On R-multiplication Gamma Acts

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Abstract: - This article introduces and studies Γ-multiplicatively closed subsets R of a Γ-monoid S, developing their foundational properties through definitions and characterization theorems and essential structural results. The work extends classical localization and ideal theory to Γ-semigroups through three principal advances: the localization of S_Γ-acts via a congruence relation, the derivation of universal factorization properties for morphisms and the Local-Global Principle for S_Γ-act structures. The concept of R_Γ-multiplication S_Γ-acts is then introduced as a generalization of multiplication S_Γ-acts with a complete characterization of their basic properties. The relationship between multiplication S_Γ-acts and R_Γ-multiplication S_Γ-The act is examined. In particular, we prove that every multiplication S_Γ-act is an R_Γ-multiplication S_Γ-act, but the converse does not always hold. Using the concept of idempotent S_Γ-subacts, we show that every S_Γ-subact is an R_Γ-multiplication S_Γ-acts. Furthermore, additional concepts such as R_Γ-cyclic and R_Γ-finite S_Γ-Acts are introduced and some of their properties are investigated. Finally, the notion of an R_Γ-finite S_Γ-Subact is used to study the relationship between R_Γ-cyclic and R_Γ-multiplication S_Γ-acts, and several important properties are discussed.

Key-Words: - Localization of S_Γ-acts, Universal Property of Localization, Γ-semigroup, multiplication S_Γ-acts, R_Γ-cyclic S_Γ-subact, R_Γ-finite S_Γ-subact, R_Γ-multiplication S_Γ-acts.

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1 Introduction

Let S and Γ be nonempty sets. The set S is called a Γ-semigroup if there exists a mapping: $S \times \Gamma \times S \to S$ denoted by $(s.\gamma.s') \mapsto s\gamma s'$, satisfying the associative property: $s\alpha(t\gamma r)=(s\alpha t)\gamma r$ for all s, $t, r \in S$ and $\alpha, \gamma \in \Gamma$, [1]. An element $s \in S$ is a left (right) identity if $s\gamma t = t$ ($t\gamma s = t$) for all $t \in S$ and $\gamma \in \Gamma$. Let S be a Γ-semigroup . An element $s \in S$ is called an identity if it is both a left and right identity. In this case, S is termed a Γ-monoid. A Γ-semigroup S is said to be commutative if $s\gamma t=t\gamma s$ for all $s, t\in$ S and $\gamma \in \Gamma$. A nonempty subset $T \subseteq S$ is called a Γ-subsemigroup if $t\gamma t' \in T$ for all $t, t' \in T$ and $\gamma\in\Gamma$.A subset $A\subseteq S$ is a left (right) Γ-ideal if $S\Gamma A\subseteq A$ ($A\Gamma S\subseteq A$), where $S\Gamma A=\{s\gamma a$: $s\in S$, $\gamma\in\Gamma$, $a\in A\}$. The term Γ-ideal refers two-sided Γ-ideal. The union of any family of Γ-ideals in a Γ-semigroup S is itself a Γ-ideal, [2]. An element $s\in S$ is γ-idempotent if there exists $\gamma\in\Gamma$ such that $s\gamma s= s$. A Γ-semigroup S is called idempotent if every element $s \in S$ is γ-idempotent for some $\gamma\in\Gamma$. A Γ-ideal B of a Γ-semigroup S is called globally idempotent if $B\Gamma B=B$, [3]. A Γ-ideal P of a Γ-semigroup S is prime if for any Γ-ideals $A,B\subseteq S$, the condition $A\Gamma B\subseteq P$ implies

$A\subseteq P$ or $B\subseteq P$. A Γ-ideal B of a Γ-semigroup S is maximal if it is proper and not properly contained in any other proper Γ-ideal. Let S be a globally idempotent Γ-semigroup (i.e., $S\Gamma S=S$). Then every maximal Γ-ideal of S is prime. An element $s \in S$ is called (left/right) invertible (or a (left/right) unit) if there exists r ∈ S such that $r\Gamma s$ =1 (left case) or $s\Gamma r$ = 1 (right case). And is called invertib**le** (or a *unit*) if it is both left and right invertible, [4]. The set of all unit elements of a Γ-semigroup S is denoted by $U_\Gamma(S)$. The notion of gamma acts over Γ-semigroups was introduced by [5] as an extension of the classical concept of acts over semigroups. A nonempty set A is called a left gamma act over a Γ-semigroup S (denoted S_Γ-act) if there exists a mapping: $S\times\Gamma\times A\to A$, written $(s,\alpha,a)\mapsto s\alpha a$, satisfying the compatibility condition: $(s\gamma t)\beta a=s\gamma(t\beta a)$for all $s,t\in S$, $\gamma,\beta\in\Gamma$ and $a\in A$. Right gamma acts are defined analogously. A subset $B\subseteq A$ is called an S_Γ-subact if $s\gamma b\in B$ for all $s\in S$, $\gamma\in\Gamma$ and $b\in B$. A nonempty subset U of S_Γ-act A is called a generating set of A if $A=\langle U\rangle_A$. We say that A is finitely generated if $A=\langle U\rangle_A$ for some finite subset U of A, [5] . Furthermore, A is a cyclic if $A=\langle\{a\}\rangle_A$ for some $a\in A$. An element $\theta\in A$ is called a zero element if $s\gamma\theta=\theta$ for all $s\in S$ and $\gamma\in\Gamma$. If S has a zero element 0, then $0\gamma a=\theta$ for all $\gamma\in\Gamma$ and $a\in A$. For an S_Γ-subact $B\subseteq A$ the annihilator of B in A is the set $[B:A]=\{s\in S\mid s\gamma a\in B\ \forall\ \gamma\in\Gamma, a\in A\}$ which forms a Γ-ideal of S. A map $f:A\to B$ between S_Γ-acts is an S_Γ-homomorphism if satisfies: $f(s\gamma a)=s\gamma f(a)$ for all $s\in S$, $\gamma\in\Gamma$ and $a\in A$. If f is surjective it is called an S_Γ-epimorphism. The kernel of S_Γ-homomorphism f is defined as: $\ker(f)=\{(a_1,a_2)\in A\times A\mid f(a_1)=f(a_2)\}$. An equivalence relation ρ on A is a congruence if $(a_1,a_2)\in\rho$ implies $(s\gamma a_1, s\gamma a_2)\in\rho$ for all $s\in S$, $\gamma\in\Gamma$. The quotient S_Γ-act A/ρ consists of equivalence classes $a\rho$. An S_Γ-act A is called a multiplication S_Γ-act if every S_Γ-subact $B\subseteq A$ satisfies $B=K\Gamma A$ for some Γ-ideal $K\subseteq S$. Equivalently, $B=[B:A]\Gamma A$ for every S_Γ-subact B of A, [6]. For a given Γ-monoid S, an S_Γ-act A is a multiplication if and only if for every maximal Γ-ideal Q of S either $A=T_Q(A)$ or A is Q-cyclic [6]. An S_Γ-act A is called faithful if, for all $a\in A$ and $\gamma\in\Gamma$, the equality $s\gamma a=t\gamma a$ implies $s=t$. Let S be a commutative Γ-monoid and A a faithful S_Γ-act. Then A is a multiplication if and only if $\bigcap_{i\in I}(K_i\Gamma A)=(\bigcap_{i\in I}K_i)\Gamma A$, for any family of Γ-ideals $\{K_i: i\in I\}$, [7]. Additional results and notions regarding S_Γ-Acts have been established. Of particular interest are multiplication S_Γ-acts, introduced by [8], [9], [10] . Let A be an S_Γ-act, and let B_1 and B_2 be S_Γ-subacts of A. The product of B_1 and B_2, denoted B_1*B_2 defined as: $B_1*B_2=([B_1:A]\Gamma[B_2:A])\Gamma A$. The product B_1*B_2 is itself an S_Γ-subact of A, as it is closed under the actions of S and Γ. Let A be a multiplication S_Γ-act. Then for any S_Γ-subacts B_1, B_2 of A, their product is defined as: $B_1*B_2=(K\Gamma L)\Gamma A$ where K and L are Γ-ideals of the a Γ-semigroup S satisfying $B_1=K\Gamma A$ and $B_2=L\Gamma A$ [11].

This work begins by defining Γ-multiplicatively closed subsets R of Γ-monoids S and establishing fundamental notation. We achieve three principal advances: generalizingclassical localization theory to Γ-semigroups, first constructing S_Γ-act localizations via congruence relations to analyze multi-operator systems, second proving a universal factorization property that extends classical homomorphism theorems, while characterizing distinct algebraic properties and third establishing structural results including a Local-Global Principle and new ideal-theoretic relations in algebraic theory. This paper aims to introduce and study the concept of R_Γ-multiplication S_Γ-acts, which generalizes the notion of multiplication S_Γ-acts. We characterize certain prime S_Γ-subacts, R_Γ-finite R_Γ-subacts, and R_Γ-cyclic S_Γ-acts in terms of these concepts. Additionally, we establish an equivalent condition, along with examples and counterexamples related to this notion. For completeness, we begin by defining Γ-multiplicatively closed subsets R of a Γ-monoid S and introducing the notation to be used throughout this paper. By employing the concept of multiplication R_Γ-acts, we examine [i]their relationship with R_Γ-multiplication R_Γ-acts, presenting several results and properties. The idempotency property of S_Γ-subacts is applied to prove that the product of S_Γ-subacts is an R_Γ-multiplication. Moreover, we show that R_Γ-multiplication S_Γ-acts are preserved under S_Γ-epimorphisms. Analogously to the fact that a cyclic S_Γ-act is a multiplication S_Γ-act, we investigate the connection between R_Γ-cyclic S_Γ-acts and R_Γ-multiplication S_Γ-acts. The notion of finitely generated S_Γ-acts plays a significant role in the theory; thus, we use it to characterize certain classes of S_Γ-acts. Furthermore, the concept of R_Γ-finite S_Γ-subacts are introduced to connect R_Γ-cyclic and R_Γ-multiplication S_Γ-acts, leading to several essential results. All Γ-semigroups considered in this work are commutative and have an identity element.

2 Main Results

Definition 2.1. Let S a Γ-monoid and R a nonempty subset of S. The set R is called Γ-multiplicatively closed if it satisfies:

1. The identity element $1\in R$,
2. For all $r_1, r_2\in R$ and $\alpha \in \Gamma$ then $r_1\alpha r_2 \in R$.

Examples 2.2.

1. Let $S=\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$, $\Gamma=\{\varnothing, \{a\}, \{a, b, c\}\}$. Then S is a Γ-monoid under the mapping: $S \times \Gamma \times S \to S$ defined by (A, B, C) $\mapsto$ A∩B∩C. If $R= \{\varnothing, \{b\}, \{a\}, \{c\}, \{a, b, c\}\}$ is a subset of S, then R is a Γ-multiplicatively closed subset of S.

2. Let $S=\{5n + 4 : n$ is a positive integer$\}$ and $\Gamma=\{5n + 1: n$ is a positive integer$\}$. Then S is a Γ-monoid under the mapping: $S \times \Gamma \times S \to S$ defined by $a\alpha b = a + \alpha + b$. If $R= \{ 5n + 9 : n$ is a positive integer$\}$ then R is a subset of S but not Γ-multiplicatively closed subset of S .

3. Let $S=\mathbb{R}$ and $\Gamma=\mathbb{N}$. If $a\in \mathbb{R}$, then a subset set R$=\{a^i, a^{i+1}, \cdots, a^n, \cdots\}$is Γ-multiplicatively closed for any positive integer i.

4. Consider $S=\Gamma=\mathbb{Z}$ (set of integers numbers). Then S is a Γ-semigroup under a multiplication of integer numbers. For $n\neq 1$, let R= $n\mathbb{Z}$ is a subset of S and its closed by multiplication of integer numbers but not Γ-multiplicatively closed subset of S.

Definition 2.3. Let A be an S_Γ-act and R a Γ-multiplicatively closed subset of S. Then A is called an R_Γ-multiplication S_Γ-act if for every S_Γ-subact B of A, there exist $r \in R$ and a Γ-ideal K of S such that $r\Gamma B \subseteq K\Gamma A \subseteq B$.

A Γ-ideal K of a Γ-monoid S is said to be R_Γ-multiplication if it is an R_Γ-multiplication S_Γ-subact of the S_Γ-act S. If every Γ-ideal S is R_Γ-multiplication, then S is called an R_Γ-multiplication Γ-monoid.

Examples.2.4.

1. Let S $= \{ \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\} \}$, $\Gamma=\{\varnothing, \{a\}\}$ and $A = S$ .Then A is an S_Γ-act by the mapping: $A\times \Gamma \times A \to A$ defined by $(H, K, L)\mapsto H\cap K\cap L$. Consider, $R=\{\varnothing, \{a, b\}\}$. Then R is Γ-multiplicatively closed subset of S. Here A is R_Γ-multiplication.

2. Let $S=\Gamma=\mathbb{N}^\infty= \mathbb{N}\cup\{\infty\}$, where for each $n \in \mathbb{N}$, $n < \infty$ with the multiplication defined by $s\alpha t=\min\{s, t\}$ for all $s, t \in S$ and $\alpha \in \Gamma$. Then S is Γ-monoid with zero and for each $s \in \mathbb{N}^\infty$and $\alpha \in \Gamma$, we have $s\alpha\infty = s$ and $s\alpha 1=1$ and hence ∞ it is the identity element and 1 is the zero element. Let $R=\mathbb{N}^\infty\setminus\{1\}$. Then R is Γ-multiplicatively closed subset of S. If K is an Γ-ideal of S, then $K= S$, $K=\mathbb{N}$, or there exists $n\in\mathbb{N}$ such that $K=\{1, 2, \ldots, n\}$. Now, let B be a S_Γ-subact of S_Γ-act $\mathbb{N}$, then there exists $r \in R$ such that $r\Gamma B \subseteq K\Gamma\mathbb{N} \subseteq B$. Thus, $\mathbb{N}$ is a R_Γ-multiplication S_Γ-act.

3. Let $S = \{x, y, \ \}$, $\Gamma=\{ \alpha, \beta, \gamma\}$ and $A=S$. Define a binary operation in A as shown in the following Table 1:

Table 1: The binary operation on A. (Source: Created by the authors.)

.	x	y	z
x	x	x	x
y	x	x	x
z	x	y	z

Then A is an S_Γ-act by the mapping: $S \times \Gamma\times A \to A$ defined by $x\alpha y = xy$ for all $x, y \in A$. Take $R=\{x, z\}$ then R is Γ-multiplicatively closed subset of S. Here $\{x, y\}$ is S_Γ-subact of A and $r\Gamma B \subseteq K\Gamma A \nsubseteq B$ for all $r \in R$ and Γ-ideal K of S. Thus A is not R_Γ-multiplication.

Proposition 2.5. An S_Γ-act A is R_Γ-multiplication if and only if for all S_Γ-subact B of A, there exists $r \in R$ such that $r\Gamma B \subseteq [B: A]\Gamma A \subseteq B$.

Proof: ($\Longleftarrow$) Clear.

($\Longrightarrow$) Let B be a S_Γ-subact of S_Γ-act A. Since A is R_Γ-multiplication, then there exists $r \in R$ and Γ-ideal K of Γ-semigroup S such that $r\Gamma B \subseteq K\Gamma A \subseteq B$ it follows that $K \subseteq [B: A]$ and hence $r\Gamma B \subseteq [B: A]\Gamma A \subseteq B$.

Example 2.6. For any Γ-multiplicatively closed subset R of a Γ-monoid S, every multiplication S_Γ-act is an R_Γ-multiplication S_Γ-act. The converse holds when $R \subseteq U_\Gamma(S)$.

We present an example of an S_Γ-act that is R_Γ-multiplication but not multiplication.

Example 2.7. Let $S = \mathbb{Z}$, $\Gamma=\mathbb{N}$. Then S is a Γ-monoid by the mapping $(z_1,n,z_2) \rightarrow z_1.n.z_2$, where $z_1.n.z_2$ is the usual multiplication. Now, let $A = \{ 1, 2, 3, 4, 5, 6 \}$and define the mapping: $S \times \Gamma \times A \rightarrow A$ by $(z, n, a) \longmapsto 2$. Then any nonempty subset of A that contains 2 is S_Γ-subact of A. In particular, if $B=\{1, 2\}$ is S_Γ-subact of A, then $B \neq \{2\}=K\Gamma A$ for each Γ-ideals K of S. Thus A is not multiplication S_Γ-act. Consider $R= \mathbb{Z}$-$\{0\}$, then R is a Γ-multiplicatively closed subset of S. Then there exists $r \in R$ such that $\Gamma B \subseteq K\Gamma A \subseteq B$. Thus A is a R_Γ-multiplication S_Γ-act.

Proposition 2.8. Let A be a multiplication S_Γ-act and B be an idempotent S_Γ-subact of A. Then B is R_Γ-multiplication.

Proof: Let L be a S_Γ-subact of B. By hypothesis A is R_Γ-multiplication, thus there exists $r \in R$ such that $r\Gamma L \subseteq [L: A]\Gamma A \subseteq L$. Thus $(r\Gamma[B: A])\Gamma L \subseteq [L: A]\Gamma [(B: A]\Gamma A) \subseteq L$, since A is a multiplication and $[L: A] \subseteq [B: A]$. Then $(r\Gamma[L: A])\Gamma L \subseteq [B: A]\Gamma B \subseteq L$. Since L is idempotent S_Γ-subact, then $r\Gamma L \subseteq [B: A]\Gamma B \subseteq L$ and hence B is R_Γ-multiplication S_Γ-subact of S_Γ-act A.

For elements x, y in a Γ-monoid S, we say x divides y (denoted $x|y$) if there exist $z \in S$ and $\alpha \in \Gamma$ such that $x\alpha z = y$.

Let R be a Γ-multiplicatively closed subset of a Γ-monoid S. Then the Γ-saturation R^* of R is defined by: $R^*=\{s\in S: s|r$ for some $r\in R\}$. The Γ-saturation R^* itself is Γ-multiplicatively closed and satisfies $R\subseteq R^*$

Proposition 2.9. Let S be a Γ-monoid and A be an S_Γ-act. Consider the following.

- **i.** If $R \subseteq T$ are Γ- multiplicatively closed subsets of S and A is an R_Γ-multiplication S_Γ-act, then A is a T_Γ-multiplication S_Γ-act.
- **ii.** An S_Γ-act A is an R_Γ-multiplication if and only if A is an $R^*{}_\Gamma$-multiplication S_Γ-act.

Proof: i. Let B be S_Γ-subact of A. Then there exists $r \in R \subseteq T$ and a Γ-ideal K of S such that $\Gamma B \subseteq K\Gamma A \subseteq B$. Thus A is a R_Γ-multiplication S_Γ-act.

ii. Let A be a R_Γ-multiplication S_Γ-act. Since $R \subseteq R^*$, by (i) A is an $R^*{}_\Gamma$-multiplication. Conversely, suppose that A is an R^*-multiplication S_Γ-act. Let B be S_Γ-subact. Then there exists $x \in R^*$ such that $\Gamma B \subseteq [B: A]\Gamma A$. Take $y \in S$ with $x\Gamma y \subseteq R$. Then $(x\Gamma y)\Gamma B \subseteq [B: A]\Gamma A$. Hence A is a R-multiplication S_Γ-act.

Theorem 2.10. Let S be a Γ-monoid and A be an S_Γ-act. Then the following statements are equivalent:

- **i.** A is a multiplication S_Γ-act.
- **ii.** A is an $(S\backslash P)_\Gamma$-multiplication S_Γ-act for each prime Γ-ideal P of S.
- **iii.** If S is a globally idempotent Γ-semigroup, then A is an $(S\backslash Q)_\Gamma$-multiplication S_Γ-act for every maximal Γ-ideal Q of S.

Proof. (i) $\Longrightarrow$ (ii) Assume A is a multiplication S_Γ-act, and let P be a prime Γ-ideal of S. Then $(S \backslash P)$ is a multiplicative closed subset of S. and A is an $(S\backslash P)_\Gamma$-multiplication S_Γ-act by Example 2.6.

(ii)$\Longrightarrow$ (iii) Assume S is globally idempotent, and let M be a maximal Γ-ideal of S. Thus, M is a prime Γ-ideal. By(ii) A is an $(S\backslash Q)_\Gamma$-multiplication S_Γ-act.

(iii)$\Longrightarrow$ (i) Let B be a S_Γ-subact of A. Define $I(B)=\{s \in S| \ s\Gamma A \subseteq B\}$, which is the largest Γ-ideal satisfying $I(B)\Gamma A \subseteq B$. By (iii), there exists $r \in S\backslash Q$ and a Γ-ideal $K\Gamma Q$ of S such that: $r\Gamma B\subseteq(K\Gamma Q)\ \Gamma A \subseteq B$. Note that $K\Gamma Q \subseteq I(B)$ by the definition of $I(B)$. Consider $\Psi=\{r\Gamma Q\}$. Then Ψ cannot be contained in any maximal Γ-ideal. By global idempotence, $1 \in \Psi$.

Thus $1= \bigcup_{i=1}^{n}(r\alpha_i Q_i)$,$\alpha_i \in \Gamma$. Take $I=\bigcup(K\Gamma Q_i)$. Then $B = 1\Gamma B \subseteq \bigcup_{i=1}^{n}(r\alpha_i Q_i)\Gamma B \subseteq \bigcup_{i=1}^{n} (K\Gamma Q_i)\Gamma A \subseteq I(B)\Gamma A \subseteq B$. Hence $B = I(B)\Gamma A$, proving A is multiplication S_Γ-act.

Corollary 2.11. Let $\boldsymbol{S}$ be a globally idempotent **Γ**-monoid and let $\boldsymbol{A}$ be an $\boldsymbol{S}_\Gamma$-act. Then the following are equivalent:

1. A is a multiplication S_Γ-act.
2. A is an $(S \setminus Q)_\Gamma$-multiplication S_Γ-act for every maximal Γ-ideal Q of S.

Proof: Since S is globally idempotent, every maximal Γ-ideal Q is prime. Thus, by Theorem 2.10, conditions (1) and (2) are equivalent.

The following proposition gives a characterization of Γ-multiplicatively closed sets.

Proposition 2.12. Let $\boldsymbol{S}$ be a **Γ**-monoid and $\boldsymbol{R \subseteq S}$ a Γ-multiplicatively closed subset. Then:

1. The intersection of an arbitrary family of Γ-multiplicatively closed subsets of S is Γ-multiplicatively closed.
2. If K is a Γ-ideal of S such that $K \cap R = \emptyset$, then there exists a maximal Γ-ideal P with respect to inclusion, satisfying $K \subseteq P$ and $P \cap R = \emptyset$

Proof.

1. Let $\{R_i\}_{i\in I}$ be a family of Γ-multiplicatively closed subsets of S , and let $R = \bigcap_{i\in I} R_i$. Since $1 \in R_i$ for all $\in I$, $1 \in R$. If $r_1, r_2 \in R$ and $\alpha \in \Gamma$, then $r_1, r_2 \in R_i$ for all $i \in I$, hence $r_1 \alpha r_2 \in$R. Thus, R is Γ-multiplicatively closed.
2. The proof follows by Zorn's Lemma. Consider the set $\mathcal{T} = \{L | L$ is a Γ-ideal, $K \subseteq L, L \cap R = \emptyset\}$, partially ordered by inclusion. Since $K \in P$, the set is nonempty. Every chain in P has an upper bound (the union of the chain), so by Zorn's Lemma, P has a maximal element P, which is the desired maximal Γ-ideal.

Let S be a Γ-monoid $R \subseteq S$, a Γ-multiplicatively closed subset and A an S_Γ-act. The localization of A at R, denoted $R^{-1}A$, is the set of equivalence classes $[a/r]\rho$, where $a \in A$, $r \in R$ and ρ is the congruence relation defined by: $(a,r)\ \rho\ (a',r')$ if and only if there exists $s \in R$ and $\alpha, \beta \in \Gamma$ such that $s\alpha(r'\beta a) = s\alpha(r\beta a')\cdot$ The set $R^{-1}A$ is an S_Γ-act under the mapping: $S \times \Gamma \times R^{-1}A \to R^{-1}A$, $(s\alpha[a/r]\rho) \mapsto [s\alpha a/r]\rho$. To verify this satisfies the definition of an S_Γ-act, we:

1. Associativity: For all $s_1, s_2 \in$S, $\alpha,\beta \in \Gamma$, and $[a/r]\rho \in R^{-1}A$:$(s_1\alpha s_2)\beta[a/r]$=$[(s_1\alpha s_2)\beta a/r]\rho$ = $[s_1\alpha(s_2\beta a)/r]\rho = s_1\alpha(s_2\beta[a/r]\rho)$, which follows from the Γ-monoid and S_Γ-act definitions.
2. Congruence Compatibility: The relation ρ respects the S_Γ-act: if $(a,r)\rho(a',r')$, then for any $s \in S$, $\alpha \in \Gamma$, $(s\alpha a,r)\rho(s\alpha a',r')$.

Two equivalence classes $[a/r]\rho$ and $[a'/r']\rho$ are considered equivalent if there exists some $s \in R$ and $\alpha \in \Gamma$ such that: $s\alpha[a/r']\rho = s\alpha[a'/r]\rho$.

The following Theorem establishes the universal property of localization in S_Γ-acts, from which we derive two important corollaries: a local-global principle for S_Γ-acts and an extension of prime avoidance for Γ-multiplicative sets.

Theorem 2.13. Let S be a Γ-monoid, $R \subseteq S$ a Γ-multiplicatively closed subset and A an S_Γ-act. Then for every S_Γ-act B and S_Γ-homomorphism f:$A \to B$ such that for each $r \in R$ and $\alpha \in \Gamma$ the map λ_r:$B \to B$, $\lambda_r(b)$=$r\alpha b$ is bijective, there exists a unique S_Γ-homomorphism $\tilde{f}$: $R^{-1}A \to B$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[d, "f"'] & R^{-1}A \arrow[dl, dashed, "\tilde{f}"] \\ B & \end{tikzcd}$$

where $\boldsymbol{t(a)} = [\boldsymbol{a/1}]\boldsymbol{\rho}$ is the canonical map.

Proof: Define $\tilde{\boldsymbol{f}}$: $\boldsymbol{R^{-1}A} \to \boldsymbol{B}$ by $\tilde{\boldsymbol{f}}([\boldsymbol{a/r}]\boldsymbol{\rho}) = \boldsymbol{\lambda_r^{-1}}\big(\boldsymbol{f(a)}\big)$ for all $[\boldsymbol{a/r}]\boldsymbol{\rho} \in \boldsymbol{R^{-1}A}$, where $\boldsymbol{\lambda_r^{-1}}$ is the inverse of the bijective map $\boldsymbol{\lambda_r}$. First, we prove that is $\tilde{\boldsymbol{f}}$ Well-Defined. If $[\boldsymbol{a_1/r_1}]\boldsymbol{\rho}$=$[\boldsymbol{a_2/r_2}]\boldsymbol{\rho}$, there exist $\boldsymbol{s \in R}$ and $\boldsymbol{\alpha,\beta} \in \Gamma$ such that $\boldsymbol{s\alpha\,(r_2\beta a_1) = s\alpha(r_1\beta a_2)}$. Apply $\boldsymbol{f}$ to both sides. $(\boldsymbol{s\alpha r_2})\boldsymbol{\beta f(a_1)}$ =$(\boldsymbol{s\alpha r_1})\boldsymbol{\beta f(a_2)}$. Since $\boldsymbol{s\alpha\, r_2}$, $\boldsymbol{s\alpha\, r_1} \in R$, the maps $\boldsymbol{\lambda_{s\alpha r_2}}$ and $\boldsymbol{\lambda_{s\alpha r_1}}$ are bijective. Apply $\boldsymbol{\lambda_{s\alpha r_2}^{-1}}$ to both sides $\boldsymbol{\lambda_{s\alpha r_2}^{-1}}\big((\boldsymbol{s\alpha r_2})\boldsymbol{\beta f(a_1)}\big)$=$\boldsymbol{\lambda_{s\alpha r_2}^{-1}}\big((\boldsymbol{s\alpha r_1})\boldsymbol{\beta f(a_2)}\big)$ $\Rightarrow$ $\boldsymbol{f(a_1)}$=$\boldsymbol{\lambda_{s\alpha r_2}^{-1}}\big((\boldsymbol{s\alpha r_1})\boldsymbol{\beta f(a_2)}\big)$. Multiply by $\boldsymbol{\lambda_{r_1}^{-1}}$ **we get** $\boldsymbol{\lambda_{r_1}^{-1}}\big(\boldsymbol{f(a_1)}\big) =$

$\boldsymbol{\lambda_{r_1}^{-1}}\Big(\boldsymbol{\lambda_{s\alpha r_2}^{-1}}\big((\boldsymbol{s\alpha r_1})\boldsymbol{\beta f(a_2)}\big)\Big) =$

$\boldsymbol{\lambda_{r_2}^{-1}}\Big(\boldsymbol{\lambda_{s\alpha r_1}^{-1}}\big((\boldsymbol{s\alpha r_1})\boldsymbol{\beta f(a_2)}\big)\Big)$ =$\boldsymbol{\lambda_{r_2}^{-1}}\big(\boldsymbol{f(a_2)}\big)$ (By $\boldsymbol{S_\Gamma}$-act properties and bijectivity) proving $\tilde{f}$ is well-defined. Now, we prove $\tilde{\boldsymbol{f}}$ is an SΓ-homomorphism. For $\boldsymbol{s} \in \boldsymbol{S}, \boldsymbol{\alpha} \in \boldsymbol{\Gamma}$ and $[\boldsymbol{a}/\boldsymbol{r}]\boldsymbol{\rho} \in \boldsymbol{R}^{-1}\boldsymbol{A}, \tilde{\boldsymbol{f}}(\boldsymbol{s}\boldsymbol{\alpha}[\boldsymbol{a}/\boldsymbol{r}]\boldsymbol{\rho})$ $=\tilde{\boldsymbol{f}}([\boldsymbol{s}\boldsymbol{\alpha}\boldsymbol{a}/\boldsymbol{r}]\boldsymbol{\rho})$ $=\boldsymbol{\lambda}_r^{-1}(\boldsymbol{f}(\boldsymbol{s}\boldsymbol{\alpha}\boldsymbol{a}))=$ $\boldsymbol{\lambda}_r^{-1}(\boldsymbol{s}\boldsymbol{\alpha}\boldsymbol{f}(\boldsymbol{a}))=$ $\boldsymbol{s}\boldsymbol{\alpha}\left(\boldsymbol{\lambda}_r^{-1}\boldsymbol{f}(\boldsymbol{a})\right)=\boldsymbol{s}\boldsymbol{\alpha}\,\tilde{\boldsymbol{f}}\left(\left[\frac{a}{r}\right]\boldsymbol{\rho}\right)$ for commutativity, $\tilde{\boldsymbol{f}}$ $(\boldsymbol{t}(\boldsymbol{a}))=\tilde{\boldsymbol{f}}([\boldsymbol{a}/\boldsymbol{1}]\boldsymbol{\rho})=\boldsymbol{\lambda}_{r_1}^{-1}(\boldsymbol{f}(\boldsymbol{a}))=\boldsymbol{f}(\boldsymbol{a})$. For uniqueness, if g: $\boldsymbol{R}^{-1}\boldsymbol{A}\rightarrow\boldsymbol{B}$ also satisfies $\boldsymbol{g}\circ\boldsymbol{t}=\boldsymbol{f}$, then: $\boldsymbol{g}([\boldsymbol{a}/\boldsymbol{r}]\boldsymbol{\rho})=\boldsymbol{g}(\boldsymbol{t}(\boldsymbol{a}).[\boldsymbol{1}/\boldsymbol{r}]\boldsymbol{\rho})$ $=$ $\boldsymbol{g}(\boldsymbol{t}(\boldsymbol{a})).\boldsymbol{g}([\boldsymbol{1}/\boldsymbol{r}]\boldsymbol{\rho})$ $=$ $\boldsymbol{f}(\boldsymbol{a})$ $.\ \boldsymbol{g}([\boldsymbol{1}/\boldsymbol{r}]\boldsymbol{\rho})$ $=\boldsymbol{f}(\boldsymbol{a}).\boldsymbol{\lambda}_r^{-1}(\boldsymbol{f}(\boldsymbol{1}_A))=$ $\boldsymbol{f}(\boldsymbol{a})$. $\tilde{\boldsymbol{f}}([\boldsymbol{1}/\boldsymbol{r}]\boldsymbol{\rho})$ $=$ $\boldsymbol{\lambda}_r^{-1}(\boldsymbol{f}(\boldsymbol{a}))=$ $\tilde{\boldsymbol{f}}([\boldsymbol{a}/\boldsymbol{r}]\boldsymbol{\rho}$. Hence $\boldsymbol{g}=\tilde{\boldsymbol{f}}$.

Corollary 2.14. Let S be a Γ-monoid and A an S_Γ-act. Then the following are equivalent:

i. A is a cyclic S_Γ-act.
ii. A_P is a cyclic $(S_P)_\Gamma$-act for every maximal Γ-ideal P, where $A_P=(S\backslash P)^{-1}A$.

Proof. i $\Longrightarrow$ ii: If A is a cyclic S_Γ-act, then $A=\langle a\rangle_A$ for some $a \in A$. Thus $A_P=\langle[a/r]\rho\rangle_A$.
ii $\Longleftarrow$ i: Suppose that A is not cyclic, there exists a maximal Γ-ideal P such that A_P is not cyclic, leading to a contradiction.

Corollary 2.15. Let $\boldsymbol{S}$ be a Γ-monoid and $\boldsymbol{R}$ a Γ-multiplicatively closed subset. If $\mathbf{K_1}, \mathbf{K_2},\ldots, \mathbf{K_n}$ are Γ-ideals of $\boldsymbol{S}$ such that at most two are not prime, then:$\boldsymbol{R}\cap(\bigcup_{\mathbf{i=1}}^{\mathbf{n}}\mathbf{K_i})=\emptyset$ **then there exists** $\boldsymbol{r} \in \boldsymbol{R}$ such that $\boldsymbol{r} \notin \bigcup_{\mathbf{i=1}}^{\mathbf{n}}\mathbf{K_i}$.

Proof. The proof follows by induction on **n** and the properties of the Γ-multiplicative closure.

Proposition 2.16. Every homomorphic image of a $\boldsymbol{R}_\Gamma$-multiplication $\boldsymbol{S}_\Gamma$-act is a $\boldsymbol{R}_\Gamma$-multiplication. Proof. Let $\boldsymbol{A}$, $\boldsymbol{A'}$ be $\boldsymbol{S}_\Gamma$-acts and $\boldsymbol{f}$: $\boldsymbol{A}\rightarrow\boldsymbol{A'}$ be an $\boldsymbol{S}_\Gamma$-epimorphism. Suppose that $\boldsymbol{A}$ is an $\boldsymbol{R}_\Gamma$-multiplication $\boldsymbol{S}_\Gamma$-act. Let $\boldsymbol{B'}$ be a $\boldsymbol{S}_\Gamma$-subact of $\boldsymbol{A'}$. Then $\boldsymbol{f}^{-1}(\boldsymbol{B'})=\boldsymbol{B}$ is a SΓ-subact of $\boldsymbol{A}$. Since $\boldsymbol{A}$ is an $\boldsymbol{R}_\Gamma$-multiplication $\boldsymbol{S}_\Gamma$-act there exists $\boldsymbol{r}\in\boldsymbol{R}$ such that $\boldsymbol{r}\Gamma\boldsymbol{B}\subseteq[\boldsymbol{B}:\boldsymbol{A}]\Gamma\boldsymbol{A}\subseteq\boldsymbol{B}$. Then $\boldsymbol{f}(\boldsymbol{r}\Gamma\boldsymbol{B})\subseteq\boldsymbol{f}([\boldsymbol{B}:\boldsymbol{A}]\Gamma\boldsymbol{A})\subseteq\boldsymbol{f}(\boldsymbol{B})$. Thus implies that $\boldsymbol{r}\Gamma\boldsymbol{B'}=\boldsymbol{r}\Gamma\boldsymbol{f}(\boldsymbol{B})\subseteq[\boldsymbol{B}:\boldsymbol{A}]\Gamma\boldsymbol{f}(\boldsymbol{A})=[\boldsymbol{A'}:\boldsymbol{A}]\Gamma\boldsymbol{B}\subseteq\boldsymbol{B'}$. Therefore, $\boldsymbol{A'}$ is an $\boldsymbol{R}_\Gamma$-multiplication $\boldsymbol{S}_\Gamma$-act.

The converse of the above Proposition is true if $f: A\rightarrow A'$ is SΓ-isomorphism.

Corollary 2.17. Let A be an R_Γ-multiplication SΓ-act and $\pi: A \longrightarrow A/\rho$ where ρ is a congruence on A. Then A/ρ is an R_Γ-multiplication S_Γ-act.

We now introduce the product of R_Γ-multiplication SΓ-subacts in multiplication acts. This construction requires the following preliminary results:

Lemma 2.18. Let A be an SΓ-act and B_1, B_2 are SΓ-subacts of A. Then $[B_1 * B_2: A]=[B_1$:A$]\Gamma[B_2$:A$]$.

Proof: Let $s \in [B_1: A]\Gamma[B_2: A]$. Then $s\Gamma A \subseteq [B_1$: A$]\Gamma[B_2$: A$]\Gamma A$.Since $B_1 * B_2=[B_1$:A$]\Gamma[B_2$:A$])\Gamma$A. Thus $s\Gamma A \subseteq B_1 * B_2$ and hence $s \in [B_1 * B_2: A]$. Conversely, let $r \in [B_1 * B_2: A] \Longrightarrow r\Gamma A \subseteq B_1 * B_2$ $\Longrightarrow$ $r\alpha a \in B_1 * B_2$ for some $\alpha \in \Gamma$ and $a \in A$, thus $r\alpha a \in ([B_1: A]\Gamma[B_2: A])\Gamma A$ and hence $r\in [B_1: A]\ \Gamma$ $[B_2: A]$, $\alpha \in \Gamma$ and $a \in A$. Therefore $[B_1 * B_2: A]=$ $[B_1$: A$]\Gamma[B_2$: A$]$.

Lemma 2.19. If B is idempotent S_Γ-subacts of multiplication SΓ-act A, then $B * B=B$.

Proof: Since B is idempotent, then $B=[B$:A$]\Gamma B$ thus $B * B=$ ($[B$:A$]\Gamma[B$:A$])\Gamma A=[B$:A$]\Gamma B=B$.

Proposition 2.20. Let A be a multiplication SΓ-act and B_1, B_2 are idempotent SΓ-subacts of A. Then $B_1 * B_2$ is an idempotent SΓ-subacts of .

Proof: $B_1 * B_2$=($[B_1$:$A]\Gamma[B_2$:$A])\Gamma$A= $[B_2$:$A]\Gamma([B_1$:$A]A)$ $=[B_2$:$A]\Gamma B_1=$ ($[B_2$:$A]\Gamma[B_1$:$A])\Gamma B_1$. $B_1 * (B_2 * B_2)=$ ($[B_2$:$A]\Gamma[B_1$:$A])\Gamma(B_1 * B_2)$. By Lemma 2.18 and Lemma 2.19, $B_1 * B_2=[B_1 * B_2: A]\Gamma(B_1 * B_2)$.

Corollary 2.21. Let A be a multiplication S_Γ-act and B_1, B_2 are idempotent SΓ-subacts of A. Then $B_1 * B_2$ is R_Γ-multiplication SΓ-subacts of A .

Proof: It is obvious by Proposition 2.20 and Proposition 2.8.

Using the notion of Γ-multiplicatively closed subsets in a Γ-monoid S, we define R_Γ-cyclic S_Γ-acts and study their fundamental properties.

Definition 2.22. Let S be a Γ-monoid and R a Γ-multiplicatively closed subset of S. An S_Γ-act A is called. R_Γ-cyclic if there exist $\in R$ and $a \in A$ such that $r\Gamma A \subseteq S\Gamma \mathrm{a} \subseteq A$.

Examples 2.23.

1. Consider $S= \Gamma=\mathbb{N}$ and $A=\mathbb{N}\backslash\{1\}$ as an S_Γ-act under standard multiplication of natural numbers. Let U be the set of all prime numbers (a generating set for A). For $n \in \mathbb{N}\backslash\{1\}$, $N=S\Gamma n$ is an S_Γ-subact. With $R = \mathbb{N}^e \cup \{1\}$ (Γ-multiplicatively closed), there exist $r \in R$ and $a \in A$ with $r\Gamma A \subseteq S\Gamma a \subseteq A$, proving A is R_Γ-cyclic.

2. Following Example 2.4(2), let $R= \mathbb{N}_\circ \cup \{\infty\}$. Then R is Γ-multiplicatively closed subset of S. Since $r\Gamma A \subseteq S\Gamma a \subseteq A$ for some $r \in R$ and $a \in A$. Consequently, A is R_Γ-cyclic.

3. Let $S= \left\{\begin{bmatrix} a & q \\ 0 & a\end{bmatrix} | a \in \mathbb{Z} \text{ and } q \in \mathbb{Q}\right\}$ and $\Gamma = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\right\}$. Then S is a Γ-monoid by the usual multiplication of matrices. If $A=\left\{\begin{bmatrix} 0 & q \\ 0 & 0\end{bmatrix} | q \in \mathbb{Q}\right\}$, then A is a left Γ-ideal of S. Take $R= \left\{\begin{bmatrix} a & 0 \\ 0 & a\end{bmatrix} | a \in \mathbb{Z}\right\}$, R is a Γ-multiplicatively closed subset of S. Clearly, A is R_Γ-cyclic.

4. Let $S=A= \{x, y, \}$ and $\Gamma=\{\alpha, \beta\}$. Define a binary operation on A via the following Table 2:

Table 2: The binary operation on A. (Source: Created by the authors.)

.	x	y	z
x	x	x	x
y	x	x	x
z	x	y	z

Then A is an S_Γ-act under the mapping: $S \times \Gamma \times A \to A$ defined by $x\alpha y=xy$ for all $x,y \in A$ and $\alpha \in \Gamma$. For $R=\{x\}$, R is Γ-multiplicatively closed subset of S. The inclusion $r\Gamma A \subseteq S\Gamma \mathrm{a}$ fails for all $r \in R$ and $a \in A$. Therefore, A is not. R_Γ-cyclic.

The following result is the analog of the fact that a cyclic S_Γ-act is a multiplication S_Γ-act.

Proposition 2.24. Let A be an S_Γ-act and R a Γ-multiplicatively closed subset of S. If A is an R_Γ-cyclic then A is an R_Γ-multiplication.

Proof. Since A is R_Γ-cyclic then $r\Gamma A \subseteq S\Gamma \mathrm{a} \subseteq A$ for some $r \in R$ and $a \in A$. Let B be a S_Γ-subact of A. Then $r\Gamma B \subseteq r\Gamma A \subseteq S\Gamma a$ and hence $r\Gamma B = K\Gamma S\Gamma a$ for some Γ-ideal K of S. Now $(r\Gamma r)\Gamma B = (r\Gamma \mathrm{K})\Gamma S\Gamma a \subseteq (r\Gamma \mathrm{K})\Gamma A = (K\Gamma r)\Gamma A \subseteq \mathrm{K}\Gamma(S\Gamma a) r\Gamma B \subseteq B$. Therefore, $(r\Gamma r)\Gamma B \subseteq (r\Gamma \mathrm{K})\Gamma A \subseteq B$ where $r\Gamma r \subseteq \mathrm{S}, r\Gamma K$ is a Γ-ideal of S. Hence A is an R_Γ-multiplication S_Γ-act.

The converse of Proposition 2.24 does not hold in general, as shown by the following counterexample.

Example 2.25. Let $S=\Gamma=\{1, x, y\}$ where $S=\{x, y\}$ is a left zero Γ-semigroup. Take $R= \{1,x\}$, then it is clear R is a Γ-multiplicatively closed subset of S. Consider the right S_Γ-act $A=\{1_1, b_1, 1_2, b_2, 1_3, b_3\}$ presented by the following multiplication Table 3:

Table 3: The binary operation on A. (Source: Created by the authors.)

.	1	x	y
1_1	1_1	b_1	b_1
b_1	b_1	b_1	b_1
1_2	1_2	b_1	b_2
b_2	b_2	b_2	b_2
1_3	1_3	b_2	b_3
b_3	b_3	b_3	b_3

For S_Γ-subact B of S_Γ-act A there exists $r \in R$ such that $\Gamma B \subseteq K\Gamma A \subseteq B$. Thus, A is a R_Γ-multiplication S_Γ-act. But $r\Gamma A \nsubseteq S\Gamma a$ for all $r \in R$ and $a \in A$. Therefore, A isn't R_Γ-cyclic S_Γ-act.

Proposition 2.26. Let S be a Γ-monoid A be a finitely generated S_Γ-act. Then A is cyclic if and only if A is P-cyclic for all maximal Γ-ideal P of S.
Proof: (⟹) Assume that A is a finitely generated and cyclic S_Γ-act. Then there exists $a \in A$, $A = S\Gamma a$. Since A is finitely generated, there exists $q \in S\backslash P$ so that $q\Gamma A \subseteq S\Gamma a$. Thus, A is P-cyclic. (⟸) It's valid for any S_Γ-act.

Proposition 2.27. Let S be a Γ-monoid and P a maximal Γ-ideal of S. An S_Γ-act A is P-cyclic if and only if A is a finitely generated multiplication S_Γ-act.
Proof.(⟹) Since A is P-cyclic for all maximal Γ-ideal P of S, then A is a multiplication S_Γ-act. For all maximal Γ-ideal P_σ, choose $q_\sigma \in S\backslash P_\sigma$ *and* $a_\sigma \in$ A such that $q_\sigma\Gamma \mathrm{A} \subseteq \mathrm{S}\Gamma a_\sigma$. Notice that $(\{q_\sigma : q_\sigma \in S\backslash P_\sigma\}) = S$ and hence there exist $\sigma_1, \sigma_2, \ldots, \sigma_n$ with $(q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_n}) = S$. Thus $A = S\Gamma A = (q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_n})\Gamma A \subseteq q_{\sigma_1}\Gamma A \cup q_{\sigma_2}\Gamma A \cup \ldots \cup q_{\sigma_n}\Gamma A \subseteq S\Gamma a_{\sigma_1} \cup S\Gamma a_{\sigma_2} \cup \ldots \cup S\Gamma a_{\sigma_n}$, and hence $A = S\Gamma a_{\sigma_1} \cup S\Gamma a_{\sigma_2} \cup \ldots \cup S\Gamma a_{\sigma_n}$ is finitely generated S_Γ-act. (⟸)Let A be a finitely generated multiplication S_Γ-act. Then A is a cyclic for all maximal Γ-ideal of S, thus A is P-cyclic.

The following result characterizes R_Γ-multiplication S_Γ-acts in terms of their cyclic S_Γ-subact.

Proposition 2.28. Let S be a Γ-monoid, R be a finite Γ-multiplicatively closed of S, and be A an S_Γ-act. The following are equivalent:
(i) A is R_Γ-multiplication.
(ii) Every cyclic $S\Gamma a \subseteq A$ satisfies $r\Gamma S\Gamma a \subseteq k\Gamma A \subseteq S\Gamma a$ for some $r \in R$ and a Γ-ideal K of S.
Proof:(i) ⟹ (ii) Clear.
(ii) ⟸ (i) Let $B = \bigcup_{a_i \in B} S\Gamma a_i$ be S_Γ-subact of A. Then by hypothesis, there exists $r_i \in R$ such that $r_i\Gamma S\Gamma a_i \subseteq [S\Gamma a_i : A]\Gamma A$. Thus $r\Gamma B = r\Gamma\left(\bigcup_{a_i \in B} S\Gamma a_i\right) = \bigcup_{a_i \in B}(r\Gamma S\Gamma a_i) \subseteq \bigcup_{a_i \in B}[S\Gamma a_i : A]\Gamma A = (\bigcup_{a_i \in B}[S\Gamma a_i : A])\Gamma A$ and hence $r\Gamma B \subseteq [B : A]\Gamma A \subseteq$ B. So, A is an R_Γ-multiplication S_Γ-act.

Corollary 2.29. An S_Γ-act A over a Γ-monoid S is a R_Γ-multiplication if and only if for every $a \in$ A, $S\Gamma a = k\Gamma A$ for some Γ-ideal K of S.
Proof. Take R={1} and apply Proposition 2.28.

Building on our previous work on R_Γ-multiplication and R_Γ-cyclic S_Γ-acts, we now introduce the fundamental concept of R_Γ-finite S_Γ-subacts to establish new connections and results.

Definition 2.30. Let R be a Γ-multiplicatively closed subset of the Γ-monoid S and A an S_Γ-act. An S_Γ-subact B of A is called R_Γ-finite if there exists $r \in R$ and a finitely generated S_Γ-subact L of S_Γ-act A such that $r\Gamma B \subseteq L \subseteq B$. And if every S_Γ-subact R_Γ-finite then A is R_Γ-finite.

Example 2.31. Let $S = \Gamma = \mathbb{Z}$ and $A = \mathbb{Q}$. Then A becomes an S_Γ-act under standard multiplication of rational numbers. For $R = \mathbb{N}$, R is a Γ-multiplicatively closed subset of S. For the S_Γ-subact $B = a\mathbb{Z}$ ($a \in$Q∗). Take $r = 2 \in R$ and $L = 2a\mathbb{Z}$. Observe $2\Gamma B = 2\mathbb{Z}a\mathbb{Z} \subseteq 2a\mathbb{Z} = L \subseteq B$. Thus, A is R_Γ-finite.

Under the assumption that A is an R_Γ-finite S_Γ-act, we eliminate the finiteness condition on ***R*** in Proposition 2.28, as shown in the following proposition.

Proposition 2.32. Let S be a Γ-monoid, R be a Γ-multiplicatively closed subset of S and A be a R_Γ-finite S_Γ-act. Then:
1. A is R_Γ-multiplication;
2. For every cyclic S_Γ-subact $S\Gamma a$ of A, there exists $r \in R$ and a Γ-ideal K of S satisfying: $r\Gamma S\Gamma a \subseteq K\Gamma A \subseteq S\Gamma a$.

Proof: (1) ⟹ (2) Clear.
(2) ⟸ (1) Assume that for each cyclic S_Γ-subact $S\Gamma a$ of A there exist $r \in R$ and Γ-ideal K of S such that $r\Gamma S\Gamma a \subseteq K\Gamma A \subseteq S\Gamma a$. Let B be an S_Γ-subact of A. Since A is an R_Γ-finite then there exist $r \in R$ and a finitely generated S_Γ-subact $L = \bigcup_{i=1}^{n} S\Gamma a_i^*$ of S_Γ-act A such that $r\Gamma B \subseteq L \subseteq B$, where $a_i^* \in A$. By

assumption, there exists $r_i^* \in R$ such that $r_i^*\Gamma S\Gamma a_i^* \subseteq [S\Gamma a_i^*: A]\Gamma A$. Put $r'= r_1^*\alpha_1 r_2^*\alpha_2 \ldots \alpha_{n-1}r_n^*$ where $\alpha_i \in \Gamma$and $r'' = r\beta r'$. Then $r'' \in R$ and $r'\Gamma S\Gamma a_i^* \subseteq [S\Gamma a_i^*: A]\Gamma A$. So $r''\Gamma B \subseteq r'\Gamma L \subseteq \bigcup_{i=1}^{n} r'\Gamma S\Gamma a_i^*$ $\subseteq\bigcup_{i=1}^{n}[S\Gamma a_i^*: A]\Gamma A$ $\subseteq[\bigcup_{i=1}^{n} S\Gamma a_i^* : A]\Gamma A = [L: A]\Gamma A$ $\subseteq[B: A]\Gamma A$ and hence $r''\Gamma B \subseteq [B: A]\Gamma A$. Therefore, A is an R_Γ-multiplication S_Γ-act.

Proposition 2.33. Let S be a Γ-monoid, R be a Γ-multiplicatively closed subset of S and let A be an R_Γ-multiplication S_Γ-act. If B is a R_Γ-finite S_Γ-subact of A, then there exist $r^*\in R$ and a finitely generated Γ-ideal K of S such that $r^*\Gamma B \subseteq K\Gamma A \subseteq B$.
Proof. Let B be a R_Γ-finite S_Γ-subact of R_Γ-multiplication S_Γ-act A. Then there exists r_1,$r_2\in R$ and finitely generated S_Γ-subact $H= \bigcup_{i=1}^{n} S\Gamma a_i^*$ such that $r_1\Gamma B \subseteq [B: A]\Gamma A \subseteq B$ and $r_2\Gamma B \subseteq \bigcup_{i=1}^{n} S\Gamma a_i^* \subseteq B$. Put r=$r_1\alpha r_2 \in R$ for some $\alpha \in \Gamma$. Then $r\Gamma B \subseteq [B: A]\Gamma A$ and $r\Gamma B \subseteq \bigcup_{i=1}^{n} S\Gamma a_i^* \subseteq B$. Thus for all $1 \leq i \leq n$, $r\gamma a_i^*\in[B: A]\Gamma A$ such that $r\in[B: A]$, $\gamma \in \Gamma$ and $a_i^* \in A$. Take r^*=$r\beta r$ and consider K= $S\Gamma r$ is a finitely generated Γ-ideal. So $K\Gamma A \subseteq [B: A]\Gamma A \subseteq B$. Now, let $a\in B$. Then $r\rho a \subseteq r\Gamma B \subseteq \bigcup_{i=1}^{n} S\Gamma a_i^*$ and so $r\rho a = t\mu a_i^*$ for some $t \in S$,$\mu \in \Gamma$ and $a_i^* \in A$. Thus $r^*\rho a$= $(r\beta r)\rho a$=$r\beta(r\rho a) = r\beta(t\mu a_i^*)$= $(r\beta t)\mu a_i^*$ $\in K\Gamma A$. Therefore, $r^*\Gamma B \subseteq K\Gamma A \subseteq B$.

Corollary 2.34. Let S be a Γ-monoid and A a multiplication S_Γ-act. If B is a finitely generated S_Γ-subact of A, then there exists a finitely generated Γ-ideal K of S such that B=$K\Gamma A$.
Proof. Take R={1}. Then A is R_Γ-multiplication S_Γ-act, and B is an R_Γ-finite S_Γ-subact. The result follows by Proposition 2.33.

Proposition 2.35. Let S be a Γ-monoid, R be a Γ-multiplicatively closed subset of S and A be an S_Γ-act where S is R_Γ-finite. If A is both R_Γ-finite and an R_Γ-multiplication S_Γ-act, then every S_Γ-subact of A is R_Γ-finite
Proof. Let B be an S_Γ-subact of A. Since S is an R_Γ-finite, there exists s_1, s_2,..., s_n $\in[B: A]$ and $r\in R$ such that $r\Gamma[B: A] \subseteq \bigcup_{i=1}^{n} S\Gamma s_i$. Since A is an R_Γ-multiplication then $r'\Gamma B \subseteq [B: A]\Gamma A$ for some $r'\in R$. Since A R_Γ-finite then $r''\Gamma A \subseteq \bigcup_{i=1}^{k} S\Gamma a_i$ for some a_1, a_2,..., $a_k\in A$ and $r''\in R$. Put r^*=$r\alpha r'\beta r''\in R$. Then $r^*\Gamma B \subseteq (r\Gamma[B: A])\Gamma(r''\Gamma A)$$\subseteq$$(\bigcup_{i=1}^{n} S\Gamma s_i)\Gamma$ $(\bigcup_{i=1}^{k} S\Gamma a_i)$=$\bigcup_{i=1}^{n}\bigcup_{i=1}^{k} S\Gamma s_i\Gamma a_i$. Also, note that $s_i\Gamma a_i \subseteq B$. Thus B is R_Γ-finite.

Proposition 2.36. Let S be a Γ-monoid, R be a Γ-multiplicatively closed subset of S. An S_Γ-act A. is R_Γ-finite if and only if $K\Gamma A$ is an R_Γ-finite S_Γ-subact for every prime Γ-ideal K of S.

3 Conclusion

This research has established a comprehensive theoretical framework for R_Γ-multiplication S_Γ-acts generalizing classical multiplication concepts to Γ-semigroup actions through systematic investigation, we have developed three fundamental advancements. First, we constructed a complete localization theory for S_Γ-acts via congruence relations enabling precise analysis of multi-operator systems, second, we proved universal factorization properties extending classical homomorphism theorems to Γ-semigroup contexts and third, we derived structural connections, most notably a Local-Global Principle establishing relationships between local and global properties.

The work provides complete characterizations of prime. S_Γ-subacts R_Γ-finite subacts and R_Γ-cyclic S_Γ-acts supported by illuminating examples and boundary-defining counterexamples. Our results demonstrate that R_Γ-multiplication S_Γ-acts preserve essential algebraic properties, including stability under S_Γ-epimorphisms and idempotency conditions. By introducing the concept of R_Γ-finite S_Γ-subacts, we revealed new structural connections between R_Γ-cyclic and R_Γ-multiplication S_Γ-acts providing powerful tools for analyzing Γ-semigroup actions. These developments were achieved within a commutative framework with identity elements while suggesting natural directions for non-commutative generalizations. This research opens multiple avenues for future investigation, including extensions to non-commutative settings, deeper categorical analysis, and applications to related algebraic structures. The foundational results and methods developed here provide a robust platform for advancing the theory of generalized multiplication properties in Γ-semigroup actions.

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