A Novel Test for the Homogeneity of Several Covariance Matrices in High-Dimensional Data: Application to Gene Expression Analysis

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Abstract: - This paper introduces a new test for the homogeneity of several covariance matrices in high-dimensional data under the *p*-variate normal distribution. The test is constructed using U-statistic-based estimators to evaluate differences among covariance matrices, and applies an inverse-variance-weighted method to quantify the relative importance of these estimators. Its distribution under the null hypothesis is derived and follows a chi-square distribution as the dimension and sample sizes increase. Simulation results indicate that the test controls the Type I error rate better than three existing methods and achieves high power. To demonstrate its practical applicability, two real gene-expression datasets involving three- and four-group comparisons are analyzed.

Key-Words: - Multivariate normal distribution, homogeneity test, covariance matrices, U-statistics, empirical Type I error rate, empirical power, autoregressive structure, gene expression data.

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1 Introduction

In multivariate analyses such as discriminant analysis or a multivariate analysis of variance (MANOVA), testing whether the covariance matrices are equal is crucial in procedures, as the accuracy of the results depends on this property before proceeding with further analyses, [1]. The concept also has various practical applications. For example, when analyzing synthetic aperture radar for change detection, finding identical covariance matrices across datasets collected at different times suggests that there hasn't been a significant change between the images, or that integration aspects such as spectral analysis and wireless camera network architectures haven't changed, [2], [3], [4], [5].

Let $\mathbf{X}_{i1}, \mathbf{X}_{i2}, ..., \mathbf{X}_{iN_i}$ be independent and identically distributed random vectors from the *i*-th *p*-variate normal population with the mean vector $\boldsymbol{\mu}_i$ and covariance matrix, $\boldsymbol{\Sigma}_i$, where $\boldsymbol{\Sigma}_i$ is a positive definite matrix, where N_i denotes the sample size from the *i*-th population, $i = 1, ..., g$. The aim is to test whether the g population covariance matrices are equal:

$$H_0 : \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \cdots = \boldsymbol{\Sigma}_g = \boldsymbol{\Sigma} \tag{1}$$

Versus H_1 : Not all $\boldsymbol{\Sigma}_i$ are equal

For testing (1), the classical method is the likelihood ratio test (LRT), [6], whose asymptotic null distribution is the chi-square distribution when p is fixed and $p < N_i$. With the advent of high-dimensional data, where $p \geq N_i$, such scenarios commonly arise in fields like genomics, finance, economics, bioinformatics, and image analysis, where thousands of variables—such as gene expressions in DNA microarray experiments, [7], [8], [9], [10], [11], or numerous indices in financial time series [12] are measured from only a few dozen or hundreds of observations. As a result, classical approaches based on the LRT are not applicable in this context because the sample covariance matrices are singular, which makes determinant-based statistics undefined, [13], [14], [15]. Consequently, alternative methods are required to address the challenges associated with testing covariance matrices in high-dimensional data.

Some researchers have introduced a method to test the hypotheses (1). [16], developed a test based on the sum of squared Frobenius norms, which measures the differences between covariance matrices. He proved that the asymptotic distribution of the test follows the standard normal distribution. [17], proposed another test based on the Frobenius norm, whose asymptotic distribution follows the chi-square distribution. [18], introduced a U-statistic-based test that employs a linear combination of consistent and location-invariant estimators of trace functions forming the norm. Importantly, this test retains its validity even when the data deviate from normality. Extending the work of, [14], [19], developed a test for the multi-sample problem using the Frobenius norm. To handle both sparse and dense covariance matrices, [20], introduced a test which mixes two norms: the Frobenius norm following, [14], and the maximum norm inspired by [21]. [22], introduced a weighted test that expands upon the test given in [19].

The main purpose of this paper is to develop a new method for testing the homogeneity of g high-dimensional covariance matrices. This method extends the U-statistic-based estimators from [23], and incorporates an inverse variance-weighted framework. The asymptotic distribution of the proposed test is derived under a mild condition, which requires the boundedness of trace ratios of covariance matrices only up to the fourth power of the covariance matrix, making it well-suited for practical data applications. The simulation results indicate that the new test exhibits distinct characteristics compared to existing tests, including those referenced in [16], [18] and [22]. We observed that the proposed test outperforms its competing counterparts in terms of both the Type I error rate and power in many cases.

The structure of this paper is as follows. Section 2 reviews existing tests, develops the proposed test statistic, and derives its asymptotic chi-square distribution. Section 3 outlines the design of the simulation study. Section 4 presents the results and compares the proposed test with existing methods. Section 5 demonstrates the application of the proposed test to gene expression datasets. Section 6 provides the conclusions.

2 Materials and Methods

This section presents test statistics proposed by, [16], [18] and [22], followed by the introduction of the proposed test and its asymptotic distribution.

2.1 Test Statistic Proposed in [16]

In 2007, [16] introduced a test statistic, denoted T_{sc}, as expressed in Equation (2):

$$T_{sc} = \sum_{i<j}^{g} \frac{tr\left(\mathbf{S}_i - \mathbf{S}_j\right)^2}{\hat{\theta}}, \tag{2}$$

where $\mathbf{S}_i$ is the i-th sample covariance matrix computed by $\mathbf{S}_i = \mathbf{V}_i / n_i,\ n_i = N_i - 1$,

$$\mathbf{V}_i = \sum_{j=1}^{N_i}\left(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i\right)\left(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i\right)^T, \quad \bar{\mathbf{X}}_i = \frac{1}{N_i}\sum_{j=1}^{N_i}\mathbf{X}_{ij},$$

$$\hat{\theta}^2 = 2\hat{a}_2\left\{\sum_{i<j}^{g}\left(\hat{c}_i + \hat{c}_j\right)^2 + (g-1)(g-2)\sum_{i<j}^{g}\hat{c}_i^2\right\},$$

$$\hat{a}_2 = \frac{1}{(n-1)(n+2)p}\left\{tr\mathbf{V}^2 - \frac{1}{n}\left(tr\mathbf{V}\right)^2\right\},$$

$\mathbf{V} = \sum_{i=1}^{g}\mathbf{V}_i$, $\hat{c}_i = p/n_i$, and $n = \sum_{i=1}^{g} n_i$. Under H_0, T_{sc} is distributed as the standard normal distribution as $(p, N_i) \to \infty$. The null hypothesis is rejected when $T_{sc} > z_\alpha$, where z_α is the upper α quantile of the standard normal distribution.

2.2 Test Statistic Proposed in [18]

In 2017, [18], suggested an alternative test statistic denoted by T_{am} as expressed in Equation (3):

$$T_{am} = \frac{p^2 T_1}{2\sqrt{\gamma\left[tr(\mathbf{\Sigma}^2)\right]^2}}, \tag{3}$$

where $T_1 = \dfrac{tr(\mathbf{\Sigma}^2)\left\lfloor (g-1)\sum_{i=1}^{g} B_i - 2\sum_{i\neq j}^{g} B_{ij}\right\rfloor}{p^2 tr(\mathbf{\Sigma}^2)}$,

$$B_i = \frac{1}{N_i\left(N_i - 1\right)}\sum_{k\neq r}^{N_i} A_{ikr}^2, \quad B_{ij} = \frac{1}{N_i N_j}\sum_{i=1}^{N_i}\sum_{j=1}^{N_j} A_{ijkl}^2,$$

$$A_{ikr} = \mathbf{X}_{ik}^T\mathbf{X}_{ir}, \quad A_{ijkl} = \mathbf{X}_{ik}^T\mathbf{X}_{jl},$$

and $\gamma = (g-1)^2\sum_{i=1}^{g}\frac{1}{N_i^2} + \sum_{i<j}^{g}\frac{2}{N_i N_j}$. Under H_0, T_{am} is distributed as the standard normal distribution as $(p, N_i) \to \infty$. The null hypothesis is rejected when $T_{am} > z_\alpha$.

2.3 Test Statistic Proposed in [22]

In 2022, [22], proposed a test statistic denoted by T_{stc} as expressed in Equation (4):

$$T_{stc} = \frac{T_2}{\hat{\sigma}_0}, \tag{4}$$

where $T_2 = \sum_{i=1}^{g}\frac{N_i\left(N - N_i\right)}{N}E_i - \sum_{i\neq j}^{g}\frac{N_i N_j}{N}G_{ij}$,

$$E_i = \frac{1}{\left(N_i\right)_2}\sum_{j,l}^{*}\left(\mathbf{X}_{ij}^T\mathbf{X}_{il}\right)^2 - \frac{2}{\left(N_i\right)_3}\sum_{j,l,f}^{*}\left(\mathbf{X}_{ij}^T\mathbf{X}_{il}\mathbf{X}_{ij}^T\mathbf{X}_{if}\right) + \frac{1}{\left(N_i\right)_4}\sum_{j,l,f,g}^{*}\left(\mathbf{X}_{ij}^T\mathbf{X}_{il}\mathbf{X}_{if}^T\mathbf{X}_{im}\right),$$

$$G_{ij} = \frac{1}{N_i N_j}\sum_{l=1}^{N_i}\sum_{m=1}^{N_j}\left(\mathbf{X}_{il}^T\mathbf{X}_{jm}\right)^2$$

$$-\frac{1}{N_i\left(N_j\right)_2}\sum_{l=1}^{N_i}\sum_{f,m}^{*}\left(\mathbf{X}_{il}^T\mathbf{X}_{jf}\mathbf{X}_{il}^T\mathbf{X}_{jm}\right)$$
$$-\frac{1}{\left(N_i\right)_2 N_j}\sum_{l=1}^{N_j}\sum_{f,m}^{*}\left(\mathbf{X}_{jl}^T\mathbf{X}_{if}\mathbf{X}_{jl}^T\mathbf{X}_{im}\right)$$
$$+\frac{1}{\left(N_i\right)_2\left(N_j\right)_2}\sum_{l,m}^{*}\sum_{f,g}^{*}\left(\mathbf{X}_{il}^T\mathbf{X}_{jf}\mathbf{X}_{im}^T\mathbf{X}_{jg}\right),$$ and
$$\hat{\sigma}_0^2=\frac{4}{N^2}\left\{\sum_{i=1}^{g}\frac{N_i\left(N-N_i\right)^2}{\left(N_i-1\right)}E_i^2+\sum_{i\neq j}^{g}2N_iN_jG_{ij}^2\right\},$$ here $\left(N_i\right)_l=N_i!/\left(N_i-l\right)!$, $N=\sum_{i=1}^{g}N_i$, and $\overset{*}{\Sigma}$ denotes the sum for all different indices. Under H_0, T_{stc} is distributed as the standard normal distribution as min $\{p,N_i\}\to\infty$. The null hypothesis is rejected when $T_{stc}>z_\alpha$.

2.4 The Proposed Test

Under $H_0:\mathbf{\Sigma}_1=\mathbf{\Sigma}_2=\cdots=\mathbf{\Sigma}_g=\mathbf{\Sigma}$, the covariance structure is identical across populations, whereas under H_1, at least one population differs. To quantify potential deviations, we propose a test based on a parameter function ψ_i that measures the relative trace ratio of the covariance matrix $\mathbf{\Sigma}_i$ as in Equation (5):

$$\psi_i=\psi_i\left(\mathbf{\Sigma}_i\right)=\frac{ptr\mathbf{\Sigma}_i^2}{\left(tr\mathbf{\Sigma}_i\right)^2},\ i=1,2,\ldots,g. \tag{5}$$

For each group, let $A_{ik}=\mathbf{X}_{ik}^T\mathbf{X}_{ik}$, be a quadratic form and $A_{ikl}=\mathbf{X}_{ik}^T\mathbf{X}_{il}$, $k\neq l$, a symmetric bilinear form. Based on these definitions, we construct the following U-statistics:

$$U_{1i}=\frac{1}{N_i\left(N_i-1\right)}\sum_{k\neq l}^{N_i}A_{ikl}^2,$$
$$U_{2i}=\frac{1}{N_i\left(N_i-1\right)}\sum_{k\neq l}^{N_i}A_{ik}A_{il},$$

which are unbiased and consistent estimators of $tr\mathbf{\Sigma}_i^2$ and $(tr\mathbf{\Sigma}_i)^2$, respectively, [24]. The parameter function ψ_i is estimated using the corresponding consistent estimators $\hat{\psi}_i$, defined by Equation (6):

$$\hat{\psi}_i=\frac{pU_{2i}}{U_{1i}},\quad i=1,2,\ldots,g. \tag{6}$$

This construction ensures that each ψ_i is estimated without bias and remains consistent even when p rapidly grows beyond N_i, following the asymptotic conditions established in [23].

Furthermore, we define that

$$a_{ri}=(1/p)tr\mathbf{\Sigma}_i^r,\ r=1,\ldots,4,$$
$$\hat{a}_{1i}=(1/p)tr\mathbf{S}_i, \tag{7}$$
$$\hat{a}_{2i}=\frac{n_i^2}{p(n_i-1)(n_i+2)}\left\{tr\mathbf{S}_i^2-\frac{1}{n_i}(tr\mathbf{S}_i)^2\right\}, \tag{8}$$

and $$\hat{a}_{4i}=\frac{1}{d_{0i}}\left(\frac{1}{p}tr\mathbf{V}_i^4-e_i\right), \tag{9}$$

where $d_{0i}=n_i(n_i^3+6n_i^2+21n_i+18)$,
$d_{1i}=2n_i(2n_i^2+6n_i+9)$, $d_{2i}=2n_i(3n_i+2)$,
$d_{3i}=n_i(2n_i^2+5n_i+7)$,
and $e_i=pd_{1i}\hat{a}_{1i}+p^2d_{2i}\hat{a}_{1i}^2\hat{a}_{2i}+pd_{3i}\hat{a}_{2i}^2+n_ip^3\hat{a}_{1i}^4$.

[25] demonstrated that $\hat{a}_{1i}$ and $\hat{a}_{2i}$ are unbiased and consistent estimators of a_{1i} and a_{2i}, respectively. Furthermore, [17] affirmed that $\hat{a}_{4i}$ is a consistent estimator of a_{4i}.

We now state the following assumptions as presented in [23].

(A1) For $p\to\infty$, let $(1/p)tr\mathbf{\Sigma}_i=O(1)$,
(A2) $tr\mathbf{\Sigma}_i^2/p^2=\delta_i$, where $0<\delta_i\leq\infty$,
(A3) For $\left(p,N_i\right)\to\infty$, $p/N_i\to c_i$, $c_i\in(0,\infty)$,
(A4) $\inf_p(tr\mathbf{\Sigma}_i^4/(tr\mathbf{\Sigma}_i^2)^2)>0$,
for $i=1,2,..,g$.

The subsequent lemma, of which the asymptotic normality of the consistent estimator $\hat{\psi}_i$ of ψ_i is presented in [23], provides the foundation for our proposed test.

Lemma 1 Let ψ_i and $\hat{\psi}_i$ be as defined in (5) and (6), and $a_{ri}=(1/p)tr\mathbf{\Sigma}_i^r$, $r=1,\ldots,4$, for $i=1,2,..,g$. Under Assumptions (A1) – (A4), then

$$\hat{\psi}_i\xrightarrow{d}N\left(\psi_i,\beta_i^2\right),$$

as $\left(p,N_i\right)\to\infty$, where $\beta_i^2=f_i\left[\left(2N_i-1\right)m_{2i}+1\right]$,

$f_i=\frac{4m_{1i}}{N_i\left(N_i-1\right)}$, $m_{1i}=a_{2i}^2/a_{1i}^4$, and $m_{2i}=a_{4i}/pa_{2i}^2$.

The symbol "$\xrightarrow{d}$" stands for "converges in distribution."

Through the application of Slutsky's theorem to Lemma 1, we derive the asymptotic normality of $\hat{\psi}_i$, as stated in Lemma 2.

Lemma 2 Under Assumptions (A1) – (A4), then

$$\sqrt{N_i}\left(\hat{\psi}_i-\psi_i\right)\xrightarrow{d}N\left(0,\beta_i^2\right),$$

as $(p, N_i) \to \infty$, where $\beta_i^2 = f_i\left[(2N_i - 1)m_{2i} + 1\right]$, $f_i = \dfrac{4m_{1i}}{N_i(N_i - 1)}$, $m_{1i} = a_{2i}^2 / a_{1i}^4$, and $m_{2i} = a_{4i} / pa_{2i}^2$.

Within the framework of the present research, under $H_0 : \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \cdots = \mathbf{\Sigma}_g = \mathbf{\Sigma}$ all populations are assumed to share an identical covariance structure, which implies that $\psi_1(\mathbf{\Sigma}_1) = \psi_2(\mathbf{\Sigma}_2) = \ldots = \psi_g(\mathbf{\Sigma}_g) = \psi$. To quantify the degree of deviation of each population parameter ψ_i from the common mean ψ, a function T is introduced, representing a novel approach compared with conventional measures of dispersion. This function assesses the variability of the population parameters relative to their common mean ψ, with each component inversely weighted by its corresponding asymptotic variance β_i^2, accordingly, the function T is defined as follows:

$$T = \sum_{i=1}^{g} \frac{(\psi_i - \psi)^2}{\beta_i^2},$$

serving as a population-level measure of heterogeneity among parameter functions.

From Lemma 1, it follows that under the null hypothesis, the consistent estimator $\hat{\psi}_i$ is asymptotically normal with mean ψ_i and variance β_i^2. Therefore, we use the consistent estimator $\hat{\psi}_i$ and estimate the common mean ψ by a weighted mean $\left(\bar{\hat{\psi}}^*\right)$ of $\hat{\psi}_1, \ldots, \hat{\psi}_g$, weighting $\hat{\psi}_i$ with the weights inversely proportional to their variances, β_i^2, that is, $\bar{\hat{\psi}}^* = \dfrac{\sum_{i=1}^{g} \hat{\psi}_i / \beta_i^2}{\sum_{i=1}^{g} 1/\beta_i^2} = \dfrac{\mathbf{1}^T \mathbf{D}\hat{\boldsymbol{\psi}}}{\mathbf{1}^T \mathbf{D}\mathbf{1}}$,

where $\mathbf{1} = (1, \ldots, 1)^T$, a $g \times 1$ vector of ones, $\mathbf{D} = diag\left(1/\beta_1^2, \ldots, 1/\beta_g^2\right)$, and $\hat{\boldsymbol{\psi}} = \left(\hat{\psi}_1, \ldots, \hat{\psi}_g\right)^T$.

Moreover, the variance β_i^2 can be estimated by plugging in the relevant consistent estimators within the framework of β_i^2 which are a_{1i}, a_{2i}, and a_{4i}, with the consistent estimators $\hat{a}_{1i}$, $\hat{a}_{2i}$, and $\hat{a}_{4i}$, outlined in Equation (7), Equation (8), and Equation (9), respectively. Consequently, based on $\hat{m}_{1i} = \hat{a}_{2i}^2 / \hat{a}_{1i}^4$ and $\hat{m}_{2i} = \hat{a}_{4i} / p\hat{a}_{2i}^2$, we have:

$$\bar{\hat{\psi}} = \frac{\sum_{i=1}^{g} \hat{\psi}_i / \hat{\beta}_i^2}{\sum_{i=1}^{g} 1/\hat{\beta}_i^2},$$

where $\hat{\beta}_i^2 = \hat{f}_i\left[(2N_i - 1)\hat{m}_{2i} + 1\right]$, $\hat{f}_i = \dfrac{4\hat{m}_{1i}}{N_i(N_i - 1)}$, $\hat{m}_{1i} = \hat{a}_{2i}^2 / \hat{a}_{1i}^4$, and $\hat{m}_{2i} = \hat{a}_{4i} / p\hat{a}_{2i}^2$.

Therefore, the proposed test statistic under the null hypothesis H_0 is formally expressed in Equation (10):

$$T_{new} = \sum_{i=1}^{g} \left(\frac{\hat{\psi}_i - \bar{\hat{\psi}}}{\hat{\beta}_i} \right)^2. \tag{10}$$

The test statistic quantifies the dispersion of group-specific estimators relative to their collective mean under the null hypothesis, weighted by each group's inverse variance.

In the following theorem, we establish the asymptotic distribution of T_{new}.

Theorem 1 Under Assumptions (A1) – (A4), then $T_{new} \xrightarrow{d} \chi_{g-1}^2$, as $(p, N_i) \to \infty$.

Proof: By Lemma 2 and the assumption of group independence, we have

$$\mathbf{D}^{1/2}\left(\hat{\boldsymbol{\psi}} - \psi\mathbf{1}\right) \xrightarrow{d} N_g\left(\mathbf{0}, \mathbf{I}_g\right) \text{ as } (p, N_i) \to \infty,$$

under the null hypothesis, where $\mathbf{0} = (0, \ldots, 0)^T$, represents a $g \times 1$ vector of zeros. Note that, $Cov\left(\hat{\boldsymbol{\psi}}\right) = \mathbf{D}^{-1}$, moreover, it follows that

$$\left(\mathbf{I}_g - \frac{\mathbf{1}\mathbf{1}^T\mathbf{D}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right)^T \mathbf{D} \left(\mathbf{I}_g - \frac{\mathbf{1}\mathbf{1}^T\mathbf{D}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right) = \mathbf{D} - \frac{\mathbf{D}\mathbf{1}\mathbf{1}^T\mathbf{D}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}.$$

Thus, we obtain

$$\begin{aligned}
T_{new} = \sum_{i=1}^{g} \left(\frac{\hat{\psi}_i - \bar{\hat{\psi}}}{\hat{\beta}_i} \right)^2 &\xrightarrow{p} \sum_{i=1}^{g} \left(\frac{\hat{\psi}_i - \bar{\hat{\psi}}^*}{\beta_i} \right)^2 \\
&= \left(\hat{\boldsymbol{\psi}} - \bar{\hat{\psi}}\mathbf{1}\right)^T \mathbf{D} \left(\hat{\boldsymbol{\psi}} - \bar{\hat{\psi}}\mathbf{1}\right) \\
&= \left(\hat{\boldsymbol{\psi}} - \frac{\mathbf{1}\mathbf{1}^T\mathbf{D}\hat{\boldsymbol{\psi}}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right)^T \mathbf{D} \left(\hat{\boldsymbol{\psi}} - \frac{\mathbf{1}\mathbf{1}^T\mathbf{D}\hat{\boldsymbol{\psi}}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right) \\
&= \hat{\boldsymbol{\psi}}^T \left(\mathbf{D} - \frac{\mathbf{D}\mathbf{1}\mathbf{1}^T\mathbf{D}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right) \hat{\boldsymbol{\psi}} \\
&= \hat{\boldsymbol{\psi}}^T \mathbf{D}^{1/2} \left(\mathbf{I}_g - \frac{\mathbf{D}^{1/2}\mathbf{1}\mathbf{1}^T\mathbf{D}^{1/2}}{\mathbf{1}^T\mathbf{D}\mathbf{1}}\right)^T \mathbf{D}^{1/2}\hat{\boldsymbol{\psi}}.
\end{aligned}$$

Hence, T_{new} is asymptotically distributed as chi-square with $g-1$ degrees of freedom, since

$\mathbf{I}_g - \left(\mathbf{D}^{1/2}\mathbf{1}\mathbf{1}^T\mathbf{D}^{1/2}/\mathbf{1}^T\mathbf{D}\mathbf{1}\right)$ is an idempotent matrix of rank $g-1$, [1]. This completes the proof.

3 Simulation Study

To evaluate the efficiency of T_{new} (Equation 10) and compare it with three existing statistics—T_{sc}, T_{am}, and T_{stc} (Equations 2–4)—we conducted a simulation study in Python 3.10. Independent p-variate normal samples were generated for three and four populations (g = 3, 4). All population mean vectors are set to zero. We set all sample sizes N_i to be equal and varied them across four levels: 20 (small), 40 (moderate), 80 (large), and 160 (very large), and $p \in \{20, 40, 80, 160, 320\}$, ensured $p \geq N_i$ (i=1,2,3,4).

To evaluate the Type I error rate and power, covariance matrices were specified under the following three scenarios.

Case 1: First-order autoregressive (AR) structure:

$$H_0^1 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_g = \mathbf{A}_0,$$

$$H_1^1 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_{g-1} = \mathbf{A}_0,\ \Sigma_g = \mathbf{A}_1,$$

where

$$\mathbf{A}_0 = \left(\sigma_{ij}\right)_{p\times p} = \left\{1 \text{ if } |i-j| = 0;\ 0.7^{|i-j|} \text{ if } |i-j| \neq 0\right\},$$

and

$$\mathbf{A}_1 = \left(\sigma_{ij}\right)_{p\times p} = \left\{1 \text{ if } |i-j| = 0;\ 0.9^{|i-j|} \text{ if } |i-j| \neq 0\right\}.$$

Case 2: Compound symmetry (CS) structure:

$$H_0^2 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_g = \mathbf{C}_0 = 0.99\mathbf{I}_p + 0.01\mathbf{1}\mathbf{1}^T,$$

$$H_1^2 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_{g-1} = \mathbf{C}_0,\ \mathbf{\Sigma}_g = \mathbf{C}_1,$$

where $\mathbf{C}_1 = 0.9\mathbf{I}_p + 0.1\mathbf{1}\mathbf{1}^T$, $\mathbf{I}_p$ is the $p\times p$ identity matrix, and $\mathbf{1} = (1,...,1)^T$ is a $p\times 1$ vector of ones.

Case 3: Variance components (VC) structure:

$$H_0^3 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_g = \mathbf{V}_0,$$

$$H_1^3 : \mathbf{\Sigma}_1 = \ldots = \mathbf{\Sigma}_{g-1} = \mathbf{V}_0,\ \mathbf{\Sigma}_g = \mathbf{V}_1,$$

where $\mathbf{V}_0 = diag(\lambda_1,...,\lambda_p)$, $\mathbf{V}_1 = diag(d_1,...,d_p)$, with $\lambda_i \overset{iid}{\sim} Unif(1,2)$, and $d_i \overset{iid}{\sim} Unif(3,4)$, $i = 1,..., p$.

Empirical Type I error rate and empirical power were computed at a nominal significance level of 0.05, using 5,000 replications as described below.

Empirical Type I error rate was calculated as the number of replications in which H_0 was rejected when it was true, divided by 5,000.

Empirical power was calculated as the number of replications in which H_0 was rejected when H_1 was true, divided by 5,000.

To gauge the overall performance, we used the average relative error (ARE) metric, [26], [27]. It is calculated as follows:

$$ARE = 100M^{-1}\sum_{j=1}^{M}\left|\hat{\alpha}_j - \alpha\right| / \alpha,$$

where $\hat{\alpha}_j$ denotes the empirical Type I error rate under M simulation settings. A test statistic with the smallest ARE value best controls the Type I error rate. All simulation results are reported in the Appendix. Furthermore, an analysis of variance (ANOVA) was conducted to examine statistical differences in the empirical Type I error rates across the test statistics.

4 Simulation Results

The empirical Type I error rates and empirical powers of T_{new} and three existing tests namely T_{sc} [16], T_{am}, [18], and T_{stc}, [22] are summarized in Appendix in Table 1, Table 2, Table 3, Table 4, Table 5 and Table 6 and illustrated in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6.

Even when the covariance structure changes among the three settings AR, CS, and VC the tests' ability to control the Type I error rate remains consistent. Their characteristics and directional behavior do not change significantly.

Across all covariance structures, the new test T_{new} yields empirical Type I error rates that closely align with the nominal significance level of 0.05 for the AR, CS, and VC structures, with observed rates ranging from 0.0442 to 0.0660. These findings indicate minimal deviation from the expected significance level. T_{am} controls the Type I error rate reasonably well; in general, they range between 0.0415 to 0.0679. In comparison, T_{sc} yields higher empirical Type I error rates, ranging from 0.0810 and 0.1396, suggesting that it is too liberal, whereas T_{stc} yields empirical Type I error rates ranging from 0.0249 to 0.0580 and is more conservative, as it often produces empirical Type I error rates below 0.04. For instance, under the AR structure with $g = 3, N_i = 20$, $p = 160$ (Table 1, Appendix), empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} are 0.0941, 0.0667, 0.0396, and 0.0545, respectively. This suggests that T_{new} controls the Type I error rate most accurately, while the other tests either reject the null hypothesis too often or not enough. Similar patterns appear for the CS and VC structures, indicating that T_{new} remains consistent across the different covariance matrix structures considered. This demonstrates excellent Type I error rate control and the most accurate performance compared with these three tests.

With respect to the ARE values, T_{new} demonstrates the most favorable results, ranging from 7.47 to 11.23, indicating excellent control of the Type I error rate and high stability. In contrast, T_{sc} is markedly liberal, exhibiting inflated empirical Type I error rates and the highest ARE values (83.04–156.86). The T_{stc} test is more conservative, with ARE values ranging from 17.48 to 31.31. Meanwhile, T_{am} controls the Type I error rate reasonably well, with ARE values between 16.00 and 28.59; however, it tends to produce higher empirical Type I error rates when $N_i \leq 40$ and $p \leq 160$. ANOVA on the empirical Type I error rates was used to evaluate methodological differences in test performance. The resulting F-statistic of 756.16 across all design configurations confirms that T_{new} significantly outperforms the alternative tests.

When comparing their empirical powers, it is essential to consider only those statistics that adequately control the Type I error rate, such as T_{am}, T_{stc}, and T_{new}. Therefore, the empirical power of T_{sc} is not reported, as it would lead to misleading conclusions. All test statistics— T_{am}, T_{stc}, and T_{new} —demonstrate strong ability to detect differences among covariance matrices under the alternative hypothesis, with empirical powers approaching 1 as the number of variables and the sample size increase, albeit at different rates. The tests T_{am} and T_{stc} display similar empirical power patterns across most scenarios and yield higher powers than T_{new} in Cases 1 (AR) and 3 (VC). The T_{new} shows higher empirical powers in Case 2 (CS) and converges to 1 faster than the others. The results for the three-group (g = 3) and four-group (g = 4) scenarios are directionally consistent—T_{new} maintains precise Type I error rate control, minimal ARE, and high empirical power across all covariance structures, confirming its stability regardless of the number of groups.

5 Application to Real Data

This section applies the proposed test statistic T_{new} and compares it with alternative statistics T_{sc}, T_{am}, and T_{stc} to analyze two real gene expression datasets from the NCBI Gene Expression Omnibus (accession codes GSE4290 and GSE57275). The objective is to examine whether the groups share similar covariance structures. Because the datasets include a large number of genes without prior preprocessing, a gene-screening procedure was implemented. Genes whose coefficients of variation lay outside the range (0.25, 0.50) were excluded from further analysis.

The first dataset (GSE4290) consists of glioma gene expression profiles from 180 observations measured across 54,613 genes (variables), classified into four diagnostic categories: oligodendroglioma $(N_1 = 50)$, glioblastoma $(N_2 = 81)$, astrocytoma $(N_3 = 26)$, and non-tumor $(N_4 = 23)$. Following the gene-screening step, 24,785 genes were retained. The resulting test statistics and their corresponding p-values were as follows: $T_{am} = 2.0469$ (0.0203), $T_{stc} = 29.3143$ (<0.0001), and $T_{new} = 6.8653$ (0.0325). At the nominal significance level of 0.05, all three tests indicate significant differences among at least two population covariance matrices.

The second dataset (GSE57275) contains 14 observations and 45,281 genes, divided into three groups: a controlled group $(N_1 = 5)$, an infected group $(N_2 = 4)$, and an infected-medication group $(N_3 = 5)$. After excluding genes with coefficients of variation outside the range (0.25, 0.50), 3,424 genes remained. The corresponding test statistics and p-values were: $T_{am} = 1.0392$ (0.1494), $T_{stc} = 3.2541$ (0.0006), and $T_{new} = 7.38$ (0.0257). At the 5% significance level, both T_{stc} and T_{new} yield statistically significant results, indicating evidence against the equality of covariance matrices across the three groups.

6 Conclusion

This study proposes a novel test, denoted T_{new}, to assess the homogeneity of covariance matrices across g high-dimensional populations. T_{new} is novel in that it improves upon U-statistic-based estimators by integrating an inverse-variance-weighted framework that increases sensitivity to variation across populations. The asymptotic distribution of the test is derived under the assumption that the trace ratios of the covariance matrices remain bounded up to only fourth order, a relatively mild condition applicable to real-world data. As both the dimension and sample sizes increase, the proposed test follows the chi-square distribution with $g-1$ degrees of freedom. In terms of computational complexity, the proposed T_{new} is based on U-statistics that require pairwise inner-product computations of p-dimensional observations, resulting in approximately $O\left(p^2\right)$ operations per group. In contrast, the LRT requires matrix inversion, incurring a higher computational cost of $O\left(p^3\right)$. Therefore, the proposed test is more computationally efficient and scalable in high-dimensional settings.

The performance of the proposed test was evaluated through extensive simulation studies. Its empirical Type I error rate and statistical power were compared with three established methods: T_{sc} [16], T_{am} [18], and T_{stc} [22]. T_{new} consistently achieves superior control of the Type I error rate. The ARE values further support this finding: T_{new} has the lowest ARE, followed by T_{am}, T_{stc}, and T_{sc}. This pattern remains consistently across all covariance structures (AR, CS, and VC) and both group configurations (g = 3 and g = 4), confirming the robustness and reliability of T_{new} in maintaining precise Type I error rates and strong statistical power across diverse high-dimensional scenarios.

Future work could extend the proposed test to non-normally distributed data, which can capture asymmetry and heavy tails, such as the beta, Weibull, or G-Bell distributions, [28]. It could provide a test of covariance homogeneity, broadening the applicability of the proposed test to diverse environmental and biological datasets.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The author wrote, reviewed and edited the content as needed and verifies that none utilized artificial intelligence (AI) tools were used. The author takes full responsibility for the content of the publication.

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APPENDIX

Table 1. Empirical Type I error rates under H_0^1 and empirical powers under H_1^1 for the AR structure with g=3

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.1172	0.0905	0.0632	0.0608	-	0.7027	0.6229	0.6090
	40	0.1008	0.0729	0.0493	0.0614	-	0.8391	0.7750	0.7619
	80	0.1028	0.0735	0.0485	0.0562	-	0.9163	0.8680	0.8284
	160	0.0941	0.0667	0.0396	0.0545	-	0.9476	0.9122	0.8312
	320	0.0917	0.0620	0.0348	0.0568	-	0.9700	0.9435	0.8540
40	40	0.1069	0.0725	0.0463	0.0546	-	0.9852	0.9797	0.9686
	80	0.0942	0.0600	0.0362	0.0517	-	0.9892	0.9886	0.9827
	160	0.0972	0.0663	0.0404	0.0556	-	1	1	0.9843
	320	0.0909	0.0546	0.0325	0.0552	-	1	1	1
80	80	0.0968	0.0600	0.0382	0.0576	-	1	1	1
	160	0.0980	0.0588	0.0352	0.0560	-	1	1	1
	320	0.0832	0.0529	0.0329	0.0519	-	1	1	1
160	160	0.0929	0.0545	0.0362	0.0535	-	1	1	1
	320	0.0845	0.0451	0.0323	0.0483	-	1	1	1
ARE		93.03	28.59	22.97	11.07				

Note: "-" means do not report the power of a test that fails to control the Type I error rate
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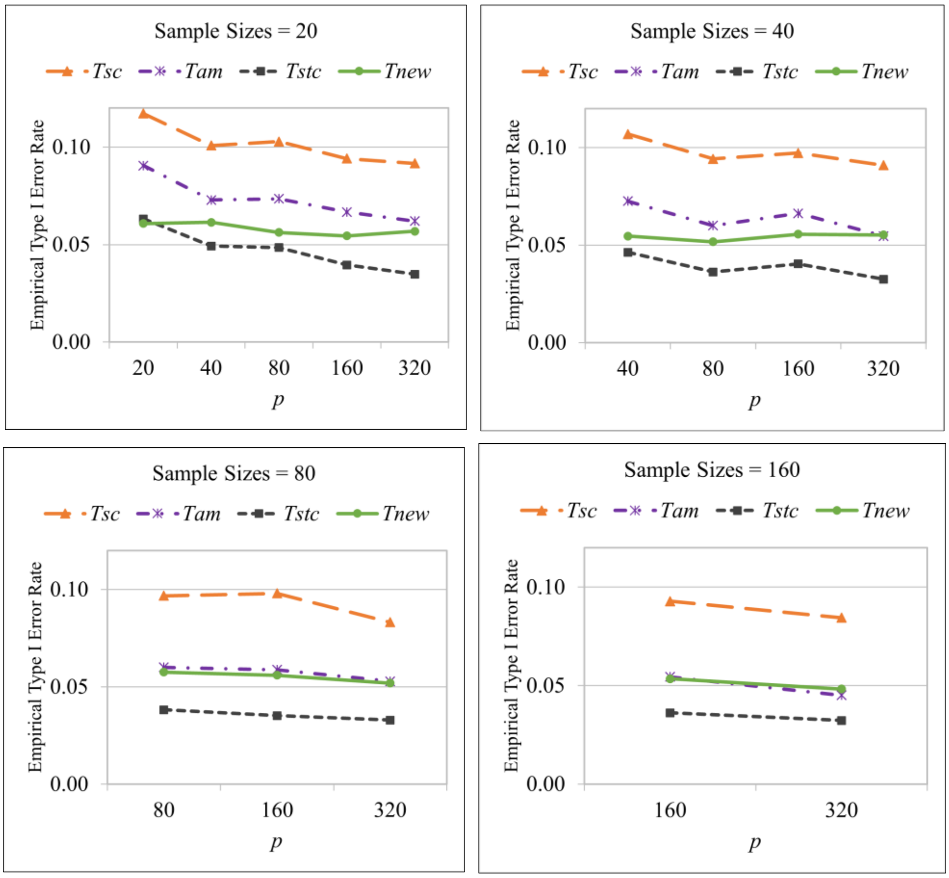

Fig. 1: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^1 the AR structure with g=3, across sample sizes of 20, 40, 80, and 160.
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Table 2. Empirical Type I error rates under H_0^1 and empirical powers under H_1^1 for the AR structure with g=4

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.1396	0.0818	0.0580	0.0654	-	0.7098	0.6292	0.6152
	40	0.1295	0.0655	0.0469	0.0562	-	0.8476	0.7828	0.7696
	80	0.1283	0.0687	0.0473	0.0536	-	0.9256	0.8768	0.8368
	160	0.1362	0.0644	0.0436	0.0518	-	0.9572	0.9214	0.8396
	320	0.1206	0.0546	0.0376	0.0538	-	0.9798	0.953	0.8626
40	40	0.1380	0.0667	0.0485	0.0604	-	0.9952	0.9896	0.9784
	80	0.1356	0.0667	0.0459	0.0532	-	0.9992	0.9986	0.9826
	160	0.1309	0.0671	0.0491	0.0516	-	1	1	0.9942
	320	0.1230	0.0519	0.0335	0.0576	-	1	1	1
80	80	0.1303	0.0592	0.0418	0.0568	-	1	1	1
	160	0.1174	0.0537	0.0402	0.0538	-	1	1	1
	320	0.1174	0.0493	0.0333	0.0442	-	1	1	1
160	160	0.1251	0.0590	0.0340	0.0538	-	1	1	1
	320	0.1261	0.0505	0.0341	0.0548	-	1	1	1
ARE		156.86	22.93	17.48	11.23				

Note: "-" means do not report the power of a test that fails to control the Type I error rate
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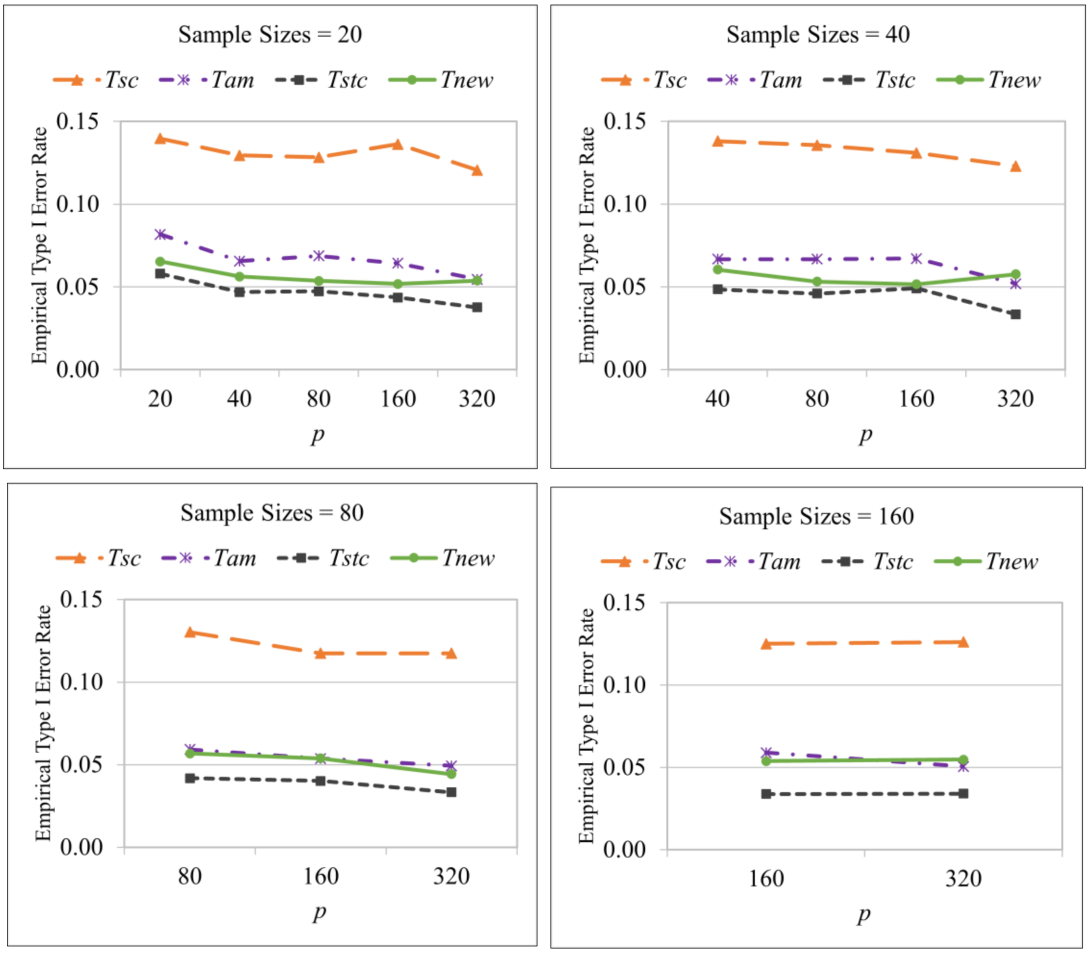

Fig. 2: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^1 the AR structure with g=4, across sample sizes of 20, 40, 80, and 160.
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Table 3. Empirical Type I error rate under H_0^2 and empirical power under H_1^2 for the CS structure with g=3

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.0913	0.0647	0.0402	0.0604	-	0.3174	0.2439	0.6124
	40	0.0903	0.0606	0.0388	0.0556	-	0.5643	0.4879	0.7316
	80	0.0913	0.0638	0.0382	0.0541	-	0.8150	0.7506	0.8805
	160	0.0891	0.0608	0.0345	0.0527	-	0.8985	0.8609	0.9872
	320	0.0853	0.0545	0.0339	0.0574	-	0.9724	0.9633	0.9988
40	40	0.0978	0.0675	0.0348	0.0564	-	0.8777	0.8360	1
	80	0.0899	0.0535	0.0348	0.0533	-	0.9795	0.9753	1
	160	0.0919	0.0604	0.0360	0.0566	-	0.9884	0.9872	1
	320	0.0899	0.0543	0.0345	0.0527	-	0.9898	0.9894	1
80	80	0.0966	0.0543	0.0323	0.0543	-	1	1	1
	160	0.0960	0.0584	0.0337	0.0550	-	1	1	1
	320	0.0895	0.0485	0.0281	0.0537	-	1	1	1
160	160	0.0931	0.0576	0.0323	0.0527	-	1	1	1
	320	0.0893	0.0501	0.0301	0.0507	-	1	1	1
ARE		83.04	16.00	31.11	9.37				

Note: "-" means do not report the power of a test that fails to control the Type I error rate

Source: Created by the author

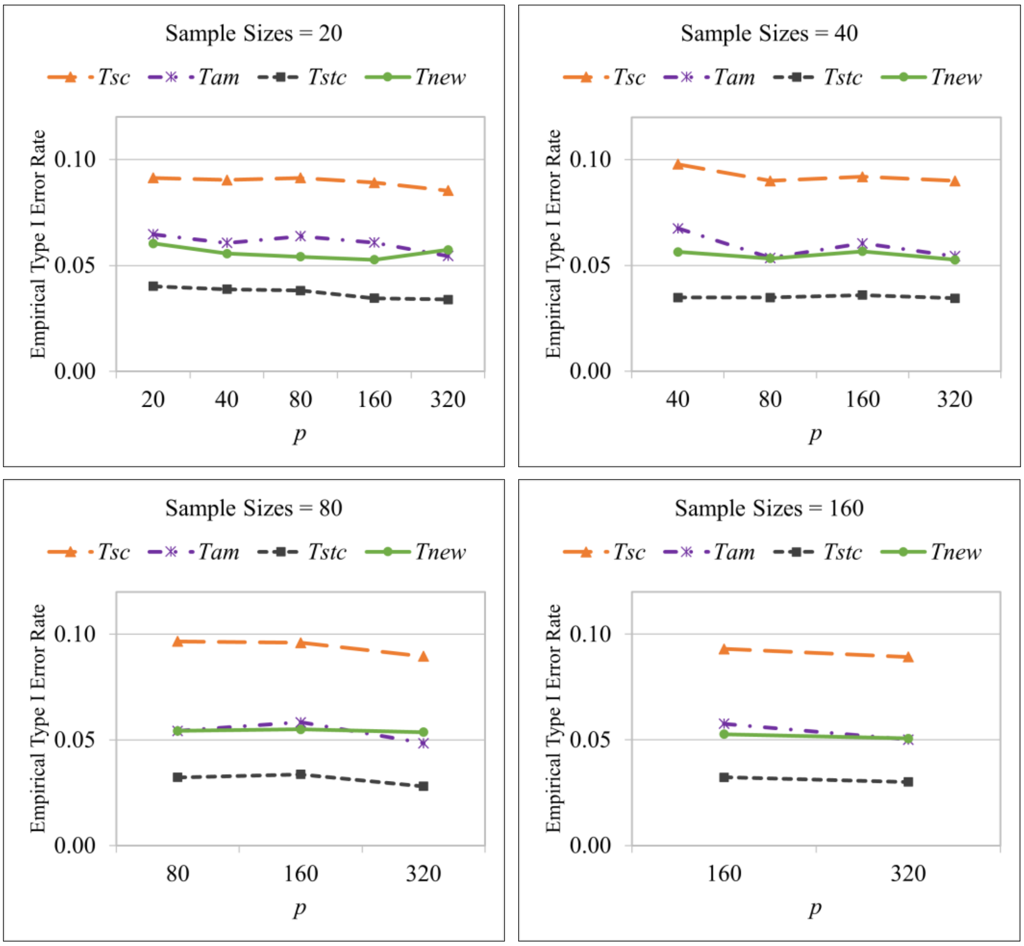

Fig. 3: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^2 the CS structure with g=3, across sample sizes of 20, 40, 80, and 160

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Table 4. Empirical Type I error rate under H_0^2 and empirical power under H_1^2 for the CS structure with g=4

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.1172	0.0598	0.0420	0.0636	-	0.6940	0.6283	0.8240
	40	0.1226	0.0679	0.0432	0.0516	-	0.8338	0.7750	0.8762
	80	0.1344	0.0689	0.0501	0.0532	-	0.9161	0.8734	0.9400
	160	0.1148	0.0600	0.0400	0.0518	-	0.9496	0.9118	0.9606
	320	0.1117	0.0535	0.0358	0.0562	-	0.9712	0.9431	0.9994
40	40	0.1342	0.0586	0.0402	0.0630	-	0.9825	0.9791	1
	80	0.1370	0.0632	0.0432	0.0568	-	0.9996	0.9990	1
	160	0.1160	0.0523	0.0386	0.0520	-	1	1	1
	320	0.1243	0.0550	0.0382	0.0526	-	1	1	1
80	80	0.1162	0.0517	0.0323	0.0536	-	1	1	1
	160	0.1255	0.0545	0.0352	0.0558	-	1	1	1
	320	0.1186	0.0509	0.0354	0.0508	-	1	1	1
160	160	0.1251	0.0545	0.0366	0.0524	-	1	1	1
	320	0.1267	0.0570	0.0378	0.0522	-	1	1	1
ARE		146.37	15.38	21.65	9.37				

Note: "-" means do not report the power of a test that fails to control the Type I error rate

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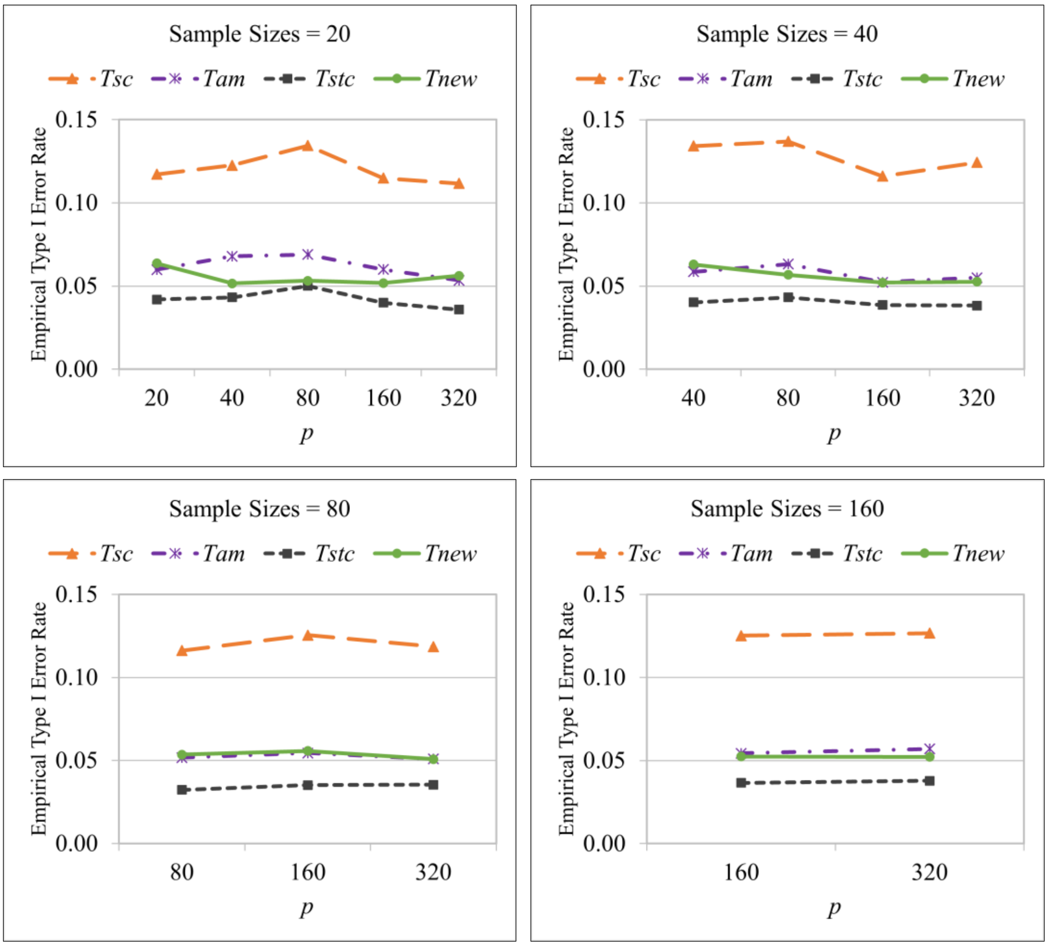

Fig. 4: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^2 the CS structure with g=4, across sample sizes of 20, 40, 80, and 160

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Table 5. Empirical Type I error rates under H_0^3 and empirical powers under H_1^3 for the VC structure with g=3

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.0909	0.0663	0.0396	0.0618	-	0.9833	0.9603	0.5745
	40	0.0942	0.0653	0.0386	0.0547	-	0.9809	0.9589	0.5525
	80	0.1000	0.0687	0.0412	0.0520	-	0.9860	0.9684	0.5682
	160	0.0851	0.0529	0.0315	0.0460	-	0.9868	0.9696	0.5654
	320	0.0810	0.0564	0.0321	0.0492	-	0.9951	0.9892	0.5587
40	40	0.0964	0.0616	0.0382	0.0549	-	1	1	0.8628
	80	0.0933	0.0582	0.0343	0.0478	-	1	1	0.9727
	160	0.0941	0.0566	0.0305	0.0502	-	1	1	0.9931
	320	0.0917	0.0578	0.0331	0.0484	-	1	1	1
80	80	0.0937	0.0578	0.0366	0.0504	-	1	1	1
	160	0.0948	0.0598	0.0374	0.0559	-	1	1	1
	320	0.0931	0.0562	0.0291	0.0446	-	1	1	1
160	160	0.0946	0.0558	0.0337	0.0534	-	1	1	1
	320	0.0873	0.0475	0.0249	0.0450	-	1	1	1
ARE		84.31	17.99	31.31	7.47				

Note: “-” means do not report the power of a test that fails to control the Type I error rate

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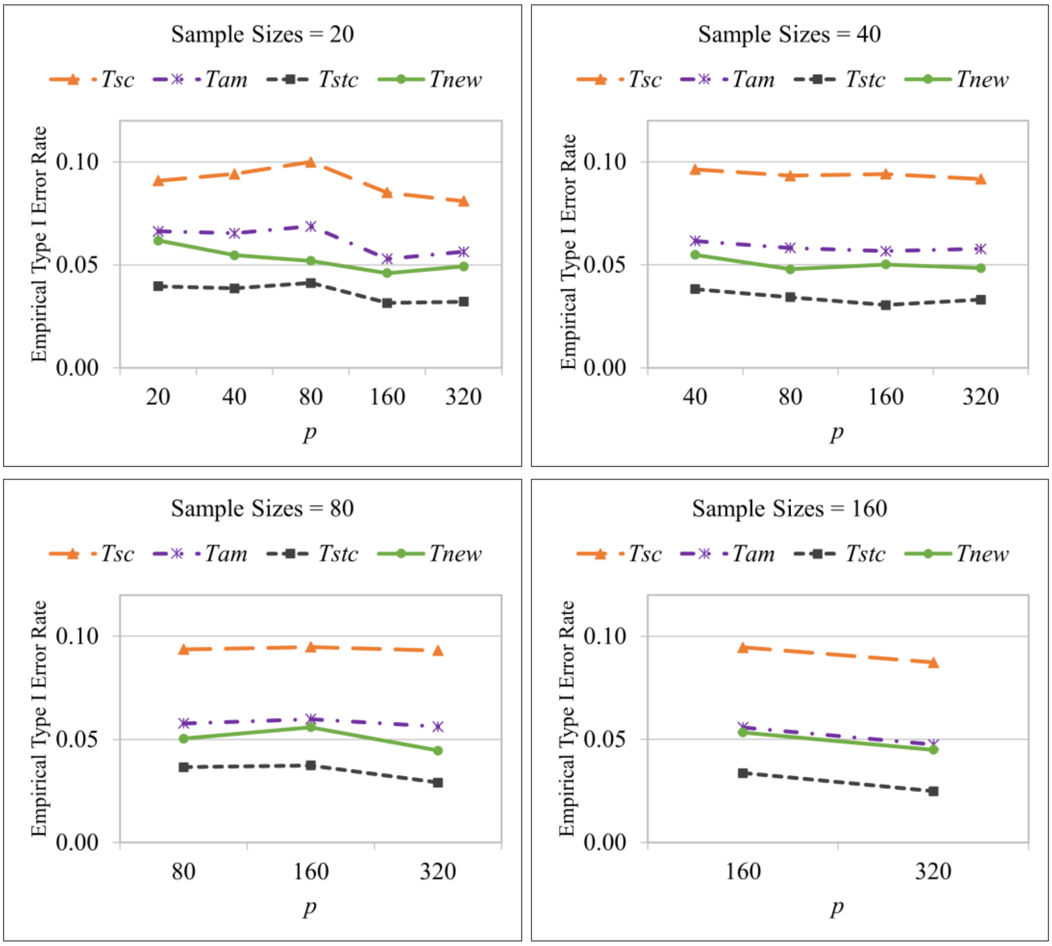

Fig. 5: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^3 the VC structure with g=3, across sample sizes of 20, 40, 80, and 160

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Table 6. Empirical Type I error rates under H_0^3 and empirical powers under H_1^3 for the VC structure with g=4

N_i	p	Empirical Type I error rate				Empirical power			
		T_{sc}	T_{am}	T_{stc}	T_{new}	T_{sc}	T_{am}	T_{stc}	T_{new}
20	20	0.1226	0.0594	0.0438	0.0660	-	0.9825	0.9674	0.8746
	40	0.1194	0.0636	0.0420	0.0564	-	0.9819	0.9589	0.9381
	80	0.1311	0.0659	0.0481	0.0542	-	0.9864	0.9710	0.9739
	160	0.1194	0.0596	0.0412	0.0548	-	0.9870	0.9744	0.9780
	320	0.1178	0.0582	0.0406	0.0534	-	0.9880	0.9760	0.9999
40	40	0.1273	0.0622	0.0420	0.0578	-	1	1	1
	80	0.1366	0.0687	0.0463	0.0508	-	1	1	1
	160	0.1204	0.0531	0.0358	0.0534	-	1	1	1
	320	0.1319	0.0604	0.0396	0.0552	-	1	1	1
80	80	0.1135	0.0511	0.0368	0.0574	-	1	1	1
	160	0.1253	0.0554	0.0364	0.0532	-	1	1	1
	320	0.1208	0.0523	0.0374	0.0538	-	1	1	1
160	160	0.1210	0.0574	0.0384	0.0540	-	1	1	1
	320	0.1204	0.0535	0.0350	0.0530	-	1	1	1
ARE		146.76	17.24	19.50	10.49				

Note: "-" means do not report the power of a test that fails to control the Type I error rate

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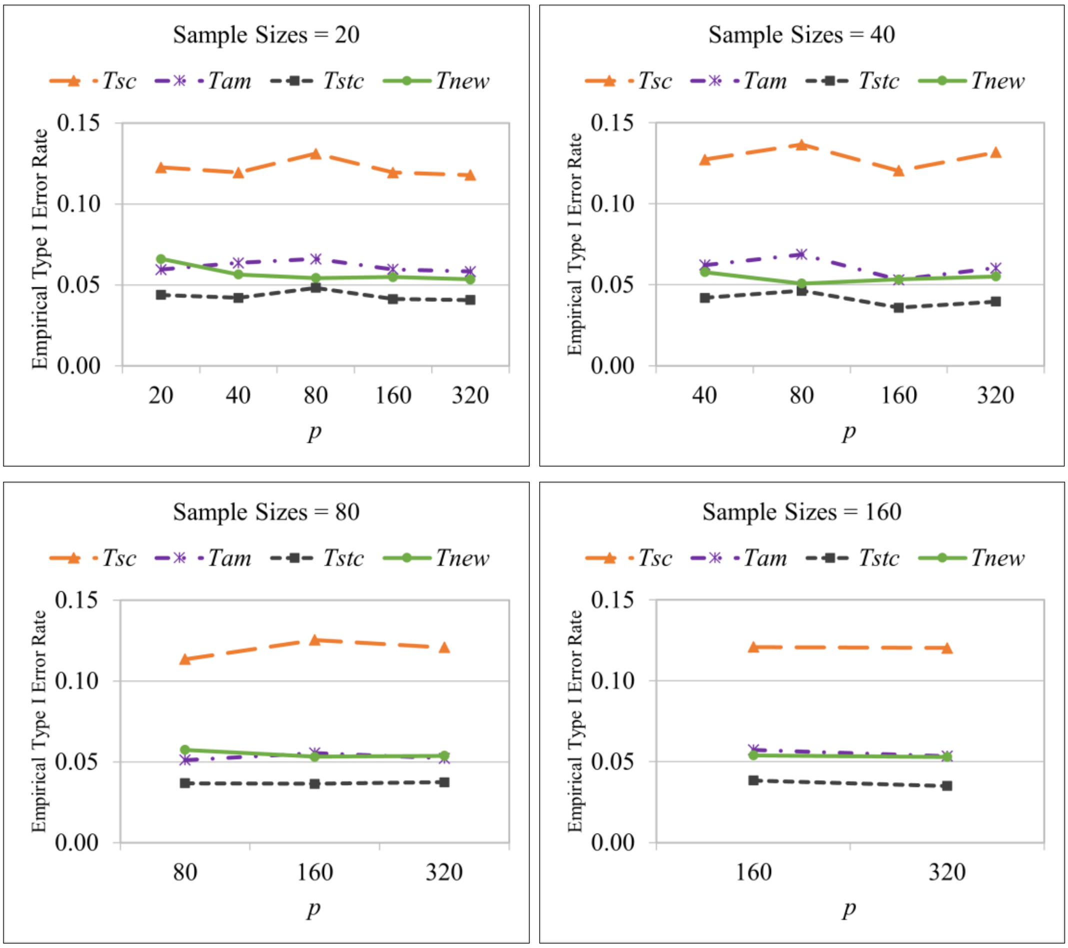

Fig. 6: Plot of the empirical Type I error rates for T_{sc}, T_{am}, T_{stc}, and T_{new} under H_0^3 the VC structure with g=4, across sample sizes of 20, 40, 80, and 160

Source: Created by the author

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Saowapa Chaipitak was responsible for data curation, formal analysis, investigation, methodology, software implementation, validation, writing the original draft, and review and editing.

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Conflict of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.