Some Properties of Unit Graphs of Rings

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Abstract: - Let R be a ring with a nonzero identity. The unit graph $G(R)$ in such a way that R is the vertex set of $G(R)$, where the adjacency of two different vertices depends on their sum being a unit in R. In this paper, the spectrum of $G(R_1)$ and $G(R_2)$ where $R_1 = Z_p\,(+)\,Z_p$ and $R_2 = Z_{p^2}\,(+)\,Z_{p^2}$ has been determined. In addition, the eigensharp property of $G(R_2)$ has been studied. Finally, the Wiener index $W(G)$ of unit graphs $G(R_1)$ and $G(R_2)$ has been evaluated.

Key-Words: - Commutative Ring, Unit Graph, Wiener index of a graph, Biclique decomposition number, Eigensharp graph.

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1 Introduction

Recently two concepts have been attracted by many researchers; the Wiener index and the biclique decomposition number. The reason why these concepts are getting more and more interesting every day is because they involved in a lot of applications in various fields long with some industrial applications.

The Wiener index was first studied in 1947, [1], who recognized the relationship between the boiling point of paraffin's and the structure of the molecule.

Wiener index has come to be one of the most well-known topological indices in chemistry, since undirected graphs are typically used to model molecules. For example, the goal of the pharmaceutical design process is to create chemical compounds with particular qualities, and this heavily depends on the chemical formula in addition to the molecular structure. This may be easily understood by looking at the molecular graph, which is the graph associated with a molecular structure. The atoms in a chemical molecule get up the vertex set, while the bonds between atoms get up the edge set. In this instance, a Wiener index is the total number of atom pairs that are encoded in a molecule using the shortest path technique and the length of the shortest bond using the bond path. More details about Wiener index and its uses in chemistry, can be found in [2], [3], [4], [5], [6].

In addition, Wiener index appears in numerous areas such as communications, plant locations, and cryptography since it is an indicator for the expected distance between two randomly selected network nodes, [7]. Moreover, the Wiener index or the average distance plays crucial role in finding the spanning tree with the smallest average distance see, for example, [2].

Finding the Wiener index of some graphs has been investigated by many authors. The authors in [8], evaluated the Wiener index of product graphs, join graphs, and composition graphs. The study, [9], computed the Wiener and hyper-Wiener indices of these composite graphs. In [10], the authors determined the Wiener index of some other operations such as Mycielskiâc construction, generalized hierarchical product, and t-th subdivision of a graph. Moreover, the Wiener index for some special families of graphs has been derived in [11], [12], [13].

The first appearance of biclique decomposition number was in 1971 in [14], to study a data communication issue in the addressing problem for loop switching. In particular, the collection of loops is considered as a graph such that each loop is a vertex and if the two loops have a mutual transfer point then the two vertices are connected. The best possible value of the length of addresses is called the biclique decomposition number. Furthermore, it is inspired through applications in numerous other fields such as combinatorial geometry, [15], communication complexity, [16], and immunology, [17].

The biclique decomposition number for some graphs families including complete bipartite graphs $K_{n,m}$, complete graphs K_n, cycles C_m with $m = 4$ or $m \neq 4k$ and several graph products are evaluated, see for example,[18], [19], [20], [21].

This paper determines the Wiener index and the biclique decomposition number for some unit graphs. Studying unit graphs has been attracted by many researchers. In [22], the authors examined basic unite graph features like connectedness, chromatic index, diameter, girth, and planarity. The authors in [23], discussed the eigensharp property for the unit Graph $G(R_1)$ where $R_1 = Z_p\,(+)\,Z_p$.

The main focus of this paper is to introduce and rigorously analyze the Wiener index and spectrum of the specific unit graphs $G(R_1)$ and $G(R_2)$. The closed-form formulas we present already reveal important structural insights and highlight novel properties of these algebraically defined graphs. While we fully acknowledge that additional comparisons with other graph families, generalizations to non-commutative rings, or explorations of higher-order idealizations are indeed interesting and worthwhile directions. We view them as promising avenues for future research that build upon the foundation established here.

In fact, the results presented in this study stand out from other approaches due to the simplicity and directness with which eigenvalues can be computed through our block-matrix decomposition method. Unlike many existing approaches that rely on more complex or case-specific techniques, our method provides a more straightforward framework for eigenvalue determination. Furthermore, to the best of our knowledge, this paper is the first to introduce both the Wiener index and the spectrum of the unit graphs $G(R_2)$. This novelty ensures that our offers new perspectives on the interplay between algebraic graph theory and unit graphs.

In this paper, the unit graphs $G(R_1)$ and $G(R_2)$ where $R_1 = Z_p\,(+)\,Z_p$ and $R_2 = Z_{p^2}\,(+)\,Z_{p^2}$ are investigated. We determine the spectrum of $G(R_1)$ and $G(R_2)$ and discuss the eigensharp property of $G(R_2)$. We also, evaluate the Wiener index of $G(R_1)$ and $G(R_2)$.

2 Preliminaries

Throughout the paper, we consider $G = (V, E)$ to be a finite simple undirected graph with $V(G)$ and $E(G)$ will denote the vertex set and the edge set of G respectively. Assume that the distance between the vertex x and the vertex y in G is denoted by $d(x, y)$. Also $d(x, y)$ can be considered as the length of a path between u and v that has a shortest length in G. The number of edges that connect a vertex x in G is its degree, which is indicated by $deg(x)$. The set of vertices adjacent to x is referred to as the set of neighborhoods of x and is denoted by $N_G(x)$. Also, if $x, y \in V(G)$ and $xy \in E(G)$, we write $x \sim y$. Let (S, T) be the induced subgraph whose vertex set is $V(S) \cup V(T)$.

The sum of all of $d(x, y)$ among all vertices of a graph $G = (V, E)$, represented as $W(G)$; is the graph's Wiener index.

$$W(G) = \sum_{\{x,y\} \subseteq V(G)} d(x, y).$$

Complete bipartite subgraphs of a graph are called bicliques. A collection $\mathcal{F}_G$ of bicliques of a graph G such that each edge in G lies in a member of $\mathcal{F}_G$ is called a biclique covering of a graph G. When the members of $\mathcal{F}_G$ are pairwise edge disjoint, then $\mathcal{F}_G$ is called a biclique decomposition of G. The biclique decomposition number $bp(G)$ is the cardinality of a minimum biclique decomposition of G.

The square matrix of order $|V(G)|$, with ij-th entry equals 1 if $x_i x_j$ is an edge of G and 0 otherwise, is the adjacency matrix of G, represented by $A(G)$. For a simple graph G, the adjacency matrix $A(G)$ is symmetric with real eigenvalues. Assume that $c_+(G)$ and $c_-(G)$ represent the numbers of positive and negative eigenvalue of the adjacency matrix $A(G)$. For any non-null graph G, we have $c_+(G) > 0$ and $c_-(G) > 0$. The study, [14], showed $\max\{c_+(G), c_-(G)\} \leq bp(G)$ for a graph G. If $bp(G) = \max\{c_+(G), c_-(G)\}$, the graph G is said to be eigensharp.

The number of linearly independent eigenvectors that are related to an eigenvalue λ_i is known as its multiplicity. If m_i is the multiplicity of the distinct eigenvalues $\lambda_i : 1 \leq i \leq j$ of the adjacency matrix $A(G)$, then the *spectrum* of G, denoted by $\sigma(A(G))$, is

$$\sigma(A(G)) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_j \\ m_1 & m_2 & ... & m_j \end{pmatrix}.$$

Assume R to be a finite ring. Let $G(R) = (V, E)$ denotes the unit graph on a ring R where $V = R$. If the sum of two distinct vertices in R is unit, then they are adjacent. Throughout the paper, all rings are considered to be commutative rings with non-zero unity. The set of all units in R will be denoted by $U(R)$. For each element u in R, $u \in U(R)$ if it has a multiplicative inverse and therefore, u is said to be a unit in R.

The ring $R(+)S = \{(l, t) : l \in R, t \in S\}$ is called the idealization of S as an $R-$module, where the operations on $R(+)S$ are defined by:

$$(l_1, t_1) + (l_2, t_2) = (l_1 + l_2, t_1 + t_2)$$
$$(l_1, t_1) \cdot (l_2, t_2) = (l_1 l_2, l_1 t_2 + l_2 t_1),$$

and $(1,0)$ is the identity under multiplication.

Throughout the paper, we consider the two finite rings $R_1 = Z_p(+)Z_p$ as the idealization of Z_p as a Z_p-module, and $R_2 = Z_{p^2}(+)Z_{p^2}$ as the idealization of Z_{p^2} as a $Z_{p^2}-$module. Note that, by performing a simple calculations, we can show that (a,b) is a unit in R_i, for $i = 1,2$, if and only if a is a unit in Z_{p^α}, for $\alpha = 1,2$.

3 The Spectrum of the Unit Graph $G\left(Z_p\left(+\right)Z_p\right)$

In this section, we study the spectrum of the unit graph $G(R_1)$ where $R_1 = Z_p\left(+\right)Z_p$ and p is a prime number. We point out that in [23], we investigated the eigensharp property of the graph $G(R_1)$ by computing the negative and positive eigenvalues of a certain subgraph of $G(R_1)$. In the current paper, we studied the eigenvalues of the graph $G(R_1)$ completely because of their importance and as a prelude to future studies on various other features.

The unit graph $G(R_1) = (V,E)$ is defined such that $V = \{(x,y) : x,y \in Z_p\}$, where two distinct vertices $(x_1,y_1),(x_2,y_2)$ are adjacent with an edge in E, if $x_1 + x_2 \neq 0 \pmod p$. Let $T = \{(0,y) : y \in Z_p\}$ and let

$$S_i = \{(i,y) : y \in Z_p, i = 1,2,\cdots,p-1\}$$

be a decomposition for V. Then, the set of vertices T generates a null subgraph. For the subgraph (S_i,T), since each vertex in T is adjacent with each vertex in S_i, then the subgraph $(S_i,T\ )$ is a complete bipartite graph $K_{p,p}$. For the same i, any two distinct vertices in S_i are adjacent the subgraph induced by S_i is a complete graph K_p. Now, for two different indices i and j, if $i+j \neq 0 \pmod p$, the subgraph (S_i,S_j) is a complete bipartite graph $K_{p,p}$. If $j = p-i$, the subgraph (S_i,S_j) will be a null graph of size p.

Remark 1 *If $p = 2$ then $G(R_1)$ is isomorphic to the cycle graph C_4. Hence,* $\sigma(C_4) = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix}$.

Let p be an odd prime number. Let $q = p^i$ and $s = p^{i+1}$. Let J_q denote the all-1 matrix of size $q \times q$, 1_q is all-1 matrix of size $q \times 1$ and N_q is all-0 matrix of size $q \times q$. Also, let 0_q be the all-0 matrix of size $q \times 1$. For $m = 1,2,\ldots,\frac{p-1}{2}$ define the partition matrix $F^{(m)}$ to be the $s \times 1$ matrix where every entry in the submatrices is 0_q, with the exception of the $(m+1)$th row, which is the submatrix 1_q, and the $(p-m+1)$th row is the submatrix -1_q. Furthermore, define the partition matrix $H^{(r)}$ for $r = 2,3,\ldots,\frac{p-1}{2}$ as the $s\times 1$ matrix, with all submatrices being 0_q except for the second and last rows, which are the submatrix 1_q and the $(r+1)$th and $(p-r+1)$th rows are the submatrix -1_q. For example, if $i = 3$ and $p = 7$, then

$$F^{(2)} = \begin{bmatrix} 0_{7^3} \\ 0_{7^3} \\ 1_{7^3} \\ 0_{7^3} \\ 0_{7^3} \\ -1_{7^3} \\ 0_{7^3} \end{bmatrix} \text{ and } H^{(2)} = \begin{bmatrix} 0_{7^3} \\ 1_{7^3} \\ -1_{7^3} \\ 0_{7^3} \\ 0_{7^3} \\ -1_{7^3} \\ 1_{7^3} \end{bmatrix}$$

Let W_q be the adjacency matrix of the complete graph K_q, then the adjacency matrix A_1 for the unit graph $G(R_1)$ can be written in the form

$$\begin{bmatrix} N_p & J_p & J_p & \cdot & \cdot & \cdot & J_p & J_p & J_p \\ J_p & W_p & J_p & J_p & \cdot & \cdot & \cdot & J_p & N_p \\ J_p & J_p & W_p & \ddots & \cdot & \cdot & \cdot & N_p & J_p \\ J_p & J_p & \ddots & \ddots & J_p & \cdot & \iddots & \cdot & \cdot \\ \cdot & \cdot & \cdot & J_p & W_p & N_p & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & N_p & \ddots & J_p & J_p & \cdot \\ \cdot & \cdot & \cdot & \iddots & \cdot & J_p & \ddots & \ddots & J_p \\ J_p & J_p & N_p & \cdot & \cdot & J_p & \ddots & W_p & J_p \\ J_p & N_p & J_p & \cdot & \cdot & \cdot & J_p & J_p & W_p \end{bmatrix}. \tag{1}$$

Now, we show that the spectrum of $G\left(Z_p\left(+\right)Z_p\right)$ is

$$\begin{pmatrix} 0 & p-1 & -(p+1) & -1 & \lambda_1 & \lambda_2 \\ p-1 & \frac{p-1}{2} & \frac{p-3}{2} & (p-1)^2 & 1 & 1 \end{pmatrix}$$

where λ_1 and λ_2 are the roots of the quadratic equation

$$x^2 - (p^2-2p-1)x + p^2(1-p) = 0$$

Step 1. Since the first block of the matrix A_1 given in (1) is the matrix N_p and the other blocks of the first column are the matrix J_p, then it is clear that the vector with zero entries everywhere except 1 in the first entry and -1 in the ith entry where $i = 2,3,\ldots,p$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$. Thus $\lambda = 0$ is an eigenvalue of multiplicity at least $p-1$.

Step 2. Using the same technique as in [23], we can show that

$$\{F^{(m)} : F^{(m)} \text{ is a } p^2 \times 1 \text{ matrix}\}$$

where $m = 1,2,\ldots,\frac{p-1}{2}$ is a set of eigenvectors that are linearly independent of the matrix A_1 corresponding to the eigenvalue $\lambda = p-1$ and

$$\{H^{(r)} : H^{(r)} \text{ is a } p^2 \times 1 \text{ matrix}\}$$

where $r = 2, 3, \ldots, \frac{p-1}{2}$ is a set of eigenvectors that are linearly independent of the matrix A_1 corresponding to the eigenvalue $\lambda = -(p+1)$.

Step 3. Since $\lambda = -1$ is an eigenvalue of W_p of multiplicity $p-1$, after which it becomes evident that $\lambda = -1$ is an eigenvalue of A_1 with multiplicity $(p-1)^2$.

Now if we combine the eigenvectors in Steps 1, 2, and 3, we get p^2-2 linearly independent eigenvectors of A_1.

Thus, the characteristic polynomial of A_1 is

$$P(\lambda) = \lambda^{p-1}(\lambda-(p-1))^{\frac{p-1}{2}}(\lambda+p+1)^{\frac{p-3}{2}} \\ (\lambda-1)^{(p-1)^2}(\lambda^2+b\lambda+c)$$

Let λ_1 and λ_2 be the eigenvalues of A_1 that are the roots of the equation

$$\lambda^2+b\lambda+c=0.$$

Then $-b = \lambda_1+\lambda_2$ and $c = \lambda_1\lambda_2$.

Since the sum of the eigenvalues of A_1 is equal to $trace(A_1) = 0$ and the coefficient of λ^{p^2-1} in the characteristic polynomial $p(\lambda)$ is the negative of the trace of A_1, we get

$$\begin{aligned} \lambda_1+\lambda_2 = -b &= 0 - \Big(-\frac{1}{2}(p-3)(p+1) \\ &+ \frac{1}{2}(p-1)(p-1) - (p-1)^2\Big) \\ &= p^2-2p-1. \end{aligned}$$

If we compute the diagonal entries of the matrix A_1^2, we get

$$\text{trace}\,(A_1^2) = (p^2-1)(p^2-p).$$

Thus

$$\begin{aligned} \lambda_1^2+\lambda_2^2 &= (p^2-1)(p^2-p) - \Big(\frac{1}{2}(p-3)(p+1)^2 \\ &+ \frac{1}{2}(p-1)(p-1)^2 + (p-1)^2\Big) \\ &= p^4-2p^3+4p+1 \end{aligned}$$

Hence

$$\begin{aligned} c = \lambda_1\lambda_2 &= \frac{(\lambda_1+\lambda_2)^2-(\lambda_1^2+\lambda_2^2)}{2} \\ &= \frac{1}{2}((p^2-2p-1)^2-(p^4-2p^3+4p+1) \\ &= p^2-p^3. \end{aligned}$$

Therefore, λ_1 and λ_2 are eigenvalues of A_1 that are the roots of the equation

$$x^2-(p^2-2p-1)x+p^2(1-p)=0$$

Note that λ_1 and λ_2 are real numbers since

$$(p^2-2p-1)^2+4(p^3-p^2) \geq 0.$$

Hence, we get the spectrum of the matrix A_1 is

$$\begin{pmatrix} 0 & p-1 & -(p+1) & -1 & \lambda_1 & \lambda_2 \\ p-1 & \frac{p-1}{2} & \frac{p-3}{2} & (p-1)^2 & 1 & 1 \end{pmatrix}$$

where λ_1 and λ_2 are the roots of the quadratic equation

$$x^2-(p^2-2p-1)x+p^2(1-p)=0.$$

4 The Spectrum of the Unit Graph $G\left(Z_{p^2}\,(+)\,Z_{p^2}\right)$

In this section, we study the spectrum of the unit graph $G(R_2)$ where $R_2 = Z_{p^2}\,(+)\,Z_{p^2}$. Moreover, we show that for any prime p, the graph $G(R_2)$ is an eigensharp. The unit graph $G(R_2)$ consists of the vertex set $V = \{(x,y) : x, y \in Z_{p^2}\}$ of order p^4, where two distinct vertices $(x_1,y_1),(x_2,y_2)$ are adjacent if $x_1+x_2 \neq 0 \pmod p$. In particular, if $p = 2$, then $G(R_2)$ is isomorphic to the complete bipartite graph $K_{8,8}$ which is an eigensharp graph since $\sigma(K_{8,8}) = \begin{pmatrix} 0 & 8 & -8 \\ 14 & 1 & 1 \end{pmatrix}$.

Now if $p > 2$, assume O_i to be a subgraph of $G(R_2)$ induced by the set of vertex $V(O_i) = \{(i+jp, b) : 0 \leq i \leq p-1, 0 \leq j \leq p-1, b \in Z_{p^2}\}$. Then $V = \bigcup_{i=0}^{p-1} V(O_i)$. In fact, O_0 is an order p^3 null subgraph. For the subgraph (O_0, O_i), $i \neq 0$, every vertex in O_0 is adjacent with every vertex in O_i and so, $(O_0, O_i) \simeq K_{p^3,p^3}$. If $r = (i+j_1p, b_1)$ and $s = (i+j_2p, b_2)$ are two distinct vertices belong to O_i for a certain $i \geq 1$, then $r \sim s$ and so, $(O_i\,, O_i) \simeq K_{p^3}$.

If $r \in O_{i_1}$, $s \in O_{i_2}$, $i_1 \neq i_2, 0 \leq i_1, i_2 \leq p-1$, then $r \sim s$ if and only if $i_2 \neq p-i_1$.Thus, if $i_2 \neq p-i_1$, then $(O_{i_1}, O_{i_2}) \simeq K_{p^3,p^3}$, and (O_{i_1}, O_{i_2}) is a null subgraph of order p^3 otherwise.

Thus, the adjacency matrix A_2 of the graph $G(R_2)$ is the matrix of size $p^4 \times p^4$ which is given by

$$\begin{bmatrix} N_d & J_d & J_d & . & . & . & J_d & J_d & J_d \\ J_d & W_d & J_d & J_d & . & . & . & J_d & N_d \\ J_d & J_d & W_d & \ddots & . & . & . & N_d & J_d \\ J_d & J_d & \ddots & \ddots & J_d & . & . & . & . \\ . & . & . & J_d & W_d & N_d & . & . & . \\ . & . & . & . & N_d & \ddots & J_d & J_d & . \\ . & . & . & \iddots & . & J_d & \ddots & \ddots & J_d \\ J_d & J_d & N_d & . & . & J_d & \ddots & W_d & J_d \\ J_d & N_d & J_d & . & . & . & J_d & J_d & W_d \end{bmatrix}.$$

where $d = p^3$.

Similar technique as we used to determine the spectrum of the matrix A_1 in Section 3, we can show that the spectrum of the unit graph $G\left(Z_{p^2}\,(+)\,Z_{p^2}\right)$ is

$$\begin{pmatrix} 0 & d-1 & -(d+1) & -1 & \lambda_i \\ d-1 & \frac{p-1}{2} & \frac{p-3}{2} & (p-1)(d-1) & 1 \end{pmatrix}.$$

where $i = 1, 2$ and λ_1 and λ_2 are the roots of the quadratic equation

$$x^2 - (p^4 - 2p^3 - 1)x + p^6(1 - p) = 0.$$

The ensuing theorem shows that $G(R_2)$ is an eigensharp graph with $bp(G(R_2))$ equals to $a_-(G(R_2))$.

Theorem 2 *The unit graph $G(R_2)$ associated with the ring $R_2 = Z_{p^2}\,(+)\,Z_{p^2}$ is eigensharp.*

Proof. Assume O_i to be a subgraph of $G(R_2)$ induced by the set of vertices $V(O_i) = \{(i + jp, b); 0 \leq i \leq p-1, 0 \leq j \leq p-1, b \in Z_{p^2}\}$. Actually, the vertex set of $G(R_2)$ can be considered as $V = \bigcup_{i=0}^{p-1} V(O_i)$ where $O_0 \simeq N_{p^3}$ and for $i \geq 1$, $O_i \simeq K_{p^3}$. We denote each vertex of $V(O_i)$ as $\left\{a_t^i : 0 \leq t \leq p^3\right\}$. Then for $i \geq 1$, the set of edges in O_i can be covered by p^3-1 bicliques, which are the stars $B_i\left(\left\{a_t^i\right\}, Q_t^i\right)$, $1 \leq t \leq p^3 - 1$ and $Q_t^i = \left\{a_k^i : t < k\right\}$. Moreover, assume $X_r = V(O_r) \cup V(O_{p-r})$ and $Y_r = V(O_0) \cup V(O_{r+1}) \cup \cdots \cup V\left(O_{p-(r+1)}\right)$ where $1 \leq r \leq \left\lfloor \frac{p}{2} \right\rfloor$. Let $B(X_r, Y_r)$ be the biclique subgraph induced by the set of vertex of X_r and Y_r respectively. In fact, $B(X_r, Y_r) \simeq K_{|X_r|,|Y_r|}$ forms a biclique decomposition for all the edges that incident with a vertex in X_r and another in Y_r.

Now, consider the set $\mathcal{F}_G$ where

$$\mathcal{F}_G = \{(B_i\{a_t^i\}, Q_t^i), B(X_r, Y_r)\},$$

where $1 \leq i \leq p-1, 1 \leq t \leq p^3 - 1$, and $1 \leq r \leq \left\lfloor \frac{p}{2} \right\rfloor$. Since $\mathcal{F}_G$ can provide cover for every edge in $E(G(R_2))$, then $\mathcal{F}_G$ is a biclique decomposition of $G(R_2)$ of cardinality $(p-1)\left(p^3-1\right) + \binom{\frac{p-1}{2}}{}$. Now, from Section 3, we can show that $c_+\left((G(R_2))\right) = \frac{p+1}{2}$ and $c_-(G(R_2)) = (p-1)\left(p^3-1\right) + \binom{\frac{p-1}{2}}{}$ and so $bp\left(G(R_2)\right) = (p-1)\left(p^3-1\right) + \binom{\frac{p-1}{2}}{}$. Thus $G(R_2)$ is an eigensharp graph. ■

5 The Wiener Index for $G\;\;(Z_p\,(+)\,Z_p)$

This section determines the Wiener index for the unit graph $G(R_1)$ where $R_1 = Z_p\,(+)\,Z_p$. According to Remark 1, the unit graph $G(R_1)$ when $p = 2$ is isomorphic to the cycle graph C_4 and so, the Wiener index is equal to 8. In the following theorem, we determine the Wiener index for the graph $G(R_1)$ when $p \geq 3$.

Theorem 3 *The Wiener index of $G(R_1)$ is equal to $\frac{p(p-1)(p+1)^2}{2}$.*

Proof. Consider V to be the vertex set of $G(R_1)$. Let $H_i = \{(i, a) : a \in Z_p\}$ be a subset of V with p vertices and $0 \leq i \leq p-1$. Then $V = \bigcup_{i=0}^{p-1} H_i$ with $H_i \cap H_j = \phi$ for $i \neq j$. If $r \in H_i$ and $s \in H_{p-i}$, then $d(r, s) > 1$. More precisely, let $k \in H_j$, for some $j \neq i$, then $r \longrightarrow k \longrightarrow s$ is the shortest path from r to s and hence $d(r, s) = 2$. Otherwise, if $r \in H_i$ and $s \in H_j$ in which $i + j \neq 0 \pmod p$, then $d(r, s) = 1$. Therefore, for every $r, s \in V$ the determination of $d(r, s)$ depends on the sets H_i from which the two vertices are to be selected. Mainly, to determine the Wiener index for $G(R_1)$, we study the following cases:

Case 1: Assume that $r, s \in H_0$, then $d(r, s) = 2$ and hence

$$\sum_{r,s \in H_0} d(r, s) = 2 \begin{pmatrix} p \\ 2 \end{pmatrix} = p \times (p-1).$$

Case 2: Assume that $r \in H_0$ and $s \in H_i$ for some $i > 0$, then $d(r, s) = 1$ and hence

$$\sum_{r \in H_0, s \in H_i} d(r, s) = (p-1)p^2.$$

Case 3: If $r, s \in H_i$ for some $i > 0$, then $d(r, s) = 1$ and so

$$\sum_{\substack{r,s \in H_i \\ 0 < i \leq p-1}} d(r, s) = (p-1)\binom{p}{2} = \frac{1}{2}p(p-1)^2.$$

Case 4: Assume that

$$r = (i, d_1) \in H_i, s = (j, d_2) \in H_j$$

for some $i, j > 0$ and $j > i$, then

$$d(r, s) = \begin{cases} 1, & \text{if } j \neq p - i \\ 2, & \text{if } j = p - i \end{cases}.$$

Now, let us describe the H_i-subsets as follows:

$$H_1 = \{(1, b) : b \in Z_p\}$$

$$H_{p-1} = \{(p-1, b) : b \in Z_p\}.$$

$$H_k = \{(k, b) : b \in Z_p\}$$

and

$$H_j = \{(j, b) : b \in Z_p\}$$

where $k = 2, 3, \ldots, \lfloor \frac{p}{2} \rfloor$ and $j = \lceil \frac{p}{2} \rceil, \ldots, p-2$. If $r \in H_i$ and $s \in H_j$ in which $j = p - i$, then

$$\sum_{\substack{r \in H_i, s \in H_j \\ j=p-i}} d(r,s) = 2\left(\frac{p-1}{2}\right)p^2.$$

Otherwise, assume that $r \in H_i, s \in H_j$ such that $j \neq p - i$.

Suppose that $1 \leq i \leq \lfloor \frac{p}{2} \rfloor$ and $q = p - 1$. Hence, we have

$$\begin{aligned}
\sum_{1 \leq i \leq \lfloor \frac{p}{2} \rfloor} d(r,s) &= \sum_{1 \leq i \leq \lfloor \frac{p}{2} \rfloor} p^2 \left(q - (i+1)\right) \\
&= p^2(q-2) + p^2(q-3) + \cdots \\
&+ p^2 \left(q - \left\lceil \frac{p}{2} \right\rceil\right).
\end{aligned}$$

Now, suppose that $\lceil \frac{p}{2} \rceil \leq i < q$, then we have

$$\begin{aligned}
\sum_{\lceil \frac{p}{2} \rceil \leq i < p-1} d(r,s) &= \sum_{\lceil \frac{p}{2} \rceil \leq i < p-1} p^2 (q-i) \\
&= p^2 \left(q - \left\lceil \frac{p}{2} \right\rceil\right) \\
&+ p^2 \left(q - \left\lceil \frac{p}{2} \right\rceil - 1\right) \\
&+ p^2 \left(q - \left\lceil \frac{p}{2} \right\rceil - 2\right) + \cdots + p^2.
\end{aligned}$$

So, from the previous cases, we conclude that

$$\begin{aligned}
\sum_{\substack{r \in H_i, s \in H_j \\ 0 < i < j \leq p-1 \\ j \neq p-i}} d(r,s) &= \sum_{1 \leq i \leq \lfloor \frac{p}{2} \rfloor} d(r,s) \\
&+ \sum_{\lceil \frac{p}{2} \rceil \leq i < p-1} d(r,s) \\
&= \frac{1}{2} p^2 q (q-2).
\end{aligned}$$

Hence, the Winner index for the graph $G(R_1)$ is equal to

$$\begin{aligned}
W(G(R_1)) &= \sum_{r,s \in H_0} d(r,s) + \sum_{r \in H_0, s \in H_i} d(r,s) \\
&+ \sum_{\substack{r,s \in H_i \\ 0<i \leq p-1}} d(r,s) \\
&+ \sum_{\substack{r \in H_i, s \in H_j \\ j=p-i}} d(r,s) + \sum_{\substack{r \in H_i, s \in H_j \\ j \neq p-i}} d(r,s) \\
&= p(p-1) + 2p^2(p-1) \\
&+ \frac{1}{2} p (p-1)^2 + \frac{1}{2} p^2 (p-1)(p-3) \\
&= \frac{p(p-1)(p+1)^2}{2}.
\end{aligned}$$

■

6 The Wiener Index of $G\left(Z_{p^2}\left(+\right)Z_{p^2}\right)$

Assume $R_2 = Z_{p^2}(+)Z_{p^2}$. In the next theorem, we determine the value of wiener index of $G(R_2)$. Note that if $p = 2$, the graph $G(R_2)$ is isomorphic to the complete bipartite graph $K_{8,8}$ and hence its Wiener index is equal to 176, see, [24].

Theorem 4 *The Wiener index of $G(R_2)$ is equal to $\frac{1}{2}p^3\left(p^2-1\right)\left(p^3+p^2+p+1\right)$.*

Proof. Let $V = \{(a, b) : a, b \in Z_{p^2}\}$ be the vertex set of $G(R_2)$ with an order p^4. Let $A_i = \{((i + jp), b)$ $; j \in Z_p, b \in Z_{p^2}\}$ with $0 \leq i, j \leq p-1$ be a subset of V with an order p^3. Then $\mathcal{F} = \{A_i : 0 \leq i \leq p-1\}$ forms a disjoint decomposition for V. Consider $u = ((i_1 + j_1 p), b)$ and $v = ((i_2 + j_2 p), d)$ in V. The sum $u + v$ is a unit in R_2 if and only if $i_1 + i_2 \neq 0$. Therefore, we get $d(u, v) = 1$ for each pair with distinct vertices, u and v, in V unless, if $i_2 = p - i_1$. In fact, if $i_2 = p - i_1$, then $u \longrightarrow k \longrightarrow v$ is the shortest path from u to v where $k = (i_0 + p, 1) \in A_{i_0}$ for some $i_0 \notin \{i_1, p - i_1\}$. Thus $d(u, v) = 2$. We note that the identification of sets A_i, from which two different vertices u and v are selected, has a major role in calculating $d(u, v)$. We consider all possible cases to determine the Winner index of the graph $G(R_2)$.

Case 1: If $u, v \in A_0$, then $d(u, v) = 2$. Hence, $\sum_{u,v \in A_0} d(u,v) = 2\binom{p^3}{2}$

Case 2: If $u, v \in A_i$ where $1 \leq i \leq p-1$, then $d(u, v) = 1$ and so

$$\sum_{u,v \in A_i} d(u,v) = (p-1)\binom{p^3}{2}.$$

Case 3: If $u \in A_0$ and $v \in A_j$ where $1 \leq j \leq p-1$, we have $d(u,v) = 1$. Therefore

$$\sum_{u \in A_0, v \in A_j} d(u,v) = (p-1)p^6.$$

Case 4: Suppose that $u \in A_i$ and $v \in A_j$ with $i < j$ and $j = p - i$. Then $d(u,v) = 2$ and hence

$$\sum_{u \in A_i, v \in A_j} d(u,v) = 2\frac{(p-1)}{2}p^6.$$

Case 5: Suppose that $u \in A_i$ and $v \in A_j$ with $0 < i < j$ and $j \neq p - i$. Then we have the following subcases with $d(u,v) = 1$ in all of cases.

1) Assume $1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor$, then,

$$\sum_{\substack{u \in A_i, v \in A_j \\ 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor}} d(u,v) = p^6 \sum_{i=1}^{\left\lfloor \frac{p}{2} \right\rfloor} ((p-2) - i)).$$

2) For $\left\lceil \frac{p}{2} \right\rceil \leq i \leq p-2$, then

$$\sum_{\substack{u \in A_i, v \in A_j \\ \left\lceil \frac{p}{2} \right\rceil \leq i \leq p-2}} d(u,v) = \ p^6 \sum_{i=\left\lceil \frac{p}{2} \right\rceil}^{p-2} ((p-1) - i)).$$

Thus,

$$\begin{aligned}\sum_{u \in A_i, v \in A_j} d(u,v) &= p^6 \sum_{k=1}^{\left\lfloor \frac{p}{2} \right\rfloor} ((p-2) - k)) \\ &\quad + p^6 \sum_{k=\left\lceil \frac{p}{2} \right\rceil}^{p-2} ((p-1) - k)) \\ &= \frac{1}{2} p^6 (p-1)(p-3).\end{aligned}$$

where $0 < i < j \leq p - 1$ and $j \neq p - i$.

Therefore, we get that the Wiener index of $G(R_2)$ is equal to:

$$\begin{aligned}W(G(R_2)) &= \sum_{u,v \in V} d(u,v) \\ &= (p+1)\binom{p^3}{2} + 2(p-1)p^6 \\ &+ \frac{1}{2}p^6 (p-1)(p-3) \\ &= \frac{1}{2}p^3 (p^2 - 1)(p^3 + p^2 + p + 1)\end{aligned}$$

■

7 Conclusion

In this paper, we investigated the **Unit graph** constructed on the rings $R_1 = \mathbb{Z}_p(+)\mathbb{Z}_p$ and $R_2 = \mathbb{Z}_{p^2}(+)\mathbb{Z}_{p^2}$. We first examined the spectra of $G(R_1)$ and $G(R_2)$, followed by an analysis of the eigensharp property of $G(R_2)$. Furthermore, since the Wiener index serves as a fundamental tool for characterizing molecular graphs and linking molecular structure to their physical and chemical properties, we have determined the Wiener indices $W(G(R_1))$ and $W(G(R_2))$ of these unit graphs.

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