key-Words: Filbert matrix, Lilbert matrix, q-analogue, LU-decomposition, inverse matrix, determinant

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1 Introduction

For any real number $p \neq 0$ and integer $n > 1$, consider the second-order linear recurrence sequences $\{U_n, V_n\}$ defined by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, U_1 = 1 \quad (1)$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, V_1 = p. \quad (2)$$

When setting $p = 1$, the sequences reduce to the familiar Fibonacci and Lucas numbers, denoted F_n and L_n, respectively. Their closed-form (Binet-style) expressions are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad (3)$$

and

$$V_n = \alpha^n + \beta^n = \alpha^n \left(1 + q^n\right), \quad (4)$$

with α, β are the roots of the characteristic equation $x^2 - px - 1 = 0$, explicitly $\alpha, \beta = \left(p \pm \sqrt{\Delta}\right)/2$ with $\Delta = p^2 + 4$. Defining the quotient $q = \beta/\alpha = -\alpha^{-2}$, it follows that $\alpha = \mathbf{i}/\sqrt{q}$.

Several earlier studies have explored various classes of combinatorial matrices, often derived from classical structures such as Hankel, Toeplitz, tridiagonal, Hessenberg, and Vandermonde matrices. The construction of these matrices often relies on particular integer sequences and, in some cases, their functional analogues. In particular, the matrix entries have been defined using Gaussian q-binomial coefficients, along with familiar numerical sequences such as natural numbers, Fibonacci numbers, and Lucas numbers. For further background and detailed examples, as detailed in the works cited in [1], [2], [3], [4], [5]. We summarize a few of these constructions below for reference.

- The Filbert matrix, denoted by $H_n = (h_{i,j})_{i,j=1}^n$ is defined through Fibonacci numbers as $h_{i,j} = \frac{1}{F_{i+j-1}}$, where F_n represents the nth Fibonacci number. This matrix serves as a Fibonacci-based analogue of the classical Hilbert matrix. The structure and properties of the Filbert matrix were first introduced and studied by [6].
- The author in [7], studied how Cameron's operator can be used to derive determinant expressions for Fibonacci numbers and their generalizations through restricted recurrence relations.
- The authors in [8], studied the bidiagonal factorization of Filbert and Lilbert matrices via Neville's elimination.
- The study, [9], demonstrated that the determinants of certain tridiagonal matrix sequences, under specific conditions, are equal to Fibonacci numbers and provided a mathematical proof of this relationship.
- The study, [10], define the nonlinear matrix $\mathcal{B} = [\mathcal{B}_{m,n}]_{m,n>0}$ as follows

  $$\mathcal{B}_{m,n} = \frac{1 + xq^{\lambda+\Phi(m,n)}}{1 + yq^{\mu+\Phi(m,n)}} \quad (5)$$

  where Φ denotes a shifted nonlinear function of the indices, which takes the following form

  $$\Phi(m,n) =: a\,(p+m)^v + b\,(r+n)^w. \quad (6)$$

The authors of the aforementioned studies work on various matrix properties such as LU-decomposition, Cholesky decomposition, determinants and inverses, ultimately deriving explicit formulas for each. To establish these matrix properties, the authors utilized q-analogue representations for the matrix entries and applied a combination of symbolic computation techniques, most notably the q-Zeilberger algorithm, along with algebraic tools such as backward induction to establish their results.

When certain characteristics of a matrix such as its eigenvalues, eigenvectors, determinant, or inverse are explicitly known, the matrix can serve as a benchmark (or test) matrix for assessing the performance and accuracy of computational algorithms, particularly in numerical linear algebra and computer science applications.

In this study, motivated by the work of [10] and using similar equations (5)-(6), we introduce a new class of matrices whose rows are determined according to the parity of their indices. Specifically, odd- and even-indexed rows are constructed through distinct rules involving the nonlinear index functions Φ and Ψ.

Consider real values x, y, q, λ, μ, a, b, p, r, v, w with $1 + xq^{\mu+a(p+m)^v+b(r+n)^w} \neq 0$ and $1 + yq^{\mu+a(p+m)^v+b(r+n)^w} \neq 0$. Under these assumptions, we construct the matrix $\mathcal{G} = [\mathcal{G}_{m,n}]_{m,n>0}$ as follows

$$\mathcal{G} = [\mathcal{G}_{m,n}] = \begin{cases} \dfrac{1 + xq^{\lambda+\Phi(m)+\Psi(n)}}{1 + xq^{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{1 + yq^{\lambda+\Phi(m)+\Psi(n)}}{1 + yq^{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is even,} \end{cases} \tag{7}$$

here

$$\Phi(i) := a(p+i)^v \text{ and } \Psi(i) := b(r+i)^w. \tag{8}$$

1.1 Our Contribution

We now provide a brief outline of the steps to be carried out:

- We begin by deriving explicit expressions for the L and U matrices, along with their respective inverses, as well as the inverse and q-form determinant of the matrix $\mathcal{G}$. Due to the asymmetry of the entries in $\mathcal{G}$, a Cholesky decomposition is not applicable. Notably, when specific values of q are chosen, the resulting formulations naturally reduce to known results involving Fibonacci and Lucas numbers. While matrix size is often immaterial in many contexts (except for inversion), the matrix $\mathcal{G}$ can be viewed as infinite-dimensional and, when needed, is truncated to its first N rows, denoted by $\mathcal{G}_N$.

- Only one of the results will be proved and the others will be completed in a similar way. Despite the fact that the results appear to be q-hypergeometric summations, they cannot be evaluated using the q-Zeilberger algorithm since they are not Gosper summable.

- Through the application of the q-form expressions of the Binet formulas to the sequences $\{U_n\}$ and $\{V_n\}$ and through appropriate selections of the parameters x and y, the resulting matrices take the form of Filbert-Lilbert matrices:

  i. When $x = y = -1$, the entries of $\mathcal{G}$ correspond to ratios formed by generalized Fibonacci numbers.

  ii. When $x = y = 1$, the matrix $\mathcal{G}$ contains elements that are ratios of generalized Lucas numbers.

  iii. For $x = -y = -1$, the consecutive rows of $\mathcal{G}$ alternate between ratios of generalized Fibonacci numbers and generalized Lucas numbers.

  iv. When $x = -y = 1$, the matrix $\mathcal{G}$ features alternating rows consisting of ratios of generalized Lucas numbers followed by generalized Fibonacci numbers.

  The good harmony in these 4 different matrix forms will be examined.

2 The Main Results

In the following section, we will provide the LU factorization of $\mathcal{G}$, based on the mathematical equations above (7)-(8), along with explicit forms of the inverse of the matrices, and U, and the inverse of the main matrix $\mathcal{G}$ denoted by $\mathcal{G}^{-1}$.

Let's define the following equation for later use:

$$\begin{aligned} &\Omega(z_1, z_2, z_3, z_4, z_5, z_6; k) \\ &= 1 + (-1)^{\delta+z_1} x^{\left\lfloor \frac{\delta+z_2}{2} \right\rfloor} y^{\left\lfloor \frac{\delta+z_3}{2} \right\rfloor} \\ &\times q^{\lambda+\mu(\delta+z_1-2)+k(\Phi(z_4)+\Psi(z_5))+\sum_{t=1}^{\delta-z_6}(\Phi(t)+\Psi(t))}. \end{aligned} \tag{9}$$

Here especially when the case $k = 0$, we easily denote $\Omega(z_1, z_2, z_3, z_4, z_5, z_6; 0)$ by $\psi(z_1, z_2, z_3, z_6)$.

Theorem 1 *Let $1 \leq n \leq m$ and $\delta = n$.
Using (9), for m is odd;*

(i) if n is odd,

$$\begin{aligned} &L_{m,n} \\ &= q^{(n-1)(\Phi(m)-\Phi(n))} \\ &\times \frac{\Omega(1,1,-1,m,n,1;1)}{\psi(1,1,-1,0)} \\ &\times \prod_{t=1}^{(n-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1-x^{-1}yq^{\Phi(2t)-\Phi(n)}} \\ &\times \prod_{t=1}^{(n-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1-q^{\Phi(2t-1)-\Phi(n)}} \\ &\times \prod_{t=1}^{n} \frac{1+xq^{\mu+\Phi(n)+\Psi(t)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}, \end{aligned} \tag{10}$$

(ii) if n is even,

$$\begin{aligned} &L_{m,n} \\ &= \left(\frac{xy^{-1}}{q^{\Phi(n)-\Phi(m)}}\right)^{n-1} \\ &\times \frac{\Omega(1,2,-2,m,n,1;1)}{\psi(1,0,0,0)} \\ &\times \prod_{t=1}^{n} \frac{1+yq^{\mu+\Phi(n)+\Psi(t)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}} \\ &\times \prod_{t=1}^{(n-2)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1-q^{\Phi(2t)-\Phi(n)}} \\ &\times \prod_{t=1}^{n/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1-xy^{-1}q^{\Phi(2t-1)-\Phi(n)}}. \end{aligned} \tag{11}$$

For m is even;
(iii) if n is odd,

$$\begin{aligned} &L_{m,n} \\ &= \left(\frac{x^{-1}y}{q^{\Phi(n)-\Phi(m)}}\right)^{n-1} \\ &\times \frac{\Omega(1,-1,1,m,n,1;1)}{\psi(1,1,-1,0)} \\ &\times \prod_{t=1}^{(n-1)/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1-x^{-1}yq^{\Phi(2t)-\Phi(n)}} \\ &\times \prod_{t=1}^{(n-1)/2} \frac{1-q^{\Phi(2t)-\Phi(m)}}{1-q^{\Phi(2t-1)-\Phi(n)}} \\ &\times \prod_{t=1}^{n} \frac{1+xq^{\mu+\Phi(n)+\Psi(t)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}, \end{aligned} \tag{12}$$

(iv) if n is even,

$$\begin{aligned} &L_{m,n} \\ &= q^{(n-1)(\Phi(m)-\Phi(n))} \\ &\times \frac{\Omega(1,0,0,m,n,1;1)}{\psi(1,0,0,0)} \\ &\times \prod_{t=1}^{n} \frac{1+yq^{\mu+\Phi(n)+\Psi(t)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}} \\ &\times \prod_{t=1}^{(n-2)/2} \frac{1-q^{\Phi(2t)-\Phi(m)}}{1-q^{\Phi(2t)-\Phi(n)}} \\ &\times \prod_{t=1}^{n/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1-xy^{-1}q^{\Phi(2t-1)-\Phi(n)}}. \end{aligned} \tag{13}$$

Theorem 2 *Let $1 \le n \le m$ and $\delta = n$,*
(i) if n is odd,

$$\begin{aligned} &U_{n,m} \\ &= \left(1-q^{\lambda-\mu}\right) \\ &\times \left(\frac{-xq^{\mu}}{q^{-(\Phi(n)+\Psi(m))}}\right)^{n-1} \\ &\times \frac{\Omega(1,1,-1,n,m,1;1)}{\psi(0,-1,-1,1)} \\ &\times \prod_{t=1}^{n-1} \frac{1-q^{\Psi(t)-\Psi(m)}}{1+xq^{\mu+\Phi(n)+\Psi(t)}} \\ &\times \prod_{t=1}^{(n+1)/2} \frac{1}{1+xq^{\mu+\Phi(2t-1)+\Psi(m)}} \\ &\times \prod_{t=1}^{(n-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(n)}}{1+yq^{\mu+\Phi(2t)+\Psi(m)}} \\ &\times \prod_{t=1}^{(n-1)/2} \left(1-x^{-1}yq^{\Phi(2t)-\Phi(n)}\right), \end{aligned} \tag{14}$$

(ii) if n is even,

$$\begin{aligned} &U_{n,m} \\ &= \left(1-q^{\lambda-\mu}\right) \\ &\times \left(\frac{-yq^{\mu}}{q^{-(\Phi(n)+\Psi(m))}}\right)^{n-1} \\ &\times \frac{\Omega(1,0,0,n,m,1;1)}{\psi(0,0,-2,1)} \end{aligned}$$

$$\times \prod_{t=1}^{n-1} \frac{1-q^{\Psi(t)-\Psi(m)}}{1+yq^{\mu+\Phi(n)+\Psi(t)}}$$
$$\times \prod_{t=1}^{(n-2)/2} \left(1-q^{\Phi(2t)-\Phi(n)}\right)$$
$$\times \prod_{t=1}^{n/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(n)}}{1+yq^{\mu+\Phi(2t)+\Psi(m)}}$$
$$\times \prod_{t=1}^{n/2} \frac{1}{1+xq^{\mu+\Phi(2t-1)+\Psi(m)}}. \quad (15)$$

We now proceed to derive the explicit expressions for the inverse of the matrices L and U.

For convenience in what follows, we recall the definition of the Iverson notation:

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Theorem 3 *Let $1 \leq n \leq m$ and $\delta = m$. Using (16), for m is odd;*
(i) if n is odd,

$$L_{m,n}^{-1}$$
$$= (-1)^{[m \neq n]} \frac{\Omega(0,-1,-1,n,m,0;-1)}{\psi(0,-1,-1,1)}$$
$$\times \prod_{t=1}^{(m-1)/2} \frac{1-xy^{-1}q^{\Phi(m)-\Phi(2t)}}{1-xy^{-1}q^{\Phi(n)-\Phi(2t)}}$$
$$\times \prod_{\substack{t=1 \\ t \neq (n+1)/2}}^{(m-1)/2} \frac{1-q^{\Phi(m)-\Phi(2t-1)}}{1-q^{\Phi(n)-\Phi(2t-1)}}$$
$$\times \prod_{t=1}^{m-1} \frac{1+xq^{\mu+\Phi(n)+\Psi(t)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}, \quad (17)$$

(ii) if n is even,

$$L_{m,n}^{-1}$$
$$= -\frac{\Omega(0,0,-2,n,m,0;-1)}{\psi(0,0,0,1)}$$
$$\times \prod_{t=1}^{(m-1)/2} \frac{1-q^{\Phi(m)-\Phi(2t-1)}}{1-x^{-1}yq^{\Phi(n)-\Phi(2t-1)}}$$
$$\prod_{\substack{t=1 \\ t \neq n/2}}^{(m-1)/2} \frac{1-xy^{-1}q^{\Phi(m)-\Phi(2t)}}{1-q^{\Phi(n)-\Phi(2t)}}$$
$$\times \prod_{t=1}^{m-1} \frac{1+yq^{\mu+\Phi(n)+\Psi(t)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}. \quad (18)$$

For even m;
(iii) if n is odd,

$$L_{m,n}^{-1}$$
$$= -\frac{\Omega(0,-2,0,n,m,0;-1)}{\psi(0,0,-2,1)}$$
$$\times \prod_{t=1}^{m-1} \frac{1+xq^{\mu+\Phi(n)+\Psi(t)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}$$
$$\times \prod_{\substack{t=1 \\ t \neq (n+1)/2}}^{m/2} \frac{1-x^{-1}yq^{\Phi(m)-\Phi(2t-1)}}{1-q^{\Phi(n)-\Phi(2t-1)}}$$
$$\times \prod_{t=1}^{(m-2)/2} \frac{1-q^{\Phi(m)-\Phi(2t)}}{1-xy^{-1}q^{\Phi(n)-\Phi(2t)}}, \quad (19)$$

(iv) if n is even,

$$L_{m,n}^{-1}$$
$$= (-1)^{[m \neq n]} \frac{\Omega(0,0,-2,n,m,0;-1)}{\psi(0,0,-2,1)}$$
$$\times \prod_{\substack{t=1 \\ t \neq n/2}}^{(m-2)/2} \frac{1-q^{\Phi(m)-\Phi(2t)}}{1-q^{\Phi(n)-\Phi(2t)}}$$
$$\times \prod_{t=1}^{m-1} \frac{1+yq^{\mu+\Phi(n)+\Psi(t)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}$$
$$\times \prod_{t=1}^{m/2} \frac{1-x^{-1}yq^{\Phi(m)-\Phi(2t-1)}}{1-x^{-1}yq^{\Phi(n)-\Phi(2t-1)}}, \quad (20)$$

where $L_{m,n}^{-1} = 0$ for $m < n$ in all cases above.

Theorem 4 *Let $1 \leq n \leq m$ and $\delta = m$;*
(i) if m is odd,

$$U_{n,m}^{-1}$$
$$= \left(1-q^{\lambda-\mu}\right)^{-1}$$
$$\times \left(\frac{q^{-(\Phi(m)+\Psi(n))}}{xq^{\mu}}\right)^{m-1}$$
$$\times \frac{\Omega(0,-1,-1,m,n,0;-1)}{\psi(1,1,-1,0)}$$

$$
\begin{aligned}
&\times \prod_{t=1}^{(m-1)/2} \frac{1+yq^{\mu+\Phi(2t)+\Psi(n)}}{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}} \\
&\times \prod_{t=1}^{(m-1)/2} \frac{1}{1-q^{\Phi(2t-1)-\Phi(m)}} \\
&\times \prod_{\substack{t=1\\t\neq n}}^{m} \frac{1+xq^{\mu+\Phi(m)+\Psi(t)}}{1-q^{\Psi(t)-\Psi(n)}} \\
&\times \prod_{t=1}^{(m+1)/2} \left(1+xq^{\mu+\Phi(2t-1)+\Psi(n)}\right), \qquad (21)
\end{aligned}
$$

(ii) if m is even,

$$
\begin{aligned}
&U_{n,m}^{-1} \\
&= -\left(1-q^{\lambda-\mu}\right)^{-1} \\
&\times \left(\frac{q^{-(\Phi(m)+\Psi(n))}}{yq^{\mu}}\right)^{m-1} \\
&\times \frac{\Omega(0,0,-2,m,n,0;-1)}{\psi(1,0,0,0)} \\
&\times \prod_{t=1}^{m/2} \frac{1+yq^{\mu+\Phi(2t)+\Psi(n)}}{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}} \\
&\times \prod_{t=1}^{m/2} \left(1+xq^{\mu+\Phi(2t-1)+\Psi(n)}\right) \\
&\times \prod_{\substack{t=1\\t\neq n}}^{m} \frac{1+yq^{\mu+\Phi(m)+\Psi(t)}}{1-q^{\Psi(t)-\Psi(n)}} \\
&\times \prod_{t=1}^{(m-2)/2} \frac{1}{1-q^{\Phi(2t)-\Phi(m)}}, \qquad (22)
\end{aligned}
$$

where $U_{n,m}^{-1}=0$ for $m<n$ in both cases above.

We now proceed to compute the inverse of $\mathcal{G}$, noting that the resulting expression is inherently dependent on the dimension of the matrix.

Theorem 5 *For $1\le m,n\le N$ and $\delta=N$;*

(i) if n is odd,

$$
\begin{aligned}
&(\mathcal{G}_N)_{m,n}^{-1} \\
&= \left(1-q^{\lambda-\mu}\right)^{-1}\left(\frac{q^{-(\Phi(n)+\Psi(m))}}{-xq^{\mu}}\right)^{N-1} \\
&\times \frac{\Omega(0,-1,0,n,m,0;-1)}{\psi(1,1,0,0)} \\
&\times \prod_{t=1}^{\lfloor N/2\rfloor} \frac{1+yq^{\mu+\Phi(2t)+\Psi(m)}}{1-x^{-1}yq^{\Phi(2t)-\Phi(n)}} \\
&\times \prod_{t=1}^{N} \left(1+xq^{\mu+\Phi(n)+\Psi(t)}\right) \\
&\times \prod_{\substack{t=1\\t\neq m}}^{N} \frac{1}{1-q^{\Psi(t)-\Psi(m)}} \\
&\times \prod_{\substack{t=1\\t\neq (n+1)/2}}^{\lfloor (N+1)/2\rfloor} \frac{1+xq^{\mu+\Phi(2t-1)+\Psi(m)}}{1-q^{\Phi(2t-1)-\Phi(n)}}, \qquad (23)
\end{aligned}
$$

(ii) if n is even,

$$
\begin{aligned}
&(\mathcal{G}_N)_{m,n}^{-1} \\
&= \left(1-q^{\lambda-\mu}\right)^{-1}\left(\frac{q^{-(\Phi(n)+\Psi(m))}}{-yq^{\mu}}\right)^{N-1} \\
&\times \frac{\Omega(0,1,-2,n,m,0;-1)}{\psi(1,1,0,0)} \\
&\times \prod_{t=1}^{\lfloor (N+1)/2\rfloor} \frac{1+xq^{\mu+\Phi(2t-1)+\Psi(m)}}{1-xy^{-1}q^{\Phi(2t-1)-\Phi(n)}} \\
&\times \prod_{t=1}^{N} \left(1+yq^{\mu+\Phi(n)+\Psi(t)}\right) \\
&\times \prod_{\substack{t=1\\t\neq m}}^{N} \frac{1}{1-q^{\Psi(t)-\Psi(m)}} \\
&\times \prod_{\substack{t=1\\t\neq n/2}}^{\lfloor N/2\rfloor} \frac{1+yq^{\mu+\Phi(2t)+\Psi(m)}}{1-q^{\Phi(2t)-\Phi(n)}}. \qquad (24)
\end{aligned}
$$

Finally, we will give the determinant of $\mathcal{G}$:

Theorem 6 *For $N \geq 1$ and $\delta = N$,*

$$\begin{aligned}
&\det \mathcal{G}_N \\
&= y^{\binom{\lfloor N/2 \rfloor}{2}} q^{\mu\binom{N}{2}} \left(1 - q^{\lambda-\mu}\right)^{N-1} \psi(1,1,0,0) \\
&\times \prod_{t=1}^{\lfloor (N-2)/2 \rfloor} \prod_{k=t}^{\lfloor (N-2)/2 \rfloor} q^{\Phi(2t)} \left(1 - q^{\Phi(2k+2)-\Phi(2t)}\right) \\
&\times \prod_{t=1}^{N-1} \prod_{k=t}^{N-1} q^{\Psi(t)} \left(1 - q^{\Psi(k+1)-\Psi(t)}\right) \\
&\times \prod_{t=1}^{\lfloor N/2 \rfloor} \prod_{k=1}^{N} \frac{1}{1 + yq^{\mu+\Phi(2t)+\Psi(k)}} \\
&\times \prod_{t=1}^{\lfloor (N+1)/2 \rfloor} \prod_{k=1}^{N} \frac{1}{1 + xq^{\mu+\Phi(2t-1)+\Psi(k)}} \\
&\times \prod_{t=1}^{\lfloor N/2 \rfloor} \prod_{k=1}^{\lfloor (N+1)/2 \rfloor} q^{\Phi(2k-1)} \left(1 - x^{-1}yq^{\Phi(2t)-\Phi(2k-1)}\right) \\
&\times \prod_{t=1}^{\lfloor (N-1)/2 \rfloor} \prod_{k=t}^{\lfloor (N-1)/2 \rfloor} q^{\Phi(2t-1)} \left(1 - q^{\Phi(2k+1)-\Phi(2t-1)}\right) \\
&\times \begin{cases} (-1)^{\lfloor N/4 \rfloor} x^{3\binom{(N+1)/2}{2}} & \text{if } N \text{ is odd,} \\ (-1)^{\lfloor (N+2)/4 \rfloor} x^{\binom{3\lfloor (N+1)/2 \rfloor}{2}/3} & \text{if } N \text{ is even.} \end{cases}
\end{aligned} \tag{25}$$

The determinant of $\mathcal{G}$ follows directly from the LU-decomposition and is given by the sequential multiplication of the diagonal components of matrix U, based on the mathematical equation above (25).

3 Proofs

The results stated above will be established using the method of backward induction. Given the length and complexity of this approach, we restrict our attention to proving the LU-decomposition of the matrix $\mathcal{G}$, while noting that the remaining results can be derived through similar arguments and techniques.

To verify the LU-decomposition of $\mathcal{G}$, it suffices to prove that:

$$\sum_{1 \leq d \leq \min(m,n)} L_{m,d} U_{d,n} = \mathcal{G}_{m,n}. \tag{26}$$

It is important to observe that, due to the structure of the matrix L, each pair of successive rows and columns is governed by four distinct formulas, based on the mathematical equations above (10)-(13). Likewise, the rows of the matrix U are defined by two different expressions depending on their positions, based on the mathematical equations above (14)-(15). As a result, establishing the LU-decomposition of $\mathcal{G}$ requires analyzing four separate cases in (26).

Furthermore, by using the equalities (10)-(13) and (17)-(20) together in the multiplication LL^{-1}, we can easily prove the results for L^{-1}. Similarly, by using the equalities (14)-(15) and (21)-(22) together in the multiplication U^{-1}, we can prove the results for U^{-1}. Using the equality $(\mathcal{G}_N)^{-1} = U^{-1}L^{-1}$ and the equations (17)-(22), we can prove the results belonging to $(\mathcal{G}_N)^{-1}$ based on the mathematical equations above (23)-(24).

Before addressing these cases, we first examine a more general scenario. Without loss of generality, we assume that $m \geq n$, and aim to establish a general expression that involves an auxiliary parameter K:

$$\mathtt{SUM}_K = \sum_{K \leq d \leq n} L_{m,d} U_{d,n}. \tag{27}$$

To establish the desired result for the LU-decomposition of $\mathcal{G}$, it is sufficient to consider the case $K = 1$ in (27), in the previously defined summations, specifically the expression denoted by $\mathtt{SUM}_1$. The parities of m and K have to be considered before this. There are four subcases of $\mathtt{SUM}_K$ given by

$$\begin{aligned}
&\mathtt{SUM}_K \\
&= \begin{cases} \mathtt{SUM}_K^{(1)} & \text{when } m \text{ and } n \text{ are both odd,} \\ \mathtt{SUM}_K^{(2)} & \text{when } m \text{ is even while } n \text{ is odd,} \\ \mathtt{SUM}_K^{(3)} & \text{when } m \text{ is odd while } n \text{ is even,} \\ \mathtt{SUM}_K^{(4)} & \text{when } m \text{ and } n \text{ are both even,} \end{cases}
\end{aligned} \tag{28}$$

where $\delta = d$,

(i) When both m and K are odd,

$$\begin{aligned}
&\mathtt{SUM}_K^{(1)}\\
&:= \left(1-q^{\lambda-\mu}\right)\\
&\times \sum_{d=K}^{\min(m,n)} \frac{\Omega(1,1,-1,d,n,1;1)}{\psi(0,-1,-1,1)}\\
&\times \frac{\Omega(1,1,-1,m,d,1;1)}{\psi(1,1,-1,0)}\\
&\times \frac{1+xq^{\mu+\Phi(d)+\Psi(d)}}{1+xq^{\mu+\Phi(m)+\Psi(d)}}\\
&\times \frac{\left(-xq^{\mu+\Phi(m)+\Psi(n)}\right)^{d-1}}{1+xq^{\mu+\Phi(d)+\Psi(n)}}\\
&\times \prod_{t=1}^{(d-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\
&\times \prod_{t=1}^{(d-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\
&\times \prod_{t=1}^{d-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}},
\end{aligned} \tag{29}$$

(ii) When m is even while K is odd,

$$\begin{aligned}
&\mathtt{SUM}_K^{(2)}\\
&:= \left(1-q^{\lambda-\mu}\right)\\
&\times \sum_{d=K}^{\min(m,n)} \frac{\Omega(1,1,-1,d,n,1;1)}{\psi(0,-1,-1,1)}\\
&\times \frac{\Omega(1,-1,1,m,d,1;1)}{\psi(1,1,-1,0)}\\
&\times \frac{1+xq^{\mu+\Phi(d)+\Psi(d)}}{1+yq^{\mu+\Phi(m)+\Psi(d)}}\\
&\times \frac{\left(yq^{\mu+\Phi(m)+\Psi(n)}\right)^{d-1}}{1+xq^{\mu+\Phi(d)+\Psi(n)}}\\
&\times \prod_{t=1}^{(d-1)/2} \frac{1-q^{\Phi(2t)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\
&\times \prod_{t=1}^{(d-1)/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\
&\times \prod_{t=1}^{d-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}},
\end{aligned} \tag{30}$$

(iii) When m is odd while K is even,

$$\begin{aligned}
&\mathtt{SUM}_K^{(3)}\\
&:= \left(1-q^{\lambda-\mu}\right)\\
&\times \sum_{d=K}^{\min(m,n)} \frac{\Omega(1,0,0,d,n,1;1)}{\psi(1,0,0,0)}\\
&\times \frac{\Omega(1,2,-2,m,d,1;1)}{\psi(0,0,-2,1)}\\
&\times \frac{1+yq^{\mu+\Phi(d)+\Psi(d)}}{1+xq^{\mu+\Phi(m)+\Psi(d)}}\\
&\times \frac{x^{d/2}\left(-q^{\mu+\Phi(m)+\Psi(n)}\right)^{d-1}}{q^{\Phi(m)}\left(1-x^{-1}yq^{\Phi(d)-\Phi(m)}\right)}\\
&\times \prod_{t=1}^{d/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\
&\times \prod_{t=1}^{d/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\
&\times \prod_{t=1}^{d-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}},
\end{aligned} \tag{31}$$

(iv) When m and K are both even,

$$\begin{aligned}
&\mathtt{SUM}_K^{(4)}\\
&:= \left(1-q^{\lambda-\mu}\right)\\
&\times \sum_{d=K}^{\min(m,n)} \frac{\Omega(1,0,0,d,n,1;1)}{\psi(0,0,-2,1)}\\
&\times \frac{\Omega(1,0,0,m,d,1;1)}{\psi(1,0,0,0)}\\
&\times \frac{\left(-yq^{\mu+\Phi(m)+\Psi(n)}\right)^{d-1}}{q^{-\Phi(m)/2}}\\
&\times \frac{1+yq^{\mu+\Phi(d)+\Psi(d)}}{1+yq^{\mu+\Phi(m)+\Psi(d)}}\\
&\times \prod_{t=1}^{(d-1)/2} \left(1-q^{\Phi(2t)-\Phi(m)}\right)\\
&\times \prod_{t=1}^{d-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}\\
&\times \prod_{t=1}^{d/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\
&\times \prod_{t=1}^{d/2} \frac{1}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}.
\end{aligned} \tag{32}$$

To facilitate the evaluation of the summations, based on the mathematical equations above (29)-(32) denoted by $\mathtt{SUM}_K^{(i)}$ with $1 \leq i \leq 4$ in (28), we present the lemma stated below.

Lemma 1 *For $\delta = K$;*
(i) if both odd m, K,

$$\begin{aligned}&\mathtt{SUM}_K^{(1)}\\&:= \left(\frac{-xq^{\mu}}{q^{-(\Phi(m)+\Psi(n))}}\right)^{K-1}\\&\times \frac{1-q^{\lambda-\mu}}{1+xq^{\mu+\Phi(m)+\Psi(n)}}\\&\times \prod_{t=1}^{(K-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\&\times \prod_{t=1}^{(K-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\&\times \frac{\Omega(1,1,-1,m,n,1;1)}{\psi(0,-1,-1,1)}\\&\times \prod_{t=1}^{K-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}, \end{aligned} \tag{33}$$

(ii) if even m, odd K,

$$\begin{aligned}&\mathtt{SUM}_K^{(2)}\\&:= \left(\frac{yq^{\mu}}{q^{-(\Phi(m)+\Psi(n))}}\right)^{K-1}\\&\times \frac{1-q^{\lambda-\mu}}{1+yq^{\mu+\Phi(m)+\Psi(n)}}\\&\times \prod_{t=1}^{(K-1)/2} \frac{1-q^{\Phi(2t)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\&\times \prod_{t=1}^{(K-1)/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\&\times \frac{\Omega(1,-1,1,m,n,1;1)}{\psi(0,-1,-1,1)}\\&\times \prod_{t=1}^{K-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}, \end{aligned} \tag{34}$$

(iii) if odd m, even K,

$$\begin{aligned}&\mathtt{SUM}_K^{(3)}\\&:= \frac{\left(-q^{\mu+\Phi(m)+\Psi(n)}\right)^{K-1}\left(1-q^{\lambda-\mu}\right)}{x^{(2-K)/2}\left(1-x^{-1}yq^{\Phi(K)-\Phi(m)}\right)}\\&\times \frac{1+yq^{\mu+\Phi(K)+\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(n)}}\\&\times \frac{\Omega(1,2,-2,m,n,1;1)}{\psi(0,0,-2,1)}\\&\times \prod_{t=1}^{K/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\&\times \prod_{t=1}^{K/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\&\times \prod_{t=1}^{K-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}, \end{aligned} \tag{35}$$

(iv) if both even m, K,

$$\begin{aligned}&\mathtt{SUM}_K^{(4)}\\&:= \left(\frac{-yq^{\mu}}{q^{\Phi(m)+\Psi(n)}}\right)^{K-1}\\&\times \frac{\left(1-q^{\lambda-\mu}\right)\left(1+yq^{\mu+\Phi(K)+\Psi(n)}\right)}{q^{-\Phi(m)/2}\left(1+yq^{\mu+\Phi(m)+\Psi(n)}\right)}\\&\times \prod_{t=1}^{K/2} \frac{1-xy^{-1}q^{\Phi(2t-1)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}\\&\times \prod_{t=1}^{K/2} \frac{1}{1+yq^{\mu+\Phi(2t)+\Psi(n)}}\\&\times \prod_{t=1}^{K-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(t)}}\\&\times \prod_{t=1}^{(K-1)/2} \left(1-q^{\Phi(2t)-\Phi(m)}\right)\\&\times \frac{\Omega(1,0,0,m,n,1;1)}{\psi(0,0,-2,1)}. \end{aligned} \tag{36}$$

Proof 1 *While proving these results, the parity of both m and K is taken into account. Initially, when m is odd, the expressions presented in equations* (33) *and* (35) *are analyzed in conjunction, due to their inherent structural connection. Likewise, for the case where m is even, equations* (34) *and* (36) *are examined together, as they exhibit a similar relationship.*

In the case where both m and K are odd, it must be shown that

$$\mathtt{SUM}_{K-1}^{(3)} = \mathtt{SUM}_K^{(1)} + S_{K-1}^{(3)}, \tag{37}$$

where $S_d^{(t)}$ *is summand term of* $\mathtt{SUM}_{\mathbf{K}}{}^{(\mathbf{t})}$ *(*$1 \le t \le 4$*).*

The following equation, which unifies both of the aforementioned relations, is established under the condition that both m, K *are odd. Since the argument remains valid in general, we may assume* $m \ge n$*; the case* $n > m$ *follows by symmetry. It is worth noting that the assertion holds trivially in the case* $K = n$*. Given that* $K - 1$ *is an even number, the step in the reverse induction process corresponding to* $\delta = K$ *proceeds as follows*:

$$\begin{aligned}
&\mathtt{SUM}_{K-1}^{(3)} \\
&= \left(\frac{-xq^{\mu}}{q^{-(\Phi(m)+\Psi(n))}}\right)^{K-1} \\
&\times \frac{1-q^{\lambda-\mu}}{1+xq^{\mu+\Phi(m)+\Psi(n)}} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}} \\
&\times \prod_{t=1}^{K-1} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}} \\
&\times \frac{\Omega(1,1,-1,m,n,1;1)}{\psi(0,-1,-1,1)} \\
&+ \frac{\Omega(0,-1,-1,K-1,n,2;1)}{\psi(0,-1,-1,1)} \\
&\times \frac{\Omega(0,1,-3,m,K-1,2;1)}{\psi(-1,-1,-3,2)} \\
&\times \frac{x^{(K-1)/2}\left(1-q^{\lambda-\mu}\right)}{q^{\Phi(m)}} \\
&\times \frac{\left(-q^{\mu+\Phi(m)+\Psi(n)}\right)^{K-2}}{1-x^{-1}yq^{\Phi(K-1)-\Phi(m)}} \\
&\times \frac{1+yq^{\mu+\Phi(K-1)+\Psi(K-1)}}{1+xq^{\mu+\Phi(m)+\Psi(K-1)}} \\
&\times \prod_{t=1}^{K-2} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}}.
\end{aligned} \tag{38}$$

Following a series of simplifications and using (37) *and* (38)*,*

$$\begin{aligned}
&\mathtt{SUM}_{K-1}^{(3)} \\
&= \frac{x^{(K-3)/2}\left(-q^{\mu+\Phi(m)+\Psi(n)}\right)^{K-2}}{q^{\Phi(m)}\left(1-x^{-1}yq^{\Phi(K-1)-\Phi(m)}\right)} \\
&\times \frac{\left(1-q^{\lambda-\mu}\right)\left(1+yq^{\mu+\Phi(K-1)+\Psi(n)}\right)}{\left(1+xq^{\mu+\Phi(m)+\Psi(n)}\right)} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-x^{-1}yq^{\Phi(2t)-\Phi(m)}}{1+xq^{\mu+\Phi(2t-1)+\Psi(n)}} \\
&\times \prod_{t=1}^{(K-1)/2} \frac{1-q^{\Phi(2t-1)-\Phi(m)}}{1+yq^{\mu+\Phi(2t)+\Psi(n)}} \\
&\times \frac{\Omega(0,1,-3,m,n,2;1)}{\psi(-1,-1,-3,2)} \\
&\times \prod_{t=1}^{K-2} \frac{1-q^{\Psi(t)-\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(t)}}.
\end{aligned} \tag{39}$$

Thus the equation (39) *finalize the proof.*

The corresponding statement for even m *can be proved through a similar manner.*

Let us now revisit the stated the findings related to the matrix $\mathcal{G}$ and give following equations:

$$\frac{1+xq^{\lambda+\Phi(m)+\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(n)}} = \sum_{1\le d\le \min(m,n)} L_{m,d}U_{d,n}$$
$$= \begin{cases} \mathtt{SUM}_1^{(1)} & \text{if } n \text{ is odd,} \\ \mathtt{SUM}_1^{(3)} & \text{if } n \text{ is even} \end{cases} \tag{40}$$

and

$$\frac{1+yq^{\lambda+\Phi(m)+\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(n)}} = \sum_{1\le d\le \min(m,n)} L_{m,d}U_{d,n}$$
$$= \begin{cases} \mathtt{SUM}_1^{(2)} & \text{if } n \text{ is odd,} \\ \mathtt{SUM}_1^{(4)} & \text{if } n \text{ is even.} \end{cases} \tag{41}$$

When $K = 1$, the equalities (33) and (35), and, (34) and (36) give us the results, based on the mathematical equations (40)-(41).

4 Conclusions

Using the equations (1)-(2), define the matrix $\mathcal{H} = [\mathcal{H}_{m,n}]$ as shown

$$\mathcal{H}_{m,n} = \begin{cases} \dfrac{U_{\lambda+\Phi(m)+\Psi(n)}}{U_{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{V_{\lambda+\Phi(m)+\Psi(n)}}{V_{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is even,} \end{cases} \tag{42}$$

where

$$\Phi(i) := a(i+p)^v \text{ and } \Psi(i) := b(i+r)^w \quad (43)$$

for considering arbitrary choices of integers λ, μ, p, r, a, b, v, w.

For instance, when the order is 3, the corresponding matrix $\mathcal{H}$, based on the mathematical equations (42)-(43), has the following structure:

$$\mathcal{H} = \begin{bmatrix} \frac{U_{\lambda+\Phi(1)+\Psi(1)}}{U_{\mu+\Phi(1)+\Psi(1)}} & \frac{U_{\lambda+\Phi(1)+\Psi(2)}}{U_{\mu+\Phi(1)+\Psi(2)}} & \frac{U_{\lambda+\Phi(1)+\Psi(3)}}{U_{\mu+\Phi(1)+\Psi(3)}} \\ \frac{V_{\lambda+\Phi(2)+\Psi(1)}}{V_{\mu+\Phi(2)+\Psi(1)}} & \frac{V_{\lambda+\Phi(2)+\Psi(2)}}{V_{\mu+\Phi(2)+\Psi(2)}} & \frac{V_{\lambda+\Phi(2)+\Psi(3)}}{V_{\mu+\Phi(2)+\Psi(3)}} \\ \frac{U_{\lambda+\Phi(3)+\Psi(1)}}{U_{\mu+\Phi(3)+\Psi(1)}} & \frac{U_{\lambda+\Phi(3)+\Psi(2)}}{U_{\mu+\Phi(3)+\Psi(2)}} & \frac{U_{\lambda+\Phi(3)+\Psi(3)}}{U_{\mu+\Phi(3)+\Psi(3)}} \end{bmatrix}. \quad (44)$$

Using the Binet formulas (3)-(4) for the sequences U_n and V_n, the matrix $\mathcal{H}$, based on the mathematical equation (44), can be equivalently expressed in an alternative form as follows:

$$\mathcal{H}_{m,n} = \alpha^{\lambda-\mu} \begin{cases} \dfrac{1+xq^{\lambda+\Phi(m)+\Psi(n)}}{1+xq^{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is odd,} \\ \dfrac{1+yq^{\lambda+\Phi(m)+\Psi(n)}}{1+yq^{\mu+\Phi(m)+\Psi(n)}} & \text{if } m \text{ is even,} \end{cases} \quad (45)$$

where $x = -1$, $y = 1$, $q = \beta/\alpha$,

$$\Phi(i) := a(p+i)^v \text{ and } \Psi(i) := b(r+i)^w. \quad (46)$$

In this paper, we defined the matrix $\mathcal{G} = [\mathcal{G}_{m,n}]$ as in (7) and derived several results for arbitrary real parameters x, y and q, among them the LU-decomposition of $\mathcal{G}$, the inverses L^{-1}, U^{-1} and the matrix inverse $\mathcal{G}^{-1}$.

It is important to observe that, based on the mathematical equations above (45)-(46), the matrix $\mathcal{H}$ represents a particular form of the matrix $\mathcal{G}$, obtained by omitting the constant multiplicative factors. As a result, various properties of the matrix $\mathcal{H}$ including its LU-decomposition, the inverses L^{-1},U^{-1} and $\mathcal{G}^{-1}$can be directly derived from the general results established for $\mathcal{G}$ in this work by taking the specific parameter values $y = -x = 1$ and $q = \beta/\alpha$.

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding this study.