Computational Methods for Optimal Affine Bounds of Continuous Functions

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Abstract: This paper presents global computational problems related to establishing lower bounds for highdimensional continuous functions over a simplex. We propose a rigorous approach to constructing affine lower bounds (ALB) that remain closely beneath the original function. By extending the least squares method for control Bernstein, we develop a convergent AB to both polynomials and rational functions. Furthermore, we demonstrate that these bounds achieve high convergence rates to the original functions. Finally, we evaluate our approach by comparing it with previous methods using error bound approximation.

Key-Words: Global optimization, Bounding functions, Bernstein expansion, Polynomials, Minimization, simplices

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1 Introduction

Methods of relaxation used to solve nonlinear systems and global optimization problems typically involve bounding the non-convexity present in the constraint or objective functions from below using linear or convex functions, see [1], [2], [3]. A discrete relaxation approach for optimization problems has some key conditions that need to be investigated, as outlined in [4]:

- Each feasible discrete point in the problem can be used by the relaxation method.
- The derived function is easier to analyze than the original problem.
- Rate of convergent to the original function is obtained.

Various relaxation techniques for global optimization applications have been explored in the literature, such as those found in [5], [6], [7]. In our work, we study the constraint problem

$$\min_{\boldsymbol{x}\in T} H(\boldsymbol{x})$$

where the set of feasible solutions T is defined as

$$T = \begin{cases} g_i(\boldsymbol{x}) \leq 0 & \text{for } 0 \leq i \leq t \\ \boldsymbol{x} \in S \end{cases} \tag{1}$$

and S is a simplex domain with real functions H, and g_i given over S.

In [4] and [8], several methods were addressed for creating lower bounds optimization for polynomials for the problem (1). These methods did not obtain the convergent rates in multidimensional case. Many other bounds with applications in different fields are given in [9], [10], [11], [12].

In this work, we utilize Bernstein control points to construct an affine bound function that optimally approximates the original polynomial and rational functions from below. Additionally, positivity analysis and stability analysis for the Bernstein form, as discussed in [10], are considered. Our second objective is to derive an affine bounding that ensures a high convergennt rate to the origin over triangles, guaranteeing that the minimum error bound approaches to zero. First, we examine various existing methods that compute lower bounds using the Bernstein approach and interval domain techniques. Then, we introduce a novel linear affine lower bound (ALB) that converges to the original multivariate function at both linear and quadratic rates. This method is based on the least squares of control points, incorporating degree elevation and the convex hull of the elevated points degree. Finally, we demonstrate that the proposed bound effectively estimates the original function, ensures convergence, and preserves the overall shape of the curve across its domain.

2 Bernstein Approach (BA)

In this section, we present some notations and existent definitions of BA.

Definition 2.1 *Consider $l_0, \dots, l_n$ be $n+1$ points of $\mathbb{R}^n$ $(n \geq 1)$. The list $L = [l_0, \dots, l_n]$ defines the general simplex of vertices $l_0, \dots, l_n$, where L is a non-degenerate simplex of $\mathbb{R}^n$. It follows that $l_0, \dots, l_n$ are linearly independent.*

Let $\chi_0, \dots, \chi_l$ be the associated barycentric coordinates to L where $\sum_{i=0}^n \chi_i(\boldsymbol{x}) = 1$ and $\boldsymbol{x} = \sum_{i=0}^n \chi_i(\boldsymbol{x}) l_i$, $\forall \boldsymbol{x} \in L$. We set the multi-index $t = (t_0, \dots, t_n) \in \mathbb{N}^{n+1}$ and $|t| = t_0 + \cdots + t_n$.

It is known that any simplex L in $\mathbb{R}^n$ can be linearly transformed to the canonical simplex $E = [e_0, e_1, \dots, e_n]$. To this end, we present our methods over E where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $e_0 = (0, ..., 0)$. If $\boldsymbol{x} = (x_1, ..., x_l) \in E$, then $(\chi_1, ..., \chi_l) = (x_1, ..., x_n)$ and $\chi_0 = 1 - \sum_{i=1}^n x_i$. For $s, t \in \mathbb{N}^{n+1}$ with $s \leq t$, we define

$$\binom{t}{s} := \prod_{i=0}^{n} \binom{t_i}{s_i}.$$

If $M \in \mathbb{N}$ so that $|t| = M$, we use

$$\binom{M}{t} := \frac{M!}{t_1! \dots t_n! . t_0!}.$$

Remark 2.1 *The BA of degree M over L is given as $(D_t^{(M)})_{|t|=M}$, where*

$$D_t^{(M)}(\chi) = \binom{M}{t} \chi^t.$$

For $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{x}^s := \prod_{i=1}^m x_i^{s_i}$, let a (monomial) polynomial function h of degree m be denoted by

$$h(x) = \sum_{|s| \leq m} c_s \boldsymbol{x}^s.$$

The Bernstein form of maximum degree $m \leq M$ for this monomial polynomial over a simplex is presented as

$$h(\boldsymbol{x}) = \sum_{|t|=M} \psi_t(h, M) D_t^{(M)},$$

where the Bernstein coefficients $\psi_t(h, M)$ are computed by

$$\psi_t(h, M) = \sum_{s \leq t} \frac{\binom{t}{s}}{\binom{M}{s}} c_s, \ \ |t| = M.$$

Definition 2.2 *The grid points associated to l are defined as*

$$l_t(M, L) = \frac{t_0 l_0 + \cdots + t_n l_n}{M} \in \mathbb{R}^n \ (|t| = M).$$

The control points for the curve of a polynomial are denoted by

$$(l_t(M, L), \ \psi_t(h, M)) \in \ \mathbb{R}^{n+1} \ (|t| = M).$$

Proposition 2.1 *[13] The following properties are obtained.*

(i) Linear precision: *If h has a degree less than or equal 1, then*

$$\psi_t(h, M) = h(l_t(M, L)), \ \forall |t| = M;$$

(ii) Interpolation: *For $(e_0, ..., e_n)$, we have*

$$\psi_{Me_i} = h(l_i), \ 0 \leq i \leq n;$$

(iii) Convex hull: *The curve of h over L is convexly optimized by its control points;*

(iv) Enclosure:

$$\min_{|t|=M} \psi_t(h, M) \leq h(\boldsymbol{x}) \leq \max_{|t|=M} \psi_t(h, M), \ \forall \boldsymbol{x} \in L.$$

3 Improved Bound Using BA

First, we extend the bounding approach from [8] to polynomials defined over simplices. We then introduce a new approach that ensures convergence properties over a simplex. Before deriving our lower bound, the following three conditions must be met:

First, the chosen control points must prevent degenerate cases where they fail to provide a single interpolating hyperplane, as discussed in [8]. Second, the appropriate lower bound is not necessarily a hyperplane. In such cases, the methods are adjusted by estimating a downward error bound. Third, the desired convergence properties must be achieved. The well-known Bernstein constant bound function offers the fastest convergence over a given domain. However, constant bounding approaches do not effectively tighten the continuous curve or surface across the entire domain.

3.1 Minimum Bernstein Constant Bound

A constant bound can be derived for polynomials from the minimum value of Bernstein coefficients over L:

$$AB^{(L)} := \min_{|s|=m} \psi_s(h,m).$$

From the enclosure bound,

$$AB^{(L)}(\boldsymbol{x}) \leq h(\boldsymbol{x}),\ \forall \boldsymbol{x} \in L,$$

defines a lower constant bound for h.

Corollary 1 *The minimum error can be bounded as follows:*

$$\min_{\boldsymbol{x}\in L}(h(\boldsymbol{x}) - AB^{(L)}(\boldsymbol{x})) \\ \leq \min_{|s|=m} \left|h(l_s(m,L)) - AB^{(L)}(l_s(m,L))\right|.$$

3.2 Linear Programming (LP)

In this approach, a linear function is computed by passing through the control points $\boldsymbol{\psi}_t$ and determining the slopes by solving LP, where the minimum Bernstein coefficient is defined at a specific index t. This method provides a more accurate error bound compared to constant functions, this was shown in [8]. However, it is computationally slow.

3.3 Linear Systems approach (LS)

Fo deriving this approach, we determine hyperplanes that pass through the selected controller $\boldsymbol{\psi}^0 = \boldsymbol{\psi}_t$ so that estimating the lower convex is increasing. Slopes, solution of LS and a set of substitutions must be determined. Let $AB_0(\boldsymbol{x}) := \psi_t$, we choose from $\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^l$ a set of tight bounds $AB_1,...,AB_n$. Ending with AB_n, a hyperplane is passing through a lower edge of the convex hull of $\boldsymbol{\psi}^0,...,\boldsymbol{\psi}^n$. This approach provide a set of independent vectors $\{n^1,...,n^l\}$, and computes slopes from $\boldsymbol{\psi}^0$ to $\boldsymbol{\psi}^j$ in direction n^j, see [4]. This approach is tighter than LP method but does not achieve convergence rates.

3.4 Minimum Value of Bernstein

Here, the minimum coefficient of Bernstein ψ_t needs to be computed with other coefficients values $\psi^1,...,\psi^l$. We define a non-degenerate bounding $S(x)$ as an interpolating for these values. Then, determine the nonnegative value v,

$$v = \max_{|t|=m}\{S(l_t(m,L)) - \psi_t\}. \tag{2}$$

For the non-zero v, a tight bound is maintained through the downward step:

$$AB^L(\boldsymbol{x}) = S(\boldsymbol{x}) - v. \tag{3}$$

This approach often fails to provide estimated error bound, as demonstrated in [8]. Additionally, it does not achieve a convergence rate.

3.5 Bounding by Slops

Similar to the above methods, the bound proposed in [4], derived from control points, is obtained by solving a linear system of polynomials. However, in more complex cases, additional challenges arise, and this bound may not hold if computed using the downward step v as in (2) and (3). In this method, a tight bound is not always guaranteed, see [8].

3.6 Bounding by Least Squares

By computing least squares function (LSF) for the control points derived from BA, we will have a tight bound for polynomials across the entire domain. We extend this approach to polynomials defined over L.

We refer to the number of coefficients of simplicial Bernstein form of $n-$variable and degree M which equals to $W := \binom{M+n}{n}$.

Let γ be the solution of a matrix of W number of control points $l_t(m,L)$, then we obtain LSF

$$S^{(L)}(\boldsymbol{x}) = \sum_{i=1}^{n} \gamma_i x_i + \gamma_{n+1}. \tag{4}$$

By computing v of (4), we calculate the downward step and derive the lower bounding

$$AB^L(\boldsymbol{x}) = S^{(L)}(\boldsymbol{x}) - v,\ \boldsymbol{x} \in L.$$

In the following section, the bound converges linearly to the original function.

3.7 Estimated Bound

The degree of BA need to be raised so that the LSF $S^{(L)}(\boldsymbol{x})$ can be of higher degree, from which the control points $\left(l_s(m+M,L), \psi_s^{(m+M)}\right)$ are of degree $M+m$. Then, we can estimate the the coefficients of Bernstein by the corresponding net values:

$$v^{(M)} := \max_{|s|=M+m} \{\psi_s^{(M+m)}(B^{(\text{Ł})}) - \psi_s^{(m+M)}(h)\}.$$

A convergent bound for polynomials (AP) holds:

$$AB^{(M)}(\boldsymbol{x}) := S^{(\text{Ł})}(\boldsymbol{x}) - v^{(M)}.$$

Finally, we have a bounding that estimates the original function well, see Figure 1

$$AB^{(M)}(\boldsymbol{x}) \leq h(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \text{Ł}.$$

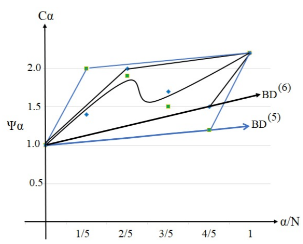

Fig. 1: A degree three polynomial, the Bernstein bound is of elevated degree to six and the linear convergent obtained, created by the authors.

Remark 3.1 *The error bound can be estimated as*

$$0 \leq \max_{\boldsymbol{x}\in Ł}\{h(\boldsymbol{x}) - AB^{(M)}(\boldsymbol{x})\}$$

$$\leq \max_{|t|=m+M}\{\psi_t(h, m+M) - AB^{(M)}(l_t(m+M))\}$$

$$\leq \max_{|s|=m}\{\psi_s(h,m) - AB^{(0)}(l_s(m, Ł))\}.$$

3.7.1 CALB Error Bound

We show that the range interval of h converges to the enclosure $AB^{(M)}$ linearly by elevating the degree. By BA monotonicity, we have $v^{(m)} \geq v^{(m+1)}$, for $m \geq 0$.

Lemma 3.1 *The error bounding estimation can be given as*

$$\min_{\boldsymbol{x}\in Ł}|h(\boldsymbol{x}) - AB^{(m+1)}(\boldsymbol{x})| \leq \min_{\boldsymbol{x}\in Ł}|h(\boldsymbol{x}) - AB^{(M)}(\boldsymbol{x})|.$$

For extending a linear rate of convergent, we recall the following definition.

Definition 3.1 *Define a simplex $L = [l_0, ..., l_n]$ of $\mathbb{R}^n$. For $|\gamma| = n-2$ and $0 \leq i < j \leq n$, let the second differences of h be given as*

$$\nabla^2\psi_{\gamma,i,j}(h,m) := \psi_{\gamma+\hat{e}_i+\hat{e}_{j-1}} + \psi_{\gamma+\hat{e}_{i-1}+\hat{e}_j}$$

$$-\psi_{\gamma+\hat{e}_{i-1}+\hat{e}_{j-1}} - \psi_{\gamma+\hat{e}_i+\hat{e}_j},$$

where $\hat{e}_{-1} := \hat{e}_n$. Then, we have

$$\nabla^2\psi_{\gamma,i,j}(h,m) := (\nabla^2\psi_{\gamma,i,j}(h,m))_{|\gamma|=m-2, 0\leq i<j\leq l}.$$

Define the maximum value as

$$||\nabla^2\psi(h,m)||_\infty := \max_{|\gamma|=n-2, 0\leq i<j\leq l} |\nabla^2\psi_{\gamma,i,j}(h,m)|.$$

Theorem 3.2 *[14] Consider h be of maximum degree $m < M$, we have*

$$\max_{|t|=M}|h(l_t(M, Ł)) - \psi_t(h,m)| \leq \frac{T}{M-1},$$

where

$$T = \frac{n(n+2)m(m-1)}{24}|| \nabla^2 \psi(h,m)||_\infty. \quad (5)$$

3.7.2 Convergence Rate

We show the linear rate of convergence by degree elevation for $AB^{(M)}(\boldsymbol{x})$ and $F(\boldsymbol{x})$ over Ł.

Theorem 3.3 *Let h be in Bernstein of degree M and $AB^{(M)}$ is the improved bounding. For $M \geq 0$, we can estimate*

$$\min_{\boldsymbol{x}\in Ł}|h(\boldsymbol{x}) - AB^{(M)}(\boldsymbol{x})| \leq \frac{T}{m+M-1},$$

where $m+M$ is the degree of BA and T is given in (5).

4 Optimal Boundary for Rationals

We extend a continuous linear lower bound for rational polynomials $Y(x) = p(x)/g(x)$ over L which optimize the original Y from bellow well. We assume that $\psi_t(g,m)$ have the same sign and non-zero, then $g(x) \neq 0$. Let the rational Bernstein coefficients of $Y(\boldsymbol{x})$ be defined as

$$\psi_t(Y,m) = \frac{\psi_t(p,m)}{\psi_t(g,m)}, \ |t| = m.$$

Subsequently, the constant

$$AB^{(\text{Ł})}(\boldsymbol{x}) := \min \psi_t(Y,m)$$

is a lower bounding for $Y(\boldsymbol{x})$ over Ł, see Figure 2.

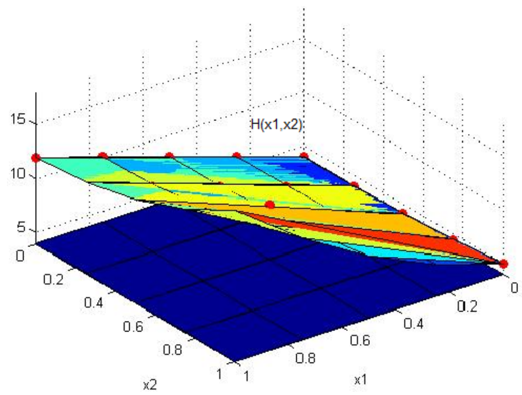

Fig. 2: The minimum Bernstein coefficient for lower bounding (lower surface) with Bernstein controls in two dimensions, created by the authors.

The estimation is given in the following lemma.

Lemma 4.1 *Let the rational $H = p/g$ be given over Ł with $AB^{(Ł)}(\boldsymbol{x}) \leq Y(x)$. Then, the error bound estimation is given by*

$$\min_{\boldsymbol{x}\in Ł}(Y(\boldsymbol{x}) - AB^{(Ł)}(\boldsymbol{x}))$$

$$\leq \min_{|s|=m} \left|Y(l_s(m, Ł)) - AB^{(Ł)}(l_s(m, Ł))\right|.$$

5 Conclusions and Applications

In this work, we examined a constrained optimization problem for polynomials and rationals over a simplex. We reviewed several existing methods designed to optimize continuous functions from below; however, these approaches failed to achieve any rate of convergence in high-dimensional simplices. To address this, we introduced a new bounding for polynomials over simplices, leveraging convex optimization principles and the least squares method for BA. This bound demonstrated high rates of convergence to the original function. Additionally, we developed a tight bounding for rational functions and estimated the error bound by reducing the upper value. These bounds effectively optimized the original functions from below while preserving the overall shape of the curve across the entire domain. Furthermore, this approach can be extended to applications such as constructing mean-risk stochastic minimum cost consensus models with asymmetric adjustment costs [15]. Another potential application of convergent bounds is in the stability analysis of nonlinear dynamical systems, particularly when the vector field is defined over simplices, $Ł = l^{[0]} \cup ... \cup l^{[n}$.

Consider the affine control system

$$\dot{x} = F_U(x) := G(x) + P(x)U(x),$$

where the vector field $F_U : \mathbb{R}^l \longrightarrow \mathbb{R}^n$ is defined by the drift $G : \mathbb{R}^l \longrightarrow \mathbb{R}^n$ and $U : \mathbb{R}^l \longrightarrow \mathbb{R}^m$ is the control with input matrix function $P : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$. If there is a pair (U, V) of polynomials such that the negative Lie derivative $\mathcal{L}_{F_U}(V)$ is positive in the monomial and Bernstein forms, then there exists a stabilizing Bernstein control for F_U. This can be investigated by using our lower bounding over a simplex.

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References

[1] Claire S. Adjiman and Christodoulos A. Floudas. Rigorous convex underestimators for general twice-differentiable problems. *Journal of Global Optimization*, 9(1):23–40, 1996. https://doi.org/10.1007/BF00121749 doi:10.1007/BF00121749.

[2] Christodoulos A. Floudas. *Nonlinear and Mixed-Integer Optimization: Fundamentals and Applications*. Oxford University Press, New York, 1995. URL: https://doi.org/10.1093/oso/9780195100563.001.0001.

[3] Christian Jansson. Convex-concave extensions. *BIT Numerical Mathematics*, 40(2):291–313, 2000. https://doi.org/10.1023/A:1022343007844 doi:10.1023/A:1022343007844.

[4] Jürgen Garloff and Andrew P. Smith. A comparison of methods for the computation of affine lower bound functions for polynomials. In Christophe Jermann, Arnold Neumaier, and Djamila Sam, editors, *Global Optimization and Constraint Satisfaction*, volume 3478 of *Lecture Notes in Computer Science*, pages 71–85. Springer, Berlin, Heidel-

berg, 2005. https://doi.org/10.1007/11425076_6.

[5] FaizA. Al-Khayyal and James E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8(2):273–286, 1983. https://doi.org/10.1287/moor.8.2.273 doi:10.1287/moor.8.2.273.

[6] Jordan Ninin, Frédéric Messine, and Pierre Hansen. A reliable affine relaxation method for global optimization. *4OR: A Quarterly Journal of Operations Research*, 13(3):247–277, 2015. https://doi.org/10.1007/s10288-014-0269-0 doi:10.1007/s10288-014-0269-0.

[7] Shaojian Qu, Kecun Zhang, and Fusheng Wang. A global optimization using linear relaxation for generalized geometric programming. *European Journal of Operational Research*, 190(2):345–356, 2008. https://doi.org/10.1016/j.ejor.2007.06.034 doi:10.1016/j.ejor.2007.06.034.

[8] Andrew P Smith. *Enclosure methods for systems of polynomial equations and inequalities*. PhD thesis, Universität Konstanz, konstanz, 2012. https://kops.uni-konstanz.de/entities/publication/d024b254-f338-47de-bd1f-3b4fdbd944b4

[9] Rida T. Farouki and V. T. Rajan. Algorithms for polynomials in Bernstein form. *Computer Aided Geometric Design*, 5(1):1–26, 1988. https://doi.org/10.1016/0167-8396(88)90016-7 doi:10.1016/0167-8396(88)90016-7.

[10] Tareq Hamadneh, Mohammed Ali, and Hassan Al-Zoubi. Linear optimization of polynomial rational functions: Applications for positivity analysis. *Mathematics*, 8(2):283, 2020. https://doi.org/10.3390/math8020283 doi:10.3390/math8020283.

[11] Anthony Narkawicz, Jürgen Garloff, Andrew P. Smith, and César A. Munoz. Bounding the range of a rational function over a box. *Reliable Computing*, 17(1):34–39, 2012.

[12] Haitham Qawaqneh, Hassan Jari, Ali Altalbe, and Ahmet Bekir. Stability analysis, modulation instability, and the analytical wave solitons to the fractional Boussinesq-Burgers system. *Physica Scripta*, 99(12):125235, 2024. https://doi.org/10.1088/1402-4896/ad8e07 doi:10.1088/1402-4896/ad8e07.

[13] Richard Leroy. Convergence under subdivision and complexity of polynomial minimization in the simplicial bernstein basis. *Reliable Computing*, 17(1):11–21, 2012. https://api.semanticscholar.org/CorpusID:11321842. Accessed: February 20, 2026.

[14] *(text in French)* Richard Leroy. *Certificats de positivité et min-imisation polynomiale dans la base de Bernstein multivariée*. PhD thesis, Université Rennes 1, Rennes, France, 2008. Accessed: February 20, 2026. URL: https://theses.hal.science/tel-00349444.

[15] Ying Ji, Huanhuan Li, and Huijie Zhang. Risk-averse two-stage stochastic minimum cost consensus models with asymmetric adjustment cost. *Group Decision and Negotiation*, 31(2):261–291, 2022. https://doi.org/10.1007/s10726-021-09752-z doi:10.1007/s10726-021-09752-z.

Contribution of Individual Authors

Tareq Hamadneh has achieved all the results and formulations.

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Conflicts of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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