A Geometric and Computer-Simulation Approach to the Study of Blaschke Products and Dirichlet Functions

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Abstract: The purpose of this article is to systematically explore the geometric properties of finite and infinite Blaschke products, as well as of the Dirichlet functions generated by them. Computer-generated graphics are intended to illustrate symmetries, to portray their fundamental domains, and to explore their global mapping properties. They are also instrumental in the extrapolation of the properties of finite Blaschke products to the case of infinite ones, where non-isolated singular points appear. The behavior of an analytic function in the neighborhood of an isolated essential singular point is described by Picard's theorem, yet this theorem is not applicable to essential non-isolated singular points. The geometric approach we are using allows us to deal with such situations in both the cases of infinite Blaschke products and the Dirichlet functions generated by these products.

Key-Words: fundamental domains, Blaschke product, Dirichlet functions, analytic continuation, conformal mapping, pre-images of lines and circles

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1 Introduction

The duality analytic-geometric in the theory of complex functions was present since its inception in 19th century by Cauchy, Riemann and Weierstass. For Cauchy a holomorphic function was essentially a complex-differentiable function for which he discovered an integral representation. Cauchy integral formula implied that an analytic function is locally a power series. For Weierstrass, the complex function theory was the theory of power series, emphasizing as well the analytic aspect of this theory. To Riemann, holomorphic functions were mappings between domains of the complex plane, leading to the Riemann Mapping Theorem. The three methodologically different avenues merged and remained foundational in modern complex analysis.

The Blaschke products and the Dirichlet functions belong rather to the Weierstrass avenue and they were treated in this way for a century. However, we realized that since the concept of fundamental domain is applicable to these functions, there can be an advantage of a geometric approach in their study. We devised methods to visualize the fundamental domains of Blaschke products and Dirichlet functions obtaining interesting results in a few previous publications. This paper is a continuation of [1], and we begin with a deeper look into a function used there.

Example 1.1 Let us start with the simplest case of a second degree Blaschke product

$$B(z) = \frac{z-z_1}{1-\overline{z}_1 z}\frac{z-z_2}{1-\overline{z}_2 z} \quad (1)$$

A detailed study of such a function is presented in [1]. In the end, we chose values for the zeros of $B(z)$, yet formulas were established for arbitrary zeros $z_k = r_k e^{i\alpha_k}$, $k = 1, 2$, $0 < r_k < 1$, $z_1 \neq z_2$. With the purpose of adding new facts to it, we complete the figure from that paper and call it here Figure 1.

The complex plane is divided in this figure into two domains and their common boundary: Ω_1, Ω_2 and $\partial\Omega_1 = \partial\Omega_2$. The domain Ω_1 is bounded, while Ω_2 is unbounded. When we traverse $\partial\Omega_1$ such that Ω_1 remains to the left, call it oriented boundary $\partial\Omega_1$, the domain Ω_2 remains to the right. Thus, $\partial\Omega_1$ and $\partial\Omega_2$ have opposite orientations.

When referring to these boundaries, it will be obvious from the context if we are talking about point sets or oriented Jordan curves. Each one of the domains Ω_1 and Ω_2 is mapped conformally by $B(z)$ onto the complex plane with the same slit alongside the positive real half axis and two rays making the same angle γ with the positive real half axis, one from 0 to $B(\zeta_1)$ and the other from $B(\zeta_2)$ to infinity, where ζ_1 and ζ_2 are the zeros of $B'(z)$. For $A = (1-r_1^2)\overline{z}_2 + (1 - r_2^2)\overline{z}_1$, the equation $B'(z) = 0$ is equivalent to $Az^2 - 2(1 - r_1^2 r_2^2)z + \overline{A} = 0$, with the solutions $\zeta_{1,2} = \frac{1}{A}[1 - r_1^2 r_2^2 \pm \sqrt{(1 - r_1^2 r_2^2)^2 - |A|^2}]$. Thus, $\zeta_1 = \rho_1 e^{i\gamma}$ and $\zeta_2 = \rho_2 e^{i\gamma}$, ρ_1 and ρ_2 are $|\zeta_{1,2}|$. Moreover, $\zeta_2 = 1/\overline{\zeta}_1$.

Figure 1 above is obtained for $z_1 = 0.4 + 0.2i$

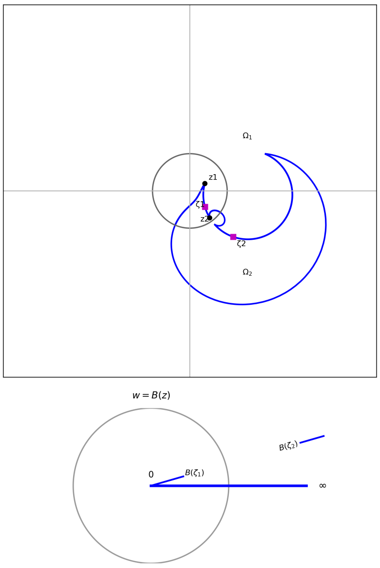

Fig. 1: An illustration of fundamental domains of a second degree Blaschke product and their conformal mapping. Source: Created by the authors.

and $z_2 = 0.53 - 0.72i$. For these values we get the approximate values $r_1 = 0.44$, $r_2 = 0.89$, $A = 0.502488 + 0.53913i$, $|A|^2 = 0.543155$, $|A| = 0.73699$, $\sin\gamma = 0.73153$, $\cos\gamma = 0.71219$, $\rho_1 = 0.495$, $\rho_2 = 2.0196$, thus $\rho_1\rho_2 = 1$, hence $\zeta_1 = 0.2487 + 0.2679i$, $\zeta_2 = 1.0148 + 1.0888i$

Let us denote by $B_{|\Omega_k}(z)$the restriction of $B(z)$ to Ω_k, $k = 1, 2$. Since $B_{|\Omega_k}(z)$ are conformal mappings, hence they map bijectively the domains Ω_k onto the complex plane with the same slit, we can define the function $\varphi(z) = B^{-1}_{|\Omega_2} \circ B_{|\Omega_1}(z)$, $z \in \Omega_1$, which maps conformally Ω_1 onto Ω_2. Then $B \circ \varphi(z) = B(z)$ for every $z \in \Omega_1$. By Riemann-Caratheodory theorem of boundary correspondence by a conformal mapping, $\varphi(z)$ can be extended to a continuous bijective mapping of $\overline{\Omega}_1 = \Omega_1 \cup \partial\Omega_1$ onto $\overline{\Omega}_2 = \Omega_2 \cup \partial\Omega_2$. We keep the notation $\varphi(z)$ for this extended function. The functions $B(z)$ and $\varphi(z)$ being continuous on $\overline{\Omega}_1$, for every $\zeta \in \partial\Omega_1$ we have $\varphi(\zeta) = \lim_{z\to\zeta} B^{-1}_{|\overline{\Omega}_2} \circ B_{|\overline{\Omega}_1}(z) = B^{-1}_{|\overline{\Omega}_2} \circ B_{|\overline{\Omega}_1}(\zeta)$, and then $B \circ \varphi(\zeta) = B(\zeta)$ for every $\zeta \in \partial\Omega_1$, i.e. $B \circ \varphi(z) = B(z)$ for every $z \in \overline{\Omega}_1$.

Since $B(z)$ maps topologically ∂D onto ∂D, if $\xi = e^{it}$ then $B(\xi) = e^{i\chi(t)}$ for some continuous real function $\chi(t)$. As shown in [2], we have $\chi'(t) > 0$, hence when t increases, ξ moves counterclockwise on the unit circle and $\chi(t)$ increases, i.e. $B(\xi)$ moves also counterclockwise on the unit circle. The function $B(z)$ maps topologically the arc of the unit circle included in Ω_1 onto the arc of the unit circle included in Ω_2. With the notations of Figure 1, this implies $B(\xi_1) = \xi_2$ and $B(\xi_2) = \xi_1$. The previous remarks imply that $\varphi(\xi_1) = \xi_2$ and $\varphi(\xi_2) = \xi_1$. Since ξ_1 and ξ_2 are at the intersection of the pre-image by $B(z)$ of the unit circle and of the real axis, we have also $B(\xi_1) = B(\xi_2) = 1$.

An easy computation shows that for every Blaschke product $B(z)$ we have:

$$B(z) = 1/\overline{B}(1/\overline{z}) \qquad (2)$$

for every $z \in \overline{\mathbb{C}}$, with the convention $\infty = 1/0$. It is enough to check (2) for a Blaschke factor. By (2) we have $\overline{B}(\xi_k)B(1/\overline{\xi}_k) = 1$, $k = 1, 2$, thus $B(1/\overline{\xi}_k) = 1$, which implies $\xi_2 = 1/\overline{\xi}_1$ and $\xi_1 = 1/\overline{\xi}_2$. Moreover, when $B_{|\overline{\Omega}_1}(z)$ moves from 1 to 0 on the real axis, the point z moves from ξ_1 to z_1 on the pre-image of that ray and when $B_{|\Omega_2}(z)$ does the same thing, the point z moves from ξ_2 to z_2, thus when z is in z_1, the point $\varphi(z)$ must be in z_2, i.e. $\varphi(z_1) = z_2$. From there, if z moves on the oriented boundary $\partial\Omega_1$, towards ζ_1, the point $\varphi(z)$ must move on the oriented boundary $\partial\Omega_2$ towards the same point and z and $\varphi(z)$ arrive simultaneously at ζ_1, i.e. $\varphi(\zeta_1) = \zeta_1$. Obviously the same happens with ζ_2, i.e. $\varphi(\zeta_2) = \zeta_2$, thus ζ_1 and ζ_2 are fixed points of $\varphi(z)$.

We can extend $\varphi(z)$ into Ω_2 in the following way. Let η be a small Jordan arc included in Ω_1 having the ends a and $b = \varphi(a)$ on $\partial\Omega_1$ and such that a, ζ_1 and b follow in a direct trigonometric order. Let $\eta' = \varphi(\eta)$.The arcs η, η' and the part Γ of $\partial\Omega_1$ between them bound the domains $D_k \subset \Omega_k$, $k = 1, 2$. By the conformal mapping theorem, $\varphi(z)$ maps conformally D_1 onto D_2 and the arc Γ onto itself, such that if $\zeta \in \Gamma$, we have $\varphi \circ \varphi(\zeta) = \zeta$. The function φ is extended to a topological mapping of $\overline{D}_1$ onto $\overline{D}_2$ having the fixed point ζ_1.

Theorem 1.1. Let $u(z) = \varphi(z)$ if $z \in \overline{\Omega}_1$ and $u(z) = \varphi^{-1}(z)$ if $z \in \overline{\Omega}_2$.The function $u(z)$ is a conformal mapping of Ω_1 onto Ω_2 and of Ω_2 onto Ω_1 and it is an invariant of $B(z)$. It maps one to one the common boundary of the two domains onto itself and it is an involution of $\widehat{\mathbb{C}}$ with two fixed points: ζ_1 and ζ_2.

Proof: Let us check that $u(z)$ is an involution of $\overline{D}_1 \cup \overline{D}_2$ having the fixed point ζ_1. Obviously, $u(z)$ maps $\overline{D}_1 \cup \overline{D}_2$ onto itself and $u \circ u(z) = z$ for every $z \in \overline{D}_1 \cup \overline{D}_2$. Thus, $u(z)$ is an involution of $\overline{D}_1 \cup \overline{D}_2$ and since $\varphi(z)$ has the fixed point ζ_1, this is also true for $u(z)$. It maps topologically the arc Γ onto itself.

We can chose a such that Γ becomes $\partial\Omega_1$. Indeed, when $a = z_2$, we have $b = z_1$ and when $a = \xi_2$, we have $b = \xi_1$.Finally, when $a = 1/\overline{z}_2$, we have

$b = 1/\overline{z}_1$. The point $\zeta_2 \in \partial\Omega_1$ is between $1/\overline{z}_1$ and $1/\overline{z}_2$. We can repeat the construction of domains D_3 and D_4 this time around ζ and conclude that $\varphi(z)$ maps conformally D_3 onto D_4 and D_4 onto D_3 and the arc Γ' between them onto itself, such that if $\zeta \in \Gamma'$, we have $\varphi \circ \varphi(\zeta) = \zeta$. The function φ is extended to a topological mapping of $\overline{D}_3$ onto $\overline{D}_4$ having the fixed point ζ_2. Continuing to move a on the oriented boundary $\partial\Omega_1$, when it arrives in $1/\overline{z}_1$ its image b must be in $1/\overline{z}_2$. Thus, when a describes that oriented boundary $\partial\Omega_1$, its image by $u(z)$ describes the oriented boundary $\partial\Omega_2$.Moreover, $u \circ u(z) = z$ for every $z \in \widehat{\mathbb{C}}$.$\Diamond$

We notice that this result completes that from [2], [3], for this particular case.

2 Foundations of Blaschke Products

In this section we recall the basic properties of finite and infinite Blaschke products: the symmetry property, critical points, the behavior on the unit circle and the motivation for a geometric analysis. In the following we will use the notation D for the open unit disc, and ∂D for the unit circle.

2.1 Finite and infinite Blaschke products

A finite Blaschke product of degree n is a function of the form

$$B(z) = e^{i\alpha}\Pi_{k=1}^{n}\frac{z-z_k}{1-\overline{z}_k z},\ z_k \in D, \alpha \in \mathbb{R} \quad (3)$$

This is a meromorphic function in the complex plane having the zeros z_k and the poles $1/\overline{z}_k$. We can extend $B(z)$ to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by setting

$$B(\infty) = \lim_{z\to\infty} B(z) = e^{i\alpha}\Pi_{k=1}^{n}\frac{(-1)^k}{\overline{z}_k} \quad (4)$$

The functions $b_k(z) = \frac{z-z_k}{1-\overline{z}_k z}$ are called *Blaschke factors* and they satisfy two important relations:

$$|b_k(e^{i\theta})| = |\frac{e^{i\theta}-z_k}{1-e^{-i\theta}\overline{z}_k}| = \frac{|e^{i\theta}-z_k|}{|e^{-i\theta}|\,|\overline{e^{i\theta}-z_k}|} = 1,\ \theta \in \mathbb{R} \quad (5)$$

$$b_k(1/\overline{z}) = \frac{1/\overline{z}-z_k}{1-z_k/\overline{z}} = \frac{1-z_k\overline{z}}{\overline{z}-z_k} = 1/\overline{b}_k(z),\ z \in \mathbb{C} \quad (6)$$

An easy computation shows that the relation (5) and (6) imply similar relations for $B(z)$, namely:

$$|B(e^{i\theta})| = 1,\ \theta \in \mathbb{R} \quad (7)$$

$$B(1/\overline{z}) = 1/\overline{B}(z) \quad (8)$$

We call (8) the *inversion relation* for $B(z)$.

Let us notice that $|b_k(z)|^2 = \frac{|z|^2-2\operatorname{Re}\overline{z}_k z+|z_k|^2}{1-2\operatorname{Re}\overline{z}_k z+|z|^2|z_k|^2}$, and since the numerator and the denominator are non-negative, the inequality $|b_k(z)|^2 < 1$ is equivalent to $|z|^2 - 2\operatorname{Re}\overline{z}_k z + |z_k|^2 <$ $1 - 2\operatorname{Re}\overline{z}_k z + |z|^2|z_k|^2$, which means $(1 - |z_k|^2)(1 - |z|^2) > 0$ and since $|z_k|^2 < 1$, this is equivalent to $|z| < 1$. Thus $|b_k(z)| < 1$ if and only if $|z| < 1$ and $|b_k(z)| > 1$ if and only if $|z| > 1$, which implies that $|B(z)| < 1$ if and only if $|z| < 1$ and $|B(z)| > 1$ if and only if $|z| > 1$. The final conclusion is that every finite Blaschke product maps D onto itself, ∂D onto itself and $\widehat{\mathbb{C}} \setminus (D \cup \partial D)$ onto itself.

An infinite Blaschke product is a function of the form:

$$B(z) = e^{i\alpha}\Pi_{n=1}^{\infty}\frac{|z_n|}{z_n}\frac{z-z_n}{1-\overline{z}_n z},\ z_n \in D, \alpha \in \mathbb{R} \quad (9)$$

It is known that the infinite product (9) converges uniformly on compact subsets of D if and only if the *Blaschke condition*

$$\Sigma_{n=1}^{\infty}(1 - |z_n|) < \infty \quad (10)$$

is satisfied. Upon this condition, the function $B(z)$ is an analytic function in D. This means that for every $z_0 \in D$, we have $B(z) = \Sigma_{k=0}^{\infty}\frac{B^{(k)}(z_0)}{k!}(z - z_0)^k$, where this series converges uniformly in a disc $|z - z_0| < r$, for some positive value of r which depends of z_0. If $r > 1 - |z_0|$, the sum of this series exists outside D and coincide with $B(z)$ in D. The extended function $B_1(z)$ obtained in this way is called *direct analytic continuation of* $B(z)$. As we will see in the next section, the set E of essential singular points of $B(z)$ is located on ∂D. Thus, if an arc $\eta \subset \partial D$ does not contain any essential singular point, such z_0 and r can be found for a direct continuation of $B(z)$ across η. Then the inversion relation (8) allows the extension of $B(z)$ as a meromorphic function in $\widehat{\mathbb{C}} \setminus E$. It also maps D onto itself, $\partial D \setminus E$ onto ∂D and $\widehat{\mathbb{C}} \setminus \overline{D}$ onto itself. Indeed, since for $|z| < 1$ we have $|b_k(z)| < 1$ for any Blaschke factor $b_k(z)$, this implies $|B(z)| < 1$ for $|z| < 1$. Also, $|z| = 1$ implies $|b_k(z)| = 1$ for every k, hence $|B(z)| = 1$. Finally, $|z| > 1$ implies $|b_k(z)|$ > 1 for every k, thus $|B(z)| > 1$.

2.2 Essential singular points of infinite Blaschke products

Since the function (9) has infinitely many zeros z_n in D, there should be at least one cluster point a of these zeros in $\overline{D} = D \cup \partial D$. The *uniqueness principle* of analytic functions asserts that whenever two analytic functions agree on a sequence of points which cluster inside their common domain of analyticity, they must agree everywhere. Thus, if $a \in D$, then $B(z)$ should be identically zero in D, which is false, hence $a \in \partial D$ and $|a| = 1$. This implies that $a = \lim_{k\to\infty} z_{n_k}$ for some subsequence (z_{n_k}) of zeros such that $\lim_{k\to\infty} |z_{n_k}| = 1$. This last equality holds if and only if $\lim_{k\to\infty} |1/\overline{z}_{n_k}| = 1$, thus $a = \lim_{k\to\infty} 1/\overline{z}_{n_k}$. For this reason, we call a *essential singular point of* $B(z)$. Indeed, when $B(z)$ is extended to $\widehat{\mathbb{C}}$, the points $1/\overline{z}_n$ are poles of the extended function, thus a appears as a limit of poles, which means that it is an non-isolated singular point. Let us denote by E the set of essential singular points of $B(z)$, which is exactly the set of cluster points of $(1/\overline{z}_n)$. Thus, if $B(z)$ has a continuation outside $\overline{D}$, that continuation is a meromorphic function in $\widehat{\mathbb{C}} \setminus E$ having the zeros z_n and the poles $1/\overline{z}_n$. It satisfies the inversion formula (8) in $\widehat{\mathbb{C}} \setminus E$, where by $1/0$ we understand ∞.

2.3 Critical points and the derivative

Let $B(z)$ be a Blaschke product of degree n. It is a rational function of degree n and therefore its derivative $B'(z)$ is also rational of degree at most $2n-2$. We have seen in Example 1 that the zeros of $B'(z)$ were a reflection in the unit circle one of each other. An elementary, but relatively tedious computation shows that for any finite Blaschke product the zeros of the derivative are two by two reflection in the unit circle one of the other, in other words, $B'(\zeta_0) = 0$ if and only if $B'(1/\overline{\zeta}_0) = 0$. Thus, $B'(z)$ has exactly $n-1$ zeros in D counted with the multiplicities, and $n-1$ zeros symmetric with the first ones with respect to ∂D. These are critical points of $B(z)$. Other critical points are the multiple poles of $B(z)$

A trivial example is $B(z) = z^k$ with $z = 0$ being a zero of order of multiplicity k. The disc $|z| < 1$ is divided by the rays $z = \rho e^{2j\pi i/k}$, $j = 0, 1, ..., k-1$ into k sectors which are conformally mapped by $B(z)$ onto the unit disc with a slit alongside the interval $(0, 1)$ of the real axis. The pre-image by $B(z)$ of this interval is formed with the k rays. If $B_1(z)$ is a Blaschke product which does not cancel at the origin, and $B(z) = z^k B_1(z)$, then the pre-image by $B(z)$ of the interval $(0, 1)$ of the real axis is formed with k Jordan arcs connecting $z = 0$ with the unit circle and having no other common point, as in Figure 2 below.

2.4 The behavior of Blaschke products on ∂D

As shown in [3], for any Blaschke product $B(z)$, we have:

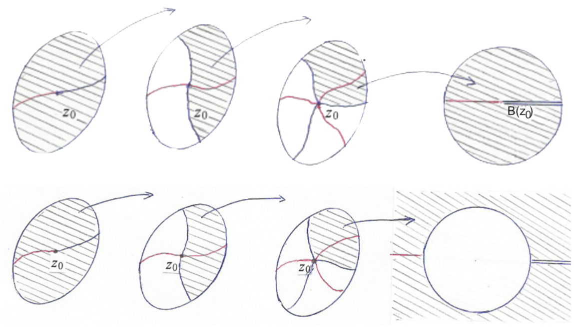

Fig. 2: The behavior of $B(z)$ in the neighborhood of a critical point order k, $k = 0, 1, 2$.Source: Created by the authors.

$$B(e^{it}) = e^{i\chi(t)},\ t \in \mathbb{R} \quad (11)$$

where $\chi(t)$ is a strictly increasing continuous real function. This implies that $B(z)$ maps $\partial D \setminus E$ onto ∂D with positive orientation. Moreover, for a Blaschke product of degree n, the equation $B(z) = 1$ has exactly n distinct solutions, no matter if the zeros of $B(z)$ are distinct or not. Thus, there is a one to one correspondence between the degree of $B(z)$ and the number of solutions of the equation $B(z) = 1$

The existence of $B(e^{it})$ for an infinite Blaschke product is not guaranteed, since $B(z)$ does not exist for $z = e^{it}$ with $z \in E$. Yet, when $e^{it} \in \partial D \setminus E$, and $B(e^{it})$ is defined, then the equality (11) holds and the equation $B(z) = 1$ must have infinitely many solutions. These solutions cannot accumulate on arcs of ∂D where $B(z)$ is defined, since this would imply that $B'(z)$ is identically zero. We conclude that they must accumulate to E.

A study of the behavior of infinite Blaschke products on ∂D can be found in [3], but only for a discrete set E. On the other hand, we can easily construct infinite Blaschke products satisfying the condition (10) and having a dense set E.

Indeed, let

$$z_{n,k} = (1 - 3^n)e^{k\alpha i/2^{n-1}},$$
$$n = 1, 2, ...; k = 1, 2, ..., 2^n, 0 < \alpha < 2\pi \quad (12)$$

It can be easily checked that

$$\Sigma_{n=1}^{\infty} \Sigma_{k=1}^{2^n} (1 - |z_{n,k}|) < \Sigma_{n=1}^{\infty} 2^n/3^n = 2 \quad (13)$$

Thus, for $B(z)$ with the zeros $z_{n,k}$, the Blaschke condition is satisfied and these zeros accumulate to almost every point of the arc of ∂D spanned by e^{it}, $0 < t < \alpha$. Hence, if $B(z)$ admits a direct analytic continuation across the complement with respect to the unit circle of this arc, the equation $B(e^{it}) = 1$ will have infinitely many solutions on it. These solutions can only accumulate to E, which means that smaller and smaller arcs of ∂D will be topologically mapped by $B(z)$ onto the whole ∂D. The conditions under which a direct continuations exists will be presented in Section 4.

2.5 Motivation for geometric analysis

The Weiersrass continuation had the purpose to reveal the maximum domain of an analytic function defined by a power series. Unfortunately, it can bring in some cases to an entity which is not really a function, called *many-valued function*. This impasse was serious and in an effort to solve it, the *abstract theory of Riemann surfaces* has been born. This theory is a masterpiece of mathematical science and it has at the basis many layers of topology.

By the time one is immerse into this theory, it can happen that one forgets about the motivation of this monumental work, which can be found in [4]. In [5], commenting on the naive representation of the Riemann surfaces attached to elementary functions, it is recalled: *Whatever the advantage of such a representation may be, the clearest picture of the Riemann surface is obtained by direct consideration of the fundamental regions in the z-plane.*

The Example 1 was given to show such a representation, i.e. to show how different conformal mappings work together to generate an analytic function. These were mappings of fundamental domains of the respective second degree Blaschke product.

Identifying those domains is a matter of geometry. We prove in this paper that it can be done for any Blaschke product, as we have done it in previous papers for different other classes of analytic functions. Relevant is here the class of Dirichlet functions, since we can obtain such functions from Blaschke products by a simple change of variable, which preserves the strip representation well studied for Dirichlet L-functions. We use extensively the tool of continuation along an arc, or lifting of an arc, [4].

3 Finite Blaschke Products: Geometry and Fundamental Domains

In this section we deal with arbitrary finite Blaschke products (4), as meromorphic functions in $\widehat{\mathbb{C}}$, describe their geometric properties, the behavior of finite Blaschke products on ∂D, the relation between their zeros and their poles, as well as between their zeros and the zeros of their derivatives. By using continuation over arcs, the geometric construction can be extended to infinite Blaschke products.

3.1 Preimages of the positive real half axis

Let $B(z)$ be the function (4) with $z_k = r_k e^{i\alpha_k}$, $0 < r_k < 1, k = 1, 2, ..., n$, $\alpha \in \mathbb{R}$

Let us denote by $\xi_1, \xi_2, ..., \xi_n$ the n distinct solutions of the equation $B(z) = 1$ described in the Section 2.4. If we perform continuation along the positive real half axis starting from every ξ_k, we get Jordan arcs γ_k connecting the zeros z_k of $B(z)$ with the corresponding poles $1/\overline{z}_k$. Each arc γ_k is mapped topologically by $B(z)$ onto the positive real half axis, with $B(z_k) = 0$, $B(\xi_k) = 1$ and $B(1/\overline{z}_k) = \infty$

Theorem 3.1. Two different arcs γ_k and $\gamma_{k'}$ can have in common only their ends.

Proof: By an extension of Gauss-Lucas theorem to finite Blaschke products, [6], the zeros of $B'(z)$ inside D are contained in the hyperbolic convex hull of the zeros of $B(z)$. If γ_k and $\gamma_{k'}$ would meet at a point z_0 inside D, this would be a branch point of $B(z)$, thus a zero of $B'(z)$. Yet, the continuations along the positive real half axis starting from zeros of $B(z)$ are outside that convex hull, thus γ_k and $\gamma_{k'}$ cannot intersect in D. The reflection formula (8) shows that they cannot intersect outside D neither$\Diamond$.

The extension of Theorem 1.1 to the case of a Blaschke product of degree n is straightforward.

Theorem 3.2. Every Blaschke product (4) partitions the complex plane into n subsets whose interiors Ω_k are fundamental domains of $B(z)$. The function $B(z)$ maps conformally every domain Ω_k onto the complex plane with a slit L_0 alongside the positive real half axis plus a part L_k of an infinite ray starting at the origin.

Proof: The partitions are not unique and there are different ways to obtain them. However, for any such partition, the branch points of $B(z)$, which are the zeros of $B'(z)$ and the multiple poles of $B(z)$ must be located on the boundaries of the fundamental domains, since inside these domains the function is injective, property which is violated at the branch points. Thus we need to chose these boundaries accordingly.

Let us count the solutions ξ_j of the equation $B(z) = 1$ starting with an arbitrary one and going counterclockwise on the unit circle. Through every point ξ_j passes a component γ_j of the pre-image of the positive real half axis connecting a zero z_j of $B(z)$

with the pole $1/\overline{z}_j$,the symmetric of z_j with respect to the unit circle. Indeed, $w = 1$ is a point on the positive real half axis and $B(\xi_j) = 1$, thus ξ_j belongs to a component γ_j of the pre-image of the positive real half axis. With the convention $\xi_{n+1} = \xi_1$ we have that for every j, the points ξ_j and ξ_{j+1} are consecutive on the unit circle. They determine adjacent arcs η_j on the unit circle, which with an end removed, are topologically mapped by $B(z)$ onto the unit circle, as shown in Section 1. In the case of multiple zeros of $B(z)$, we count those zeros repeatedly to match the counting of ξ_j and that of γ_j, in other words if z_j is a zero of order q_j of $B(z)$, then we use q_j notations for it which fit with those of ξ_j and η_j. Consecutive arcs γ_j and γ_{j+1} which meet at the same multiple zero z_k and the same multiple pole of $B(z)$, the symmetric of z_k with respect to the unit circle, are the boundaries of fundamental domains Ω_j of $B(z)$. Indeed, by Gauss-Lucas theorem, Ω_j does not contain any zero of $B'(z)$, i.e. there are no branch points of $B(z)$ in Ω_j, so $B(z)$ is a conformal mapping of Ω_j onto the complex plane with a slit alongside the positive real half axis.

An elementary computation shows that $B'(z)$ has exactly $n - 1$ zeros ζ_k in the unit disc, counted with multiplicities and $n - 1$ zeros $1/\overline{\zeta}_k$, symmetric of ζ_k with respect to the unit circle. These zeros are branch points of $B(z)$, thus they should be located on the boundaries of the fundamental domains. This happens obviously in the case when the zeros z_k are multiple ones. Suppose that $B(z)$ has exactly $p \geq 0$ multiple zeros z_k of order of multiplicity $q_1, q_2, ..., q_p$ such that $0 \leq q = \Sigma_{j=1}^{p} q_j \leq n$. Then it has also $n - q$ simple zeros. On the other hand, $B'(z)$ has p zeros of order of multiplicity respectively $q_1 - 1, q_2 - 1, ..., q_p - 1$ which are multiple zeros of $B(z)$ and counting them with multiplicities, we obtain $q - p$ such zeros of $B'(z)$. Then $n - 1 - q + p$ are simple zeros ζ_k of $B'(z)$ in the unit disc, different from the zeros of $B(z)$.

To obtain fundamental domains of $B(z)$ having these zeros on their boundaries, we proceed as follows. We connect the points $w_k = B(\zeta_k)$ with the origin by a ray, and continue this ray from $B(1/\overline{\zeta}_k)$ to infinity. Let L_k be the slit alongside this part of the ray. The pre-image of L_k contains arcs connecting every zero of $B(z)$ with ζ_k and every pole of $B(z)$ with $1/\overline{\zeta}_k$. We denote by χ'_k the arc connecting the $n - q + p$ consecutive (simple or multiple) zeros of $B(z)$, passing through the closest ζ_k and being part of this pre-image and by χ''_k the symmetric of χ'_k with respect to the unit circle. These curves are mapped two to one by $B(z)$ onto the slit L_k. It can be easily seen that χ_k and χ'_k together with γ_k and γ_{k+1} determine domains which are conformally mapped by $B(z)$ onto the complex plane with a slit alongside the positive real half axis and a part L_k of the ray passing through $B(\zeta_k)$.

The arithmetic above shows that there are exactly n fundamental domains, all except one, bounded.$\Diamond$

If $H \subset \widehat{\mathbb{C}}$ is any point set and $f(z)$ is a function defined on a subset of $\widehat{\mathbb{C}}$, we will use freely the notations $f(E) = \{f(z) \mid z \in E\}$ and $f^{-1}(E) = \{z \mid f(z) \in E\}$. This convention cannot produce any confusion.

3.2 Reduced fundamental domains

Let $B(z)$ be a Blaschke product of degree n with the fundamental domains Ω_k, which are each one conformally mapped by $B(z)$ onto the complex plane with a slit alongside the positive real half axis L_0 and the part L_k of a ray from from origin passing through $B(\zeta_k)$. We look at these slits as point sets and we define $L_B = \cup_{k=0}^{n} L_k$ and $\Gamma_B = B^{-1}(L_B)$. We notice that Γ_B is a mesh in $\mathbb{C}$ whose eyes are the simply connected domains $\Omega_k^{(r)} = \Omega_k \backslash \Gamma_B$ which are mapped each one conformally by $B(z)$ onto $\mathbb{C} \setminus L$. They are still a fundamental regions of $B(z)$, as defined in [6]. We shell call them *reduced fundamental domains* of $B(z)$.The following affirmation is a straightforward corollary of the Theorem 3.

Theorem 3.3. Every Blaschke product $B(z)$ of degree n partitions the complex plane into n sets whose interior $\Omega_k^{(r)}$ are reduced fundamental domains of $B(z)$. Each one of them is conformally mapped by $B(z)$ onto the complex plane with the same slit L.$\Diamond$

3.3 Transition maps and invariants

A construction similar to that of Theorem 1.1 can be made. Namely, let $u_{k,j}(z) = B^{-1}_{|\Omega_k^{(r)}} \circ B_{|\Omega_j^{(r)}}(z)$. This is a conformal mapping of $\Omega_j^{(r)}$ onto $\Omega_k^{(r)}$. It can be extended to a continuous mapping of $\overline{\Omega_j^{(r)}}$ onto $\overline{\Omega_k^{(r)}}$ such that $\partial\Omega_j^{(r)}$ is mapped topologically onto $\partial\Omega_k^{(r)}$.

The functions $u_{k,j}$ are invariants of $B(z)$, as defined in [3], yet they are extended this time to the whole reduced fundamental domain $\Omega_j^{(r)}$ and $B \circ u_{j,k}(z) = B(z)$, $z \in \Omega_j^{(r)}$.

3.4 An illustration of the theory

Example 3.1. We illustrate the result of Theorem 3.3 with the example of a six degree Blaschke product having the zeros:

$z_1 = z_2 = 0.8$, $z_3 = 0.8i$, $z_4 = z_5 = -0.8$ and $z_6 = -0.8i$. In this example $n = 6$ and $p = 4$.

$$B(z) = \frac{P(z)}{Q(z)}$$

$$= \frac{(z-0.8)^2}{(1-0.8z)^2}\frac{(z+0.8)^2}{(1+0.8z)^2}\frac{(z-0.8i)}{(1+0.8iz)}\frac{(z+0.8i)}{(1-0.8iz)}$$

$$= \frac{(z^2-0.64)^2}{(1-0.64z^2)^2}\frac{z^2+0.64}{1+0.64z^2}$$

$$= \frac{z^6-0.64z^4-04096z^2+0.262144}{1-0.64z^2-0.4096z^4+0.262144z^6} \quad (14)$$

Let us notice that we have the approximate values: $B(0) = 0.262144$ and $B(\infty) = 1/B(0) = 3.8147$

Let us notice that ± 0.8 are also zeros of $B'(z)$, which we denote by ζ_1 and ζ_2. An elementary computation allows us to find the remaining three zeros of $B'(z)$ located in the unit disc: $\zeta_3 = -0.43i$, $\zeta_4 = 0$ and $\zeta_5 = 0.43i$. It is known that the symmetric of the zeros of $B(z)$ with respect to the unit circle are poles of $B(z)$, therefore also of $B'(z)$. They are: ± 1.25, $\pm 2.33i$ and ∞. We have $B(\pm 0.43i) = 0.28$ and $B(\pm 2.33i\ ) = 3.75$.

For r small enough, the pre-image by $B(z)$ of the circle $|w| = r$ (color red) is formed with closed curves (the same color) around the zeros $\pm\ 0.8$ and $\pm\ 0.8i$ of $B(z)$. Two points on every curve around $\pm\ 0.8$ are mapped by $B(z)$ onto $w = r$ and each arc determined by them, with an end removed is topologically mapped by $B(z)$ onto the circle $|w| = r$. The curves around $\pm\ 0.8i$ are also topologically mapped by $B(z)$ onto the same circle. When r increases, these curves expand and when $r = B(0) = 0.262144$, the curves around $\pm\ 0.8$ touch each other at $z = 0$ (the orange curves), after which they fuse into a unique closed curve containing both zeros. This curve will touch the curves around $\pm\ 0.8i$ when $r = B(\zeta_3) = B(\zeta_5) = 0.28$ (the magenta curves) at $\pm 0.43i$ and then all three will fuse into a unique closed curve around all zeros (green curve), which will become the unit circle when $r = 1$ (black curve).

We keep the same colors for the symmetric curves with respect to the unit circle, which are pre-images by $B(z)$ of the circles $|w| = 1/r$. These are unbounded curves around ± 1.25 when $r = 0.262144$. They fuse into a unique closed curve for bigger values of r. This curve touches the pre-image of the circle $|w| = 1/0.28$ at $\pm i/0.43$ when $r = 0.28$ (orange curve) and fuse with it into a unique closed curve around the unit circle leaving the poles outside (the green curve) and becoming the unit circle when $r = 1$.These affirmations are illustrated in Figure 3 below

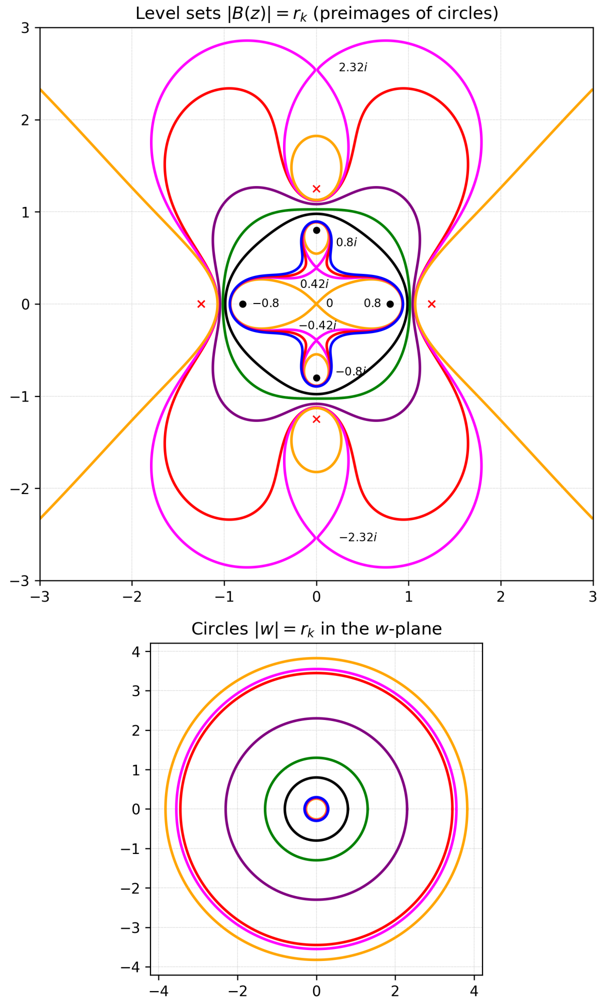

Fig. 3: Pre-images by a six degree Blaschke product of some important circles centered at the origin. Source: Created by the authors.

Following this example, we can conceive a way to deal with an arbitrary finite Blaschke product $B(z) = \Pi_{k=1}^{q}(\frac{z-z_k}{1-\bar{z}_k})^{p_k}$, $p_1 + p_2 + ... + p_q = n$, where z_k are distinct zeros of $B(z)$. For r small enough, the pre-image by $B(z)$ of the circle $|w| = r$ is formed with disjoint closed Jordan curves η_k around the zeros z_k. By the Section 2.3, there are p_k distinct points on every η_k which are mapped by $B(z)$ onto $w = 1$ and the arcs determined by them on η_k, with an end removed, are topologically mapped by $B(z)$ onto the circle $|w| = r$.

When r increases, these curves expand and they touch two by two at the zeros ζ_j of $B'(z)$, after which they fuse into a unique curve containing both zeros. For bigger r, more zeros will be included into the fused curves and there is a value $r < 1$ such that the pre-image by $B(z)$ of the circle $|w| = r$ is a closed Jordan curve containing all the zeros of $B(z)$.

The symmetric of these curves with respect to the unit circle are Jordan curves around the poles of $B(z)$ and they follow the same rules as those interior to the unit circle. These are simple evident topological facts.

We can use the Blaschke product of the Example 2 to illustrate also the Theorem 3. The pre-image by $B(z)$ of the positive real half axis is formed with a couple of two Jordan arcs connecting the double zeros ± 0.8 of $B(z)$ with the symmetric poles ± 1.25 with respect to the unit circle. Indeed, it can be easily checked that $\operatorname{Im} z = 0$ implies $\operatorname{Im} B(z) = 0$ and since $B(\pm 0.8) = 0$ and $B(\pm 1.25) = \infty$, the intervals $(0.8, 1.25)$ and $(-1.25, -0.8)$ of the real axis are mapped by $B(z)$ onto $(0, +\infty)$. Moreover, since ± 0.8 are double zeros of $B(z)$, two different arcs starting at these points are mapped onto the real axis in a neighborhood of ± 0.8, [5]. Continuations by $B(z)$ of these arcs over the real axis produce the second couple of Jordan arcs connecting ± 0.8 and ± 1.25 and which are mapped by $B(z)$ onto the positive real half axis. The domains bounded by these arcs are fundamental domains of $B(z)$ which are mapped conformally by $B(z)$ onto the complex plane with a slit alongside the positive real half axis. They are denoted Ω_1 and Ω_4 in Figure 4 below.

It can be also easily checked that $\operatorname{Re} z = 0$ implies $\operatorname{Im} B(z) = 0$, i.e. the imaginary axis is mapped by $B(z)$ also onto the real axis, which with a little analysis reveals the other four fundamental domains of $B(z)$, as shown in Figure 4.

4 Dirichlet Functions Generated by Final Blaschke Products

In this section we are studying functions of the form:

$$w = \zeta_{A,\mathbb{N}}(s) = \frac{1}{B(0)} B(e^{-s}), \;\; s = \sigma + it \quad (15)$$

where $B(z)$ is a Blaschke product (4) with $z_k \neq 0$, $k = 1, 2, ..., n$. This is a normalized Dirichlet function in the sense that

$$\lim_{\sigma \to +\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1, \quad (16)$$

for every $t \in \mathbb{R}$. Moreover, we can also check that

$$\lim_{\sigma \to -\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = \frac{1}{|B(0)|^2} \quad (17)$$

for every $t \in \mathbb{R}$.

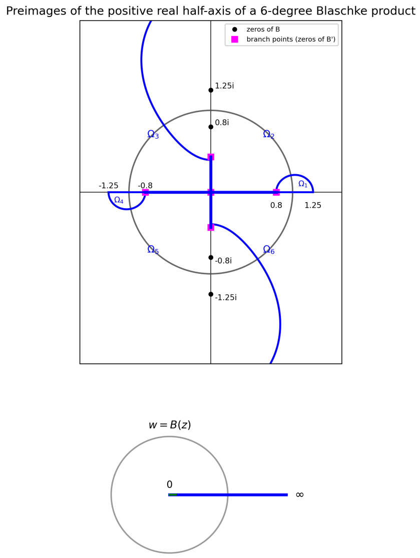

Fig. 4: Illustrating the fundamental domains of a particular 6 degree Blaschke product. Source: Created by the authors.

Theorem 4.1. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle contains infinitely many unbounded curves.

Proof: To make this affirmation obvious, we need to imagine graphics similar to those of Figure 3 in this general context. For every $\rho > 0$, the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle $C_\rho : |w| = \rho$ is formed with closed curves around the zeros of $\zeta_{A,\mathbb{N}}(s)$. If $z_k = r_k e^{i\alpha_k}$ is a zero of $B(z)$, then $s_{m,k} = -\ln r_k + (2m\pi - \alpha_k)i$, $m \in \mathbb{Z}$ are all zeros of $\zeta_{A,\mathbb{N}}(s)$ and those closed curves around $s_{m,k}$ are topologically mapped by the function $z = e^{-s}$ onto a unique closed curve around z_k. Let us denote $\Sigma_m = \{s = \sigma + it \mid \sigma \in \mathbb{R}, 0 \leq t < 2\pi\}$, $m \in \mathbb{Z}$. Every strip Σ_m is mapped conformally by e^{-s} onto the complex plane with a slit along the real axis from $-\infty$ to -1 and from 1 to $+\infty$ and it contains exactly n zeros $s_{m,k}$ such that $e^{-s_{m,k}} = z_k$, and therefore n components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the circle C_ρ. We argue that when $\rho = 1$, every strip Σ_m must contain an unbounded component of the pre-image of C_ρ around one of the zeros $s_{m,k}$. We can interpret (16) as a relation in $\widehat{\mathbb{C}}$ and then when $\sigma \to +\infty$ we have $s = \sigma + it \to \infty$, thus the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of $w = 1$ must contain the point $s = \infty$. This means that a component of the pre-image of the circle C_ρ with $\rho = 1$ passes through $s = \infty$. Since there are infinitely many strips Σ_m, there must be infinitely many unbounded components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle. $\Diamond$

Theorem 4.2. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the interval $(1, 1/|B(0)|^2)$ contains unbounded components Γ'_m which tend asymptotically to the branches of the infinite components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle.

Proof: Suppose that Ψ_m is an unbounded component of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle C_1 containing the zero $s_{m,0}$ of $\zeta_{A,\mathbb{N}}(s)$. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of a ray in C_1 making a small angle α with the positive real half axis is a curve γ_α starting at $s_{m,0}$ and intersectingΨ_m at a point s_α.

We cannot have $\sigma \to +\infty$ on γ_α since that ray does not pass through $w = 1$. Let γ'_α be the part of γ_α exterior to Ψ_m and γ''_α the part of γ_α interior to Ψ_m. When $\alpha \to 0$, the ray approaches the positive real half axis and the point $s_\alpha = \sigma_\alpha + it_\alpha \in \Psi_m$ tends to ∞ with σ_α tending to $+\infty$. Thus, when $\alpha \to 0$, the curve γ'_α tends asymptotically the lower branch of Ψ_m if $\alpha < 0$ and to the upper branch of Ψ_m if $\alpha > 0$. We will use the notations Γ'_m and Γ'_{m+1} for these curves. Due to (16) and (17), they are components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the interval $(1, 1/|B(0)|^2)$.$\Diamond$

Theorem 4.3. Consecutive curves Γ'_m and Γ'_{m+1} border infinite strips S_m whose closures are mapped by $\zeta_{A,\mathbb{N}}(s)$ onto the whole complex plane.

Let us notice that for $t \in \mathbb{R}$ we have $\zeta_{A,\mathbb{N}}(it) = \frac{1}{B(0)}B(e^{-it}) = \frac{1}{B(0)}e^{i\psi(t)}$ for a continuous real function $\psi(t)$. Thus $|\,\zeta_{A,\mathbb{N}}(it)| = \frac{1}{|B(0)|}$, which means that $\zeta_{A,\mathbb{N}}(s)$ maps the imaginary axis onto the circle centered at the origin and of radius $1/|B(0)|$. Then there is $\tau_m \in \mathbb{R}$ such that $\zeta_{A,\mathbb{N}}(i\tau_m) = 1/|B(0)|$, where $i\tau_m \in \Gamma'_m$. The continuation by $\zeta_{A,\mathbb{N}}(s)$ from $i\tau_m$ over the interval $(1, 1/|B(0)|^2)$ is exactly Γ'_m. The relations (16) and (17) show that Γ'_m extends for σ between $-\infty$ and $+\infty$.The curves Γ'_m and Γ'_{m+1} cannot intersect each other. Indeed, if they met at a point s_0 , one of the domains bounded by them will be mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the whole complex plain with a slit alongside the interval $(1, \zeta_{A,\mathbb{N}}(s_0))$. That domain should contain a pole of $\zeta_{A,\mathbb{N}}(s)$, [7], [8], which is impossible. Thus Γ'_m and Γ'_{m+1} cannot intersect each other and they bound infinite strips. Every point of the interval $(1, 1/|B(0)|^2)$ is the image by $\zeta_{A,\mathbb{N}}(s)$ of a point on Γ'_m and every point of $\mathbb{C} \setminus (1, 1/|B(0)|^2)$ is the image by $\zeta_{A,\mathbb{N}}(s)$ of a point from the strip, thus the image of the closed strip covers the whole complex plane.$\Diamond$

Theorem 4.4. There are infinitely many strips S_m which cover the whole complex plane.

Suppose that for an m_0 there is no curve Γ'_m above Γ'_{m_0}. Let s_0 be a point above Γ'_{m_0} and let $w_0 = \zeta_{A,\mathbb{N}}(s_0)$. There is a Jordan arc η' connecting w_0 and the complex conjugate $\overline{w}_0$, which does not intersect the interval $(1, 1/|B(0)|^2)$ and a Jordan arc η'' connecting the two points and intersecting the interval $(1, 1/|B(0)|^2)$. Continuation over η'' from w_0 brings to a point s'_0 below Γ'_{m_0} and continuation over η' should not intersect Γ'_{m_0} and brings us to the same point s'_0. Yet, this last situation is impossible, therefore above any Γ'_m there is a curve Γ'_{m+1} and similarly, below any Γ'_m there is a curve Γ'_{m-1}. Thus, there are infinitely many strips $\overline{S}_m$ covering the complex plane. We count them for $m \in \mathbb{Z}$ such that $0 \in S_0$.$\Diamond$

5 Infinite Blaschke Products

In [2], the behavior on the unit circle of an infinite Blaschke products $B(z)$ with finitely many essential singular points is studied. It is shown that the intervals on the unit circle between these points admit subdivisions which are mapped topologically by $B(z)$ onto the unit circle and which can be mapped topologically one on each other. The paper studies the groups of these mappings.

A different approach will allow us to obtain similar results by using the theory of fundamental domains of Blaschke products and show that the respective mappings admit analytic continuations to the reduced fundamental domains. Let

$$B(z) = \Pi_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z},$$

$$z_n = r_n e^{i\alpha_n}, 0 < r_n < 1, \alpha_n \in \mathbb{R} \quad (18)$$

be an infinite Blaschke product and let E be the set of essential singular points of $B(z)$. Every point of E is the limit of a subsequence (z_{n_k}) of (z_n). We suppose that the Blaschke condition (10) for the convergence of $B(z)$ in the unit disc is satisfied, $\widehat{\mathbb{C}} \setminus E$ contains at least one open arc of the unit circle and $B(z)$ admits a continuation to $\widehat{\mathbb{C}} \setminus E$, which, by the formula (1), is a meromorphic function in $\widehat{\mathbb{C}} \setminus E$ having the poles $1/\overline{z}_n$.

Then, $\Pi_{n=1}^{\infty} r_n$ makes sense and it is $B(0)$.

Moreover, $\lim_{z\to\infty} B(z)$ exists and it is $1/\Pi_{n=1}^{\infty} r_n$. We will denote it by $B(\infty)$.

We notice that for any convergent infinite Blaschke product (18), which does not cancel at the origin, $0 < B(0) < 1$.

5.1 Radial limits and Frostman's condition

In [9], infinite Blaschke products were studied under some conditions stronger than the Blaschke condition, (10). These ideas were extended later in many other publications. One of the Frostman conditions is related to the function:

$$\varphi_B(\theta) = \Sigma_{k=1}^{\infty}(e^{i\theta} - |z_k|) \ / \ |e^{i\theta} - z_k|, \ \theta \in \mathbb{R} \quad (19)$$

He proved that a necessary and sufficient condition for $B(z)$ and for all its sub-products to have radial limits of modulus 1 at $e^{i\theta}$ is that the series (19) converges. We call the inequality $\varphi_B(\theta) < \infty$ the *Frostman condition.*

In [10], to a Blaschke product for which the series (18) converges, the function

$$B^*(e^{i\theta}) = \lim_{r\to 1^-} B(re^{i\theta}) \quad (20)$$

is associated, and it is shown that Blaschke condition is sufficient for $B(z)$ to have the limit (20) a.e. on the unit circle and $|B^*(e^{i\theta})| = 1$ when $B^*(e^{i\theta})$ exists.

5.2 Boundary analyticity of infinite Blaschke products

To extend Theorem 3.3 to an infinite Blaschke product $B(z)$, we need to make sure that the Blaschke condition of convergence of $B(z)$ is satisfied and also that there is an arc of the unit circle disjoint of the set E of essential singular points of $B(z)$. Then $B(z)$ is a meromorphic function in $\widehat{\mathbb{C}} \ / \ E$ having infinitely many zeros z_k and infinitely many poles $1/\overline{z}_k$. The points of E are exactly the cluster points of the zeros, and due to (15) also of the poles of $B(z)$. Moreover, the zeros and the poles of all the derivatives of $B(z)$ must accumulate to E, since otherwise we would have essential singular points of $B(z)$ off the unit circle, which is excluded.

Theorem 5.1. Let K be a closed subset of the unit circle and let $e^{i\theta} \in K$. If $\varphi_B(\theta) < \infty$ and φ_B is continuous at θ, then the function $B(z)$ is analytic on every open arc γ_k connecting the points z_k and $1/\overline{z}_k$ and which is mapped topologically by $B(z)$ onto the positive real half axis. Moreover, if $B(z)$ is analytic on every one of these arcs intersecting K, then the series (19) converges uniformly on K, thus $\varphi_B(\theta)$ is finite-valued and continuous at every point θ with $e^{i\theta} \in K$.

Proof: Indeed, $B(z)$ is analytic at every point of γ_k, with the possible exception of $e^{i\theta}$. Yet, by [11], if $\varphi_B(\theta) < \infty$ and φ_B is continuous at θ, then $B(z)$ is analytic also at $e^{i\theta}$, thus $B(z)$ is analytic at every point of γ_k. Reciprocally, if $B(z)$ is analytic at every point of an arc connecting z_k and $1/\overline{z}_k$ and intersecting K, then it is analytic at any $e^{i\theta} \in K$, thus by [11], $\varphi_B(\theta)$ is finite-valued and continuous at every point θ with $e^{i\theta} \in K.\Diamond$

Theorem 5.2. Suppose that $B(z)$ satisfies the Frostman condition at every point of the unit circle. Then for every zero z_k of $B(z)$, a Jordan arc γ_k exists connecting z_k and the pole $1/\overline{z}_k$ and which is topologically mapped by $B(z)$ onto the positive real half axis.

Proof: Let z_k be an arbitrary zero of $B(z)$ and let us perform continuation by $B(z)$ over the positive real half axis starting from a zero z_k. Since this continuation can meet branch points of $B(z)$, it might not be unique, yet one of the existing continuations must bring us to $1/\overline{z}_k$ and we obtain a Jordan arc γ_k connecting the points z_k and $1/\overline{z}_k$. Let us notice that even if γ_k contains infinitely many branch points of $B(z)$, converging to a point $e^{i\theta}$,by Frostman condition the radial limit exists at $e^{i\theta}$ and the arc γ_k connecting z_k and $1/\overline{z}_k$ is well defined. This arc is mapped one to one by $B(z)$ onto the positive real half axis, with limit of $B(z)$ as $z \to e^{i\theta}$ on γ_k equal to 1. Thus, if the Frostman condition is satisfied, then to every zero z_k of $B(z)$, a Jordan arc γ_k connecting z_k and $1/\overline{z}_k$ exists, which is topologically mapped by $B(z)$ onto the positive real half axis and intersects the unit circle at a point ξ_k with $\lim_{z\in\gamma_k, z\to\xi_k} B(z) = 1.\Diamond$

Let us notice that, due to (1), we should read the limit (20) as $\lim_{r\to 1} B(re^{i\theta})$ and call *radial* this limit of $B(z)$ at $e^{i\theta}$. Indeed, if $z = re^{i\theta}$ and $r \to 1^-$, then $1/\overline{z} = \frac{1}{r}e^{i\theta}$, and when $r \to 1^-$ we have $1/r \to 1^+$ and if $B^*(e^{i\theta}) = e^{i\varphi}$ then $\lim_{r\to 1^-} B(re^{i\theta}) = \lim_{1/r\to 1^+} 1/\overline{B(\frac{1}{r}e^{i\theta})} = 1/\overline{e^{i\varphi}} = e^{i\varphi}$. In other words, we get the same value when we take $\lim_{r\to 1^-} B(re^{i\theta})$ and $\lim_{r\to 1^+} B(re^{i\theta})$.

To adapt the technique of the previous section to infinite Blaschke products $B(z)$ which admit continuations in $\widehat{\mathbb{C}} \setminus E$ to meromorphic functions in $\widehat{\mathbb{C}} \setminus E$, it is necessary one more step. Namely, given $e^{i\theta}$, instead of $\lim_{r\to 1} B(r^{i\theta})$, let us deal with $\lim_{r\to 1} B(re^{i\alpha(r)})$, where $\alpha(r)$ is such that $B(re^{i\alpha(r)}) > 0$ and call this limit *orthogonal limit* $B^o(e^{i\theta})$ of $B(z)$ at e$^{i\theta}$.

$B^o(\theta) = \lim_{r\to 1} B(re^{i\alpha(r)}), \quad B(re^{i\alpha(r)}) > 0$ (21)

We can prove:

Lemma 5.1. The orthogonal limit of $B(z)$ at $e^{i\theta}$ exists if and only if the radial limit of $B(z)$ at $e^{i\theta}$ exists, and if they exist, they coincide.

Proof: Indeed, we have $B(re^{i\theta}) = B(re^{i\alpha(r)}) + [B(re^{i\theta}) - B(re^{i\alpha(r)})]$ and due to the continuity of $B(z)$ in the unit disc, to every $\epsilon > 0$ corresponds $\delta > 0$ such that $|re^{i\theta} - re^{i\alpha(r)}| < \delta$,implies $|B(re^{i\theta}) - B(re^{i\alpha(r)})| < \epsilon$, i.e. $|e^{i\theta} - e^{i\alpha(r)}| < \delta/r$ implies $|B(re^{i\theta}) - B(re^{i\alpha(r)})| < \epsilon$, thus $\lim_{\alpha(r)\to\theta} re^{i\alpha(r)} = re^{i\theta}$ implies $\lim_{r\to 1} B(re^{i\theta}) = \lim_{r\to 1} B(re^{i\alpha(r)})$, which proves the affirmation of the Lemma1$\diamondsuit$.

By Riesz theorem, for any Blaschke product $B(z)$ satisfying the Blaschke condition, the orthogonal limit exists a.e. at the points of the unit circle.

Lemma 5.2. The function $B^o(\theta)$ is defined everywhere in $\partial D \setminus E$ and $|B^o(\theta)| = 1$ for every $\theta, e^{i\theta} \in \partial D \setminus E$.

Proof: Indeed, $B(z)$ is meromorphic in $\widehat{\mathbb{C}} \setminus E$ and there are no poles of $B(z)$ on ∂D, thus $B(z)$ is analytic on $\partial D \setminus E$, which guarantees the existence of the limit (21), and that it is $B(e^{i\theta})$ of modulus $1\diamondsuit$.

Theorem 5.3. Suppose that $B(z)$ given by (9) satisfies the Blaschke condition (10) and it can be continued to a meromorphic function in $\widehat{\mathbb{C}} \setminus E$. Then, for every zero z_n of $w = B(z)$, the pre-image by $B(z)$ of the positive real half axis contains one or several Jordan arcs connecting z_n and $1/\overline{z}_n$ and intersecting the unit circle at distinct points of $\partial D \setminus E$.

Proof: We notice that the points of E are also the only essential singular points of $B'(z)$. Suppose that z_1 is a zero of order p_1 of $B(z)$ and let D_ρ be the disc $|w| < \rho$ for a small radius ρ such that the component of the pre-image Ω'_1 by $B(z)$ of D_ρ containing z_1 does not intersect other components of this pre-image. By [5], Ω'_1 is divided into p_1 subsets by Jordan arcs $\gamma'_{1,j}$ connecting z_1 with $\partial\ \Omega'_1$ and which are topologically mapped by $B(z)$ onto the interval $[0, \rho)$ of the real axis. The interiors $\Omega'_{1,j}, j = 1, 2, ..., p_1$ of these subsets are conformally mapped by $B(z)$ onto D_ρ with a slit alongside the interval $[0, \rho)$ of the real axis. The image of Ω'_1 by $B(z)$ covers p_1 times $D_\rho\backslash\{0\}$.The symmetric domain Ω''_1 of Ω'_1 with respect to the unit circle is divided by the symmetric arcs $\gamma''_{1,j}$ of $\gamma'_{1,j}$ with respect to the unit circle into p_1 subsets whose interiors $\Omega''_{1,j}$ are conformally mapped by $B(z)$ onto the domain $|w| > 1/\rho$ with a slit alongside the interval $(1/\rho, +\infty)$ of the real axis. The image by $B(z)$ of Ω''_1 covers p_1 times the domain $|w| > 1/\rho$. If the continuation over the positive real half axis of the arcs $\gamma'_{1,j}$ and $\gamma''_{1,j}$, do not meet branch points in their way, they must arrive due to (2), at the same points $\xi_{1,j} \in \partial D \setminus E$ when w arrives in 1. The points $\xi_{1,j}$ are distinct since $B'(z) \neq 0$ on the unit circle.

If such a continuation meets a branch point of $B(z)$, then from there, the continuation will follow different paths, some of them involving the turn back to a zero of $B(z)$. The branch point is a zero of $B'(z)$ and its image by $B(z)$ is on the real axis. Such a fenomenon can happen only finitely many times. Indeed, the component of the pre-image by $B(z)$ of the interval $(\rho, 1/\rho)$ containing these zeros is a compact set and infinitely many zeros of $B'(z)$ belonging to it would imply that $B'(z)$ is identically zero, which is false.

Thus, the continuations along the real axis of every arc $\gamma'_{1,j}$ and $\gamma''_{1,j}$ produce Jordan arcs connecting z_1 and $1/\widetilde{z}_1$ and intersectingthe unit circle into distinct points $\xi_{1,j}$.

Let us notice, see, [3], that the arcs $\eta_{1,j}$ of the unit circle between $\xi_{1,j}$ and $\xi_{1,,j+1}$ with an end removed are topologically mapped by $B(z)$ onto the unit circle. This implies that points ξ associated with a different zero of $B(z)$ cannot belong to any $\eta_{1,j}$, i.e. different zeros z_n will produce different arcs $\eta_{n,j}$, $j = 1, 2, ..., p_n$. $\diamondsuit$

Since $B(z)$ maps the unit circle with E removed onto the unit circle, there must be a zero, call it z_2, of $B(z)$ such that the arc of the unit circle between ξ_{1,p_1} and $\xi_{2,1}$ with an end removed is mapped topologically by $B(z)$ onto the unit circle and if z_2 has the order of multiplicity p_2, then we get $p_1 + p_2 - 1$ sub-arcs of the unit circle between $\xi_{1,1}$ and ξ_{2,p_2} which, with an and removed, are mapped each one topologically onto the unit circle.

If we continue this process indefinitely, we get a subsequence of (z_{n_k}) of (z_n) with $e_1 = \lim_{n_k\to\infty} z_{n_k} \in E$ such that every z_{n_k} is connected with $1/\overline{z}_{n_k}$ by p_k Jordan arcs $\gamma_{n_k,j}$, $j = 1, 2, ..., p_k$ which, with $1/\overline{z}_{n_k}$ removed, are mapped topologically by $B(z)$ onto the positive real half axis. For the same n_k, if $p_k \geq 2$, consecutive arcs $\gamma_{n_k,j}$ and $\gamma_{n_k,j+1}$ bound $p_k - 1$ simply connected domains $\Omega_{n_k,j}, j = 1, 2, ..., p_k - 1$ which are mapped conformally by $B(z)$ onto the complex plane with a slit alongside the positive real half axis. Obviously, we have also $\lim_{n_k\to\infty} 1/\overline{z}_{n_k} = e_1$. Then all the arcs $\gamma_{n_k,j}$ squeeze to e_1 as $n_k \to \infty$ and therefore the domains $\Omega_{n_k,j}$ squeezes to e_1 as $n_k \to \infty$.This implies that the arcs $\eta_{n,j}$ squeeze also to e_1.

Lemma 5.4. There are infinitely many zeros of $B'(z)$ in D, different of the zeros of $B(z)$, all located in $D\setminus \cup_{n_k=1}^{\infty} \Omega_{n_k}$.

Proof: Indeed, it is obvious that adding a new distinct zero to finite Blaschke product, implies adding a new distinct zero in the unit disc to its derivative. Since $B(z)$ given by (5) has infinitely many distinct zeros, $B'(z)$ must have infinitely many distinct zeros in the unit disc too. No zero of $B'(z)$ belongs to $\cup_{n_k=1}^{\infty}\Omega_{n_k}$, since $B(z)$ is univalent in every Ω_{n_k}, thus all these zeros belong to.$D\setminus \cup_{n_k=1}^{\infty}$ Ω_{n_k}.

Lemma 5.5. If $B'(\zeta_0) = 0$ then $B'(1/\overline{\zeta}_0) = 0$.

Proof: Indeed, an easy computation shows that

$B'(z) = B(z)\Phi(z)$ (22),
where
$\Phi(z) = \Sigma_{n=1}^{\infty} \frac{1-|z_n|^2}{(z-z_n)(1-\overline{z}_n z)}$ (23)

If $\Phi(\zeta_0) = 0$, then $\overline{\Phi}(\zeta_0) = 0$ and we have

$$\Phi(1/\overline{\zeta}_0) = \Sigma_{n=1}^{\infty} \frac{1-|z_n|^2}{(1/\overline{\zeta}_0 - z_n)(1-\overline{z}_n/\overline{\zeta}_0)}$$
$$= \overline{\zeta}_0^{2} \Sigma_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-z_n\overline{\zeta}_0)(\overline{\zeta}_0-\overline{z}_n)}$$

$$= \overline{\zeta}_0^{2}\overline{\Phi}(\zeta_0) = 0,$$

thus $\Phi(\zeta_0) = 0$, and then $B'(1/\overline{\zeta}_0) = 0$. This means that the zeros of $B'(z)$ are two by two symmetric with respect to the unit circle.

Let us notice that there are infinitely many Jordan arcs connecting points ζ_j with a zeros z_n of $B(z)$ which are topologically mapped by $B(z)$ onto rays starting at $w = 0$ and ending at $B(\zeta_j)$. Suppose that ζ_1 is the closest zero of $B'(z)$ to the simple zero z_1 and let us denote δ_1' the Jordan arc connecting z_1 and ζ_1 and which is mapped topologically by $B(z)$ onto the ray from 0 to $B(\zeta_1)$. Let δ_2' be the Jordan arc connecting ζ_1 with z_2 and which is topologically mapped by $B(z)$ onto the same ray and let δ_1 be the sum of these two arcs. The symmetric $\widetilde{\delta}_1$ of δ_1 with respect to the unit circle is a Jordan arc connecting $1/\overline{z}_1$ and $1/\overline{z}_2$, passing through $1/\overline{\zeta}_1$ and which is mapped by $B(z)$ two to one onto the infinite ray starting at $B(1/\overline{\zeta}_1)$. Then the domain bounded by $\delta_1, \widetilde{\delta}_1$ γ_1 and γ_2 is mapped conformally by $B(z)$ onto the complex plane with a slit alongside the positive real half axis L_0 and alongside L_1 formed with the two rays. We can do a similar construction for the next gap obtaining a slit L_2 and continue indefinitely. We get infinitely many slits

$L_k = \{w = \rho e^{i\gamma_k} \mid \rho \geq 0,$

$\rho \notin (|B(\zeta_k)|, |B(1/\overline{\zeta}_k)|\}$

Let $L = \cup_{k=0}^{\infty} L_k$. The domain $\mathbb{C} \setminus L$ is a simply connected domain such that every reduced fundamental domain $\Omega_{n_k}^{(r)} = \Omega_{n_k} \setminus B^{-1}(L)$ is mapped conformally by $B(z)$ onto $\mathbb{C} \setminus L$. Let us notice that $B^{-1}(L)$ is formed with infinitely many Jordan arcs included in the unit circle connecting every zero z_k of $B(z)$ with $B(\zeta_k)$, as well as with the symmetric of these arcs with respect to the unit circle and included in every Ω_{n_k}. The arcs $\eta_{n,j}$ can be put together into a unique arc starting at $\xi_{1,1}$ and ending at e_1.

When a point ξ traverses this arc counterclockwise, the point $B(\xi)$ winds counterclockwise on the unit circle infinitely many times. The point e_1 is a singularity of type 1_a, as described in [3].

Obviously, we can have a singularity of type 1_b, when the respective arc starts in e_1 and ends in $\xi_{1,1}$ and , finally a singularity of type 2 when two distinct subsequences of (z_n) are such that the initial points ξ_{1,p_1} and ξ_{2,p_2} are located on different sides of an orthogonal arc to the unit circle at e_1. It is also known that the Blaschke condition (6) can be satisfied even if E covers the whole unit circle except for some open sub-arcs.

Our geometric approach works also in these extreme situations. For example, if a sequence of essential singular points (e_n) converges to e_0, every arc of the unit circle between e_n and e_{n+1} will contain infinitely many sub-arcs η_{n_k} which with an end removed are topologically mapped by $B(z)$ onto the unit circle.

The ends ξ_{n_k} and ξ_{n_k+1} of these arcs are the intersection points with the unit circle of Jordan arcs connecting zeros z_{n_k} and poles $1/\overline{z}_{n_k}$ of $B(z)$ and which are topologically mapped by $B(z)$ onto the positive real half axis.

Countable many fundamental domains of $B(z)$ can be built, as previously for every couple e_n, e_{n+1}. These fundamental domains squeeze to every e_n and to e_0.

For the points of a dense subset of E, such domains do not exist any more, yet every neighborhood of these points cover by $B(z)$ infinitely many times the complex plane. This scenario is a logic conclusion of the method and we can state.

Theorem 5.4. For any infinite Blaschke product (9) satisfying the Blaschke condition (10) and which is a meromorphic function in $\widehat{\mathbb{C}} \setminus E$, there are countable many subsets of $\widehat{\mathbb{C}}$ whose interiors $\Omega_n^{(r)}$ are mapped conformally by $B(z)$ onto the complex plane with the same slit L.

Proof: The slit L is $\cup_{k=0}^{\infty} L_k$, as defined above and

$\cup_{n=1}^{\infty} \Omega_n^{(r)} \subset \widehat{\mathbb{C}} \setminus (E \cup B^{-1}(L))$.

Every domain $\Omega_n^{(r)}$ is a reduced fundamental domain of $B(z)$.

Let us notice that $\cup \overline{\Omega}_n^{(r)}$ does not cover necessarily the complex plane, as in the case of a finite Blaschke product. Indeed, every point $e \in E$ has a neighborhood V which is not included in any reduced fundamental domain, since otherwise $w = B(z)$ would be analytic at e.

Suppose $z_0 \in V$, $|z_0| < 1$ and $B'(z_0) \neq 0$. Then $B(z)$ is locally injective at z_0 and it maps conformally a neighborhood U of z_0 included in V onto a domain of the w-plane included in the unit disc.

We can expand U such that it is mapped conformally by $B(z)$ onto the unit disc. Indeed, if $|w_1| < 1$, let us connect w_1 and $w_0 = B(z_0)$ with a Jordan arc η avoiding the zeros of $B'(z)$.

Continuation by $B(z)$ over η starting from z_0 brings to a point z_1 with $B(z_1) = w_1$ and $B(z)$ maps conformally a neighborhood of this arc onto a domain of $|w| < 1$ containing w_1. By doing this construction for every point w_1 we get the expended neighborhood. Let us keep the notation U for it.

A component of the pre-image of the interval $(0, 1)$ of the real axis is included in U and it is a Jordan arc orthogonal to the unit circle at e. The set $\widetilde{U}$ symmetric with U with respect to the unit circle is mapped conformally by $B(z)$ onto the exterior of the unit circle. The set $U \cup \widetilde{U}$ is conformally mapped by $B(z)$ onto $\widehat{\mathbb{C}} \setminus \partial D$. It can be extended to the boundaries except for the point e, where $B(z)$ is discontinuous. The set $U \cup \widetilde{U}$ keeps now the place of a reduced fundamental domain.

Obviously, there can be just countable many points $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{Q}$ and therefore countable many new such domains of $B(z)$.

6 Dirichlet Functions Generated by Infinite Blaschke Products

In this section we establish a connection between *general Dirichlet series* and Blashcke products, showing that any Blaschke product can be written as a Dirichlet series upon a change of variable, thus some *Dirichlet functions,* which are functions obtained by analytic continuation of general Dirichlet series, inherit geometric properties of Blaschke products.

A special class of Dirichlet functions, namely the functions obtained from the so called *Dirichlet L-series* received an intense scrutiny in the last half a century, due to their applications in number theory, [13] . The *Riemann Zeta function* is one of them and the famous *Riemann Hypothesis,* concerns a deep property of prime numbers.

A comparative study of Dirichlet L-functions and Dirichlet functions generated by Blaschke products reveals a complexity of Dirichlet functions hardly imaginable up to now. For the first time, we realize that the Dirichlet functions can have infinitely many poles, thus essential non-isolated singular points. Moreover, the necessary and sufficient conditions for the continuation of infinite Blaschke product to exist, reflect on similar conditions unknown up to now, for the continuation of Dirichlet functions across the convergence line.

6.1 General Dirichlet series and L-series

A *general Dirichlet series* is an expression of the form

$$\zeta_{A,\Lambda}(s) = \Sigma_{n=0}^{\infty} a_n e^{-\lambda_n s} \quad (24)$$

where $A = (a_n)$ is an arbitrary infinite sequence of complex numbers and $\Lambda = \{0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ...\}$ is a non decreasing sequence of non negative numbers such that $\lim_{n\to\infty} \lambda_n = +\infty$ and $s = \sigma + it$ is a complex variable.

For every series (24) there is a number $\sigma_a \leq +\infty$, called the *abscissa of convergence* of the series, such that the series converges uniformly on compact subsets for $\sigma > \sigma_a$ and it diverges for $\sigma < \sigma_a$, [12]. Obviously, if $\sigma_a = +\infty$ the series does not converge anywhere. The line $s = \sigma_a + it$, $t \in \mathbb{R}$ is called the *convergence line* of the series.

The function $\zeta_{A,\Lambda}(s)$ defined by (24) is an analytic function in the half plane Re $s > \sigma_a$. Some Dirichlet series admit *continuation* across the convergence line to analytic functions in the whole complex plane except for some poles. We keep the notation $\zeta_{A,\Lambda}(s)$ for these functions and call them *Dirichlet functions.*

A Dirichlet character $\chi(n)$ of some modulus $q \geq 1$ is an arithmetic periodic function of period q. It defines a Dirichlet series

$$L(s;\chi) = \Sigma_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (25)$$

called Dirichlet L-series.

It is known, [13], that for any Dirichlet L-series we have $\sigma_a = 1$ if $\chi(n)$ has modulus$q = 1$ and $\sigma_a = 0$ if $\chi(n)$ has modulus $q > 1$. Also, any Dirichlet L-series admits a continuation across the convergence line to a meromorphic function in $\mathbb{C}$ having a unique pole $s = 1$ when $q = 1$ and $s = 0$ when $q > 1$. For the Riemann Zeta function, $q = 1$, thus it has the unique pole $s = 1$.

Figure 5 below illustrates the landscape of the pre-image by Dirichlet L-functions defined one by a real Dirichlet character and the other one by a complex Dirichlet character.

Comparing Figure 4 with Figure 5 we realize how much complex such a landscape can be when the Dirichlet function is generated by a Blaschke product.

The derivative

$$L'(s;\chi) = -\Sigma_{k=2}^{\infty} \frac{\chi(k) \ln k}{k^s} \quad (26)$$

is no more a Dirichlet L-series. However, it is a Dirichlet series having the same abscissa of convergence like $L(s;\chi)$.

It can be continued to a meromorphic function in $\mathbb{C}$ having a double pole 0, or 1. We keep the notation $L'(s;\chi)$ for this function.

6.2 Generating Dirichlet functions with Blaschke products

The class of Dirichlet functions

$$\zeta_{A,\mathbb{N}}(s) = B(e^{-s})/B(0) \quad (27)$$

generated by Blaschke products $B(z) = \Pi_{n=1}^{N\leq\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z}_n z}$, where $B(0) \neq 0$, have been studied in some previous papers. We supposed that, in the infinite case, the Blaschke condition $\Sigma_{n=1}^{\infty}(1 - |z_n|)$ for the convergence of $B(z)$ in the unit disc was satisfied and $B(z)$ admitted a meromorphic continuation to $\widehat{\mathbb{C}} \setminus E$, where E is the set of cluster points of the zeros of $B(z)$ and $\widehat{\mathbb{C}} \setminus E$ contains at least one open arc of the unit circle. It was found that for such functions, $s_0 = \sigma_0 + it_0$ is a zero if and only if $z_0 = e^{-s_0}$ is a zero of $B(z)$, thus the zeros of $\zeta_{A,\mathbb{N}}(s)$ are $-\ln r_n + (2k\pi - \alpha_n)i$, $k \in \mathbb{Z}$, where $z_n = r_n e^{i\alpha_n}$ are the zeros of $B(z)$. Moreover, the poles of $\zeta_{A,\mathbb{N}}(s)$ are $\ln r_n + (2k\pi - \alpha_n)i$, $k \in \mathbb{Z}$, thus to every zero of $\zeta_{A,\mathbb{N}}(s)$ corresponds a symmetric pole with respect to the imaginary axis. Then, performing continuation by $\zeta_{A,\mathbb{N}}(s)$ over the real axis starting from a zero we can arrive at the symmetric pole in two different ways, according to the fact that we went along the positive or the negative real half axis and the continuations were unlimited. This implies that the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis contains curves $\Gamma_{m,k}$ connecting the pairs of symmetric zeros and poles. These are closed curves which are mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto the whole real axis when the continuation reaches the same pole going along the positive and the negative real half axis, or infinite curves when the continuation is limited. This last situation happens when $\lim_{\sigma\to-\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1/[B(0)]^2$ with $\sigma + it$ belonging to that curve. Notice that $B(0) = \Pi_{n=1}^{N\leq\infty} r_n > 0$. Besides these components, the pre-image of the real axis contains also disjoint unbounded curves Γ'_m, $m \in \mathbb{Z}$, on which σ varies from $-\infty$ to $+\infty$ and which are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(1, 1/[B(0)]^2)$ of the real axis and such that consecutive curves Γ'_m and Γ'_{m+1} bound infinite strips S_m covering the whole complex plane. Every strip S_m contains a unique unbounded curve $\Gamma_{m,0}$ which is mapped one to one by the function onto $(-\infty, 1) \cup (1/[B(0)]^2, +\infty)$.

We point out the dissimilarity with the well studied Dirichlet L-functions for which all the curves $\Gamma_{m,k}$ are unbounded, the curves $\Gamma_{m,0}$ are each one mapped one to one onto the interval $(-\infty, 1)$,and the curves Γ'_m are each one mapped one to one onto the interval $(1, +\infty)$ of the real axis.

In both cases, these curves and the zeros of $\zeta'_{A,\mathbb{N}}(s)$ help building fundamental domains of the function and reveal its global mapping properties.

In the following, we will study the behavior of these functions and of their derivatives on the imaginary axis, to deepen the study of the derivatives and to explore as well their global mapping properties. The behavior of $\zeta_{A,\mathbb{N}}(s)$ on the imaginary axis is a reflection of the behavior of $B(z)$ on the unit circle, which has been extensively studied in the last century. In turn, knowing the behavior of $\zeta_{A,\mathbb{N}}(s)$ on the imaginary axis, we can draw conclusions about the behavior of infinite Blaschke products on the unit circle, completing this century old theory.

When $a_0 = 1$ we say that the series (1) is *normalized*. For all normalized Dirichlet series with an abscissa of convergence less than $+\infty$ we have

$$\lim_{\sigma\to+\infty} \zeta_{A,\Lambda}(\sigma + it) = 1 \quad (28)$$

for every $t \in \mathbb{R}$. This simple equality has very important implications regarding the landscape of the pre-image by the Dirichlet function $\zeta_{A,\Lambda}(s)$ of the real axis.

The case of the *Dirichlet L-series*, where the coefficients a_n are *Dirichlet characters*, has been well studied and it has been shown that they can be analytically continued to the whole complex plane except for one pole. The function $\zeta_{A,\Lambda}(s)$ obtained in this way satisfies a *Riemann type of functional equation* and it can be also written as an *Euler product.* It is known that for the Dirichlet functions generated by such Dirichlet series, the components of the pre-image of the real axis are of three distinct types:

a). There are infinitely many unbounded curves Γ'_m, $m \in \mathbb{Z}$ which are mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the interval $(1, +\infty)$ of the real axis. These curves do not intersect each other and two consecutive ones bound infinite strips S_m with σ varying from $-\infty$ to $+\infty$, whose closures cover the whole complex plane. They are counted from $m = -\infty$ to $m = +\infty$ such that $0 \in S_0$.

b). Every strip S_m contains a unique unbounded curve $\Gamma_{m,0}$, with σ varying on the whole real axis, which does not intersect any Γ'_m and is mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the interval $(-\infty, 1)$ of the real axis.

c). Every strip S_m can contain several parabola shaped curves $\Gamma_{m,k}$, $k \neq 0$, on which σ varies from $-\infty$ to a finite value and which are mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the whole real axis.

Figure 5 below illustrates these affirmations for the Dirichlet L-functions $L(14, 4, s)$, which has real coefficients and $L(14, 2, s)$ whose coefficients are complex.

For Dirichlet functions generated by Blaschke products, the curves $\Gamma_{m,k}$, $k \neq 0$ are closed curves passing each one through a zero and a pole of the function.

For a series (12) we have:

$$\zeta'_{A,\Lambda}(s) = -\Sigma_{n=1}^{\infty} \lambda_n a_n e^{-\lambda_n s} \tag{29}$$

This is still a Dirichlet series, except that is not normalized.

It is known, [12], that $\zeta'_{A,\Lambda}(s)$ has the same abscissa of convergence as $\zeta_{A,\Lambda}(s)$, and the two series admit simultaneously analytic continuations to the whole complex plane. Moreover, the derivative of the Dirichlet function $\zeta_{A,\Lambda}(s)$ is the analytic continuation of the series (15). In the following we will be studying also the landscape of the pre-image by $\zeta'_{A,\Lambda}(s)$ of the real axis.

Let us notice that if $\sigma_a < +\infty$, then for the series (15) we have:

$$\lim_{\sigma \to +\infty} \zeta'_{A,\Lambda}(\sigma + it) = 0 \quad (30)$$

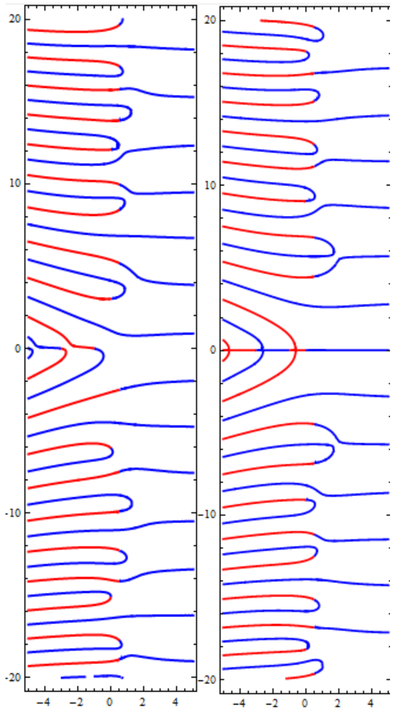

Fig. 5: Dirichlet function $L(14\quad, 2, s)$ with complex character and $L(14, 4, s)$ with real character. Source: Created by the authors.

for every $t \in \mathbb{R}$. This implies the following obvious topological properties:

For every $\delta > 0$, the pre-image by $\zeta'_{A,\Lambda}(s)$ of a circle $|w| = r$ with r small enough is formed with disjoint closed curves around the zeros $s_n = \sigma_n + it_n$ of $\zeta'_{A,\Lambda}(s)$ with $\sigma_n > \delta$, as well as with an unbounded curve whose points $s = \sigma + it$ have big positive values for σ. When r increases, some of these curves will touch each other at points s' where $\zeta''_{A,\Lambda}(s') = 0$ and then they will fuse into a unique closed curve containing both zeros. With increasing r more zeros will be included into those bounded curves. Also, the unbounded curve moves to the left touching some of the closed curves at points s' where $\zeta''_{A,\Lambda}(s') = 0$ and fusing with them into a unique unbounded curve. All the zeros of $\zeta'_{A,\Lambda}(s)$ included in the respective bounded curves will remain at the right of the new unbounded curve. If there is $\delta \in \mathbb{R}$ such that for any zero $s' = \sigma' + it'$ of $\zeta'_{A,\Lambda}(s)$ we have $\sigma' > \delta$, then there is r_0 such that all the zeros of $\zeta'_{A,\Lambda}(s)$ will be located at the right of the pre-image of the circle $|w| = r_0$. This is the case of Dirichlet functions generated by finite Blaschke products for which $\zeta'_{A,\Lambda}(s)$ has a finite number of the zeros in every strip S_m all located in the right half plane.

In a similar way, the pre-image by $\zeta'_{A,\Lambda}(s)$ of a circle $|w| = R$ with R big enough is formed with disjoint closed curves around the poles of $\zeta'_{A,\Lambda}(s)$. When R decreases, those curves expand and some of them will touch each other at points s' where $\zeta''_{A,\Lambda}(s') = 0$, after which they will fuse into a unique closed curve containing both poles. If there is $\delta \in \mathbb{R}$ such that for any pole $s' = \sigma' + it'$ of $\zeta'_{A,\Lambda}(s)$ we have $\sigma' < \delta$, then there is R_0 such that all the poles of $\zeta'_{A,\Lambda}(s)$ will be located at the left of the pre-image of the circle $|w| = R_0$.This is the case of Dirichlet functions generated by finite Blaschke products for which $\zeta'_{A,\Lambda}(s)$ has a finite number of poles in every strip S_m, all located in the left half plane.

7 The Pre-Image of the Real Axis by $\zeta'_{A,\mathbb{N}}(s)$

We call *curves of type* Γ the components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis, and *curves of type* Ψ the components of the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the real axis. If a curve of type Γ intersects a curve of type Ψ at a point other that a multiple zero or a multiple pole we say that the two curves *intertwine.* There is a one to one correspondence between intertwining curves and this is such that every curve Γ'_m intertwines with a unique curve Ψ'_m and with no curve $\Psi_{m,k}$. Also, every curve $\Gamma_{m,k}$ intertwines with a unique curve $\Psi_{m,k}$ and with no curve Ψ'_m. Moreover, at the point a curve of type Γ intersects a curve of type Ψ, the tangent to the curve of type Γ is horizontal and if the tangent to a curve of type Γ is horizontal at a given point, through that point passes also a curve of type Ψ.

Let us suppose that we color differently, say blue and yellow the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the positive and negative real half axes and use green and orange for the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the positive and negative real half axes and we visit the respective components following the pre-image of a circle centered at the origin. Since on such a circle the positive and the negative half axes alternate, so must happen with the two colors on the Γ and Ψ type of curves. This simple topological fact will be called *color alternation rule.* The following *color matching rule* is also valid for every Dirichlet L-function: blue meets always orange and yellow meets always green, except for $\Gamma_{m,0}$ and $\Psi_{m,0}$, where blue can meet also green. The same reasoning allows us to conclude that these rules are also valid for the Dirichlet functions generated by Blaschke products.

It can be easily seen that, since $\lim_{\sigma\to+\infty} \zeta_{A,\mathbb{N}}(\sigma+it) = 1$ and $\lim_{\sigma\to-\infty} \zeta_{A,\mathbb{N}}(\sigma + it) = 1/B^2(0)$,the components Γ'_m are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(1, 1/B^2(0))$. We conclude that $\Gamma_{m,0}$ is mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto $\mathbb{R} \setminus [1, 1/B^2(0)]$. and we mention again that $\Gamma_{m,k}$,$k \neq 0$, are closed curves passing through a zero and the symmetric pole with respect to the imaginary axis and are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the whole real axis.

Having in view the *color matching rule,* and the fact that $\lim_{\sigma\to+\infty} \zeta'_{A,\mathbb{N}}(\sigma + it) = 0$ and $\lim_{\sigma\to-\infty} \zeta'_{A,\mathbb{N}}(\sigma + it) = +\infty$, we can state:

Theorem 7.1. The pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the real axis has components Ψ'_m, which are mapped one to one by $\zeta'_{A,\mathbb{N}}(s)$ onto the negative real half axis. They intertwine with Γ'_m and consecutive curves Ψ'_m and Ψ'_{m+1} bound infinite strips Σ_m containing some other components of the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the real axis. These are components $\Psi_{m,k}$, $k \neq 0$, which are closed curves passing through a zero and a pole of $\zeta'_{A,\mathbb{N}}(s)$ and which are mapped by $\zeta'_{A,\mathbb{N}}(s)$ onto the whole real axis and a unique unbounded components $\Psi_{m,0}$ containing a zero and a pole of $\zeta'_{A,\mathbb{N}}(s)$. Through this last pole passes also an unbounded component $\widetilde{\Psi}_{m,0}$ which is mapped one to one onto the whole real axis. It does not intertwine with any curve of type Γ. The existence of this last curve is guaranteed by the fact that the simple poles of $\zeta_{A,\mathbb{N}}(s)$ are double poles for $\zeta'_{A,\mathbb{N}}(s)$, thus branch points of $\zeta'_{A,\mathbb{N}}(s)$, meaning that there should be two orthogonal curves of type Ψ issuing from them.

Proof: For $\zeta'_{A,\Lambda}(s)$ we have the equality (17), which implies that there must be an interval $[0, u]$ of the real axis whose pre-image by $\zeta'_{A,\Lambda}(s)$ contains an unbounded curve. Indeed, let $s_u = \sigma_u + it_u$ $\sigma_u > \sigma_a$ be a point of the pre-image by $\zeta'_{A,\Lambda}(s)$ of $w = u$. Since there is no singular point of $\zeta'_{A,\Lambda}(\sigma+it)$ for $\sigma > \sigma_a$, we can do unlimited continuation over the interval $[0, u]$ starting from s_u. This is an unbounded curve $\Psi^-_{s_u}$ starting at s_u and on which $\lim_{\sigma\to+\infty} \zeta'_{A,\Lambda}(\sigma + it) = 0$. Also, if the continuation over the interval $(u, +\infty)$ does not meet any singular point of $\zeta'_{A,\Lambda}(s)$ for $\sigma < \sigma_u$, then $\lim_{\sigma\to-\infty} \zeta'_{A,\Lambda}(\sigma + it) = +\infty$, and we get another unbounded curve $\Psi^+_{s_u}$. Since the singular points of $\zeta'_{A,\Lambda}(s)$ form a discrete set, and there are infinitely many points in the pre-image of $w = u$, such a curve $\Psi^+_{s_u}$ can always be found. Thus, the union Ψ'_{s_u} of two curves is mapped one to one by $\zeta'_{A,\Lambda}(s)$ onto the positive real half axis, or in other words, the pre-image by $\zeta'_{A,\Lambda}(s)$ of the real axis contains components extending for σ from $-\infty$ to $+\infty$, which are mapped one to one by $\zeta'_{A,\Lambda}(s)$ onto the positive real half axis.

If the continuation over the interval $(u, +\infty)$ meets a pole $s' = \sigma' + it'$, then in a neighborhood of s' we have $\lim_{\sigma \to \sigma'} \zeta'_{A,\Lambda}(\sigma + it) = +\infty$ for $\sigma + it \in \Psi^+_{s_u}$ for every real t. At the same time, there must exist a curve Ψ'', component of the pre-image of the negative real half axis, such that $\lim_{\sigma \to \sigma'} \zeta'_{A,\Lambda}(\sigma + it) = -\infty$ for $\sigma + it \in \Psi''$. Analogously, if $s_{-u} = \sigma_{-u} + it_{-u}$, belongs to the pre-image by $\zeta'_{A,\Lambda}(s)$ of $w = -u$, we either can do unlimited continuation starting from s_{-u} over the negative real half axis and then we obtain an unbounded curve Ψ'' extending for σ varying from $-\infty$ to $+\infty$, or Ψ'' meets a pole $s' = \sigma' + it'$ and in a neighborhood of s' we have $\lim_{\sigma \to \sigma'} \zeta'_{A,\Lambda}(\sigma + it) = -\infty$ for $\sigma + it \in \Psi''$. In this last case, there must exist a curve Ψ', component of the pre-image of the positive real half axis such that $\lim_{\sigma \to \sigma'} \zeta'_{A,\Lambda}(\sigma + it) = +\infty$ for $\sigma + it \in \Psi'$.

Lemma 7.1. Let $f(s)$ be an analytic function in an open subset Ω of $\mathbb{C}$ such that $f(\Omega) \cap \mathbb{R} \neq \varnothing$ and $f'(\Omega) \cap \mathbb{R} \neq \varnothing$. Let Γ be the pre-image by $f(s)$ of the real axis and Ψ be the pre-image by $f'(s)$ of the real axis. Then $s_0 \in \Gamma \cap \Psi$, if and only if the tangent to Γ at s_0 is horizontal.

Proof: Let $s = s(t)$ be the equation of Γ such that $f(s(t)) = t$ and let t_0 such that $s(t_0) = s_0$. Then $f'(s_0)s'(t_0) = 1$ and $\operatorname{Im} f'(s_0) = 0$ if and only if $\operatorname{Im} s'(t_0) = 0$ and this last equality takes place if and only if the tangent to Γ at s_0 is horizontal. Moreover, if the tangent to Γ at s_0 is horizontal, i.e. $\operatorname{Im} s'(t_0) = 0$, then there must be a component Ψ of the pre-image by $f'(s)$ of the real axis passing through s_0, since $f'(s_0)$ is real.

Theorem 7.2. Every curve Ψ'_m intersects a unique curve Γ'_m and every curve $\Psi_{m,k}$ intersects a unique curve $\Gamma_{m,k}$ such that green meets always blue and orange meets always red, except for $\Psi_{m,0}$ which can intersect $\Gamma_{m,0}$ in both, the red and the blue part. The curves $\Psi^+_{m,k}$ and $\Psi^-_{m,k}$ do not intertwine with any curve Γ.

Proof: Let us notice first that there is no zero or pole of $\zeta_{A,\mathbb{N}}(s)$ on Γ'_m, since this curve belongs to the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the open interval $(1, +\infty)$ of the real axis. Also, there is no zero or pole of $\zeta'_{A,\mathbb{N}}(s)$ on Ψ'_m since this curve belongs to the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the open interval $(-\infty, 0)$ of the real axis. Thus s_0 from the Lemma 7.1 is not a branch point of $\zeta_{A,\mathbb{N}}(s)$ or $\zeta'_{A,\mathbb{N}}(s)$. Moreover, if the equation of the curve Γ'_m is $s = s(\tau)$, $\tau > 1$, where $\zeta_{A,\mathbb{N}}(s(\tau)) = \tau$, $s_0 = s(\tau_0)$, then $\zeta'_{A,\Lambda}(s_0)$ and $s'(\tau_0)$ are well defined and

$$\zeta'_{A,\mathbb{N}}(s(\tau_0))s'(\tau_0) = 1 \quad (33)$$

When τ varies from 1 to $+\infty$ on the real axis, its image $s(\tau) = \sigma(\tau) + it(\tau)$ moves on Γ'_m such that $\sigma(\tau)$ varies from $+\infty$ to $-\infty$, thus $\sigma'(\tau) < 0$. Since, by Theorem 2, $t'(\tau_0) = 0$, the equality (5) can be satisfied if and only if $\zeta'_{A,\mathbb{N}}(s_0) < 0$, i.e. if and only if s_0 belongs to the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the negative real half axis. If it were on a curve $\Psi_{m,k}$, then Γ'_m would meet the pre-image of the positive real half axis, which is excluded, hence s_0 belongs to a curve Ψ'_m. Consecutive curves Ψ'_m and Ψ'_{m+1} form infinite strips Σ_m which almost coincide with S_m. By the color alternation rule, the curves $\Psi_{m,k}$ must intertwine with the corresponding curves $\Gamma_{m,k}$, and satisfy the color matching rule. These affirmations are illustrated by Figure 6 below for the Dirichlet function $L(14, 2, s)$.

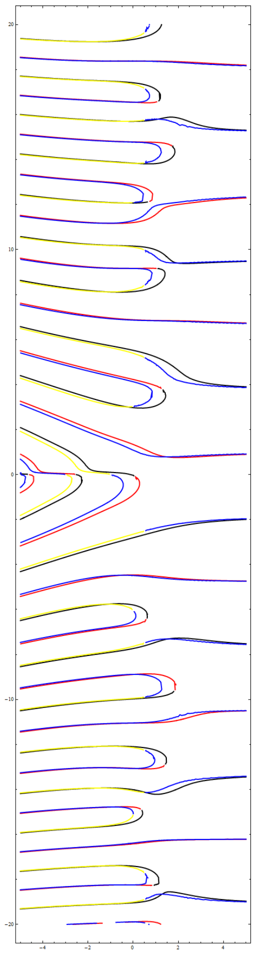

Fig. 6: Intertwining curves of Dirichlet function $L(14, 2, s)$. Source: Created by the authors.

Let us notice that although the strips S_m and Σ_m are very similar and almost coincide, the landscapes of pre-images by $\zeta_{A,\mathbb{N}}(s)$ and $\zeta'_{A,\mathbb{N}}(s)$ of the real axis are different in some ways. This difference is due to the fact that the poles of order p of $\zeta_{A,\mathbb{N}}(s)$ become poles of order $p+1$ for $\zeta'_{A,\mathbb{N}}(s)$, and also $\zeta'_{A,\mathbb{N}}(s)$ has more zeros, which implies more components of the pre-image of the real axis by this last function and more fundamental domains.

8 The Behavior of $\zeta_{A,\mathbb{N}}(s)$ On The Imaginary Axis

We deal only with infinite Blaschke products (9) which do not cancel at the origin.

We have shown in [1], that for every such Blaschke product $B(z)$, the function (12) is a Dirichlet function

having the expansion

$$\zeta_{A,\mathbb{N}}(s) = \frac{1}{B(0)}\Sigma_{k=0}^{\infty}\frac{B^{(k)}(0)}{k!}e^{-ks} \quad (34)$$

convergent for $\operatorname{Re} s > 0$. Let E be the set of cluster points of the zeros of $B(z)$, which is located on the unit circle. The points of E are non-isolated essential singular points of $B(z)$, since they are also cluster points of the poles of $B(z)$. Let E_s be the set of essential singular points of $\zeta_{A,\mathbb{N}}(s)$. We will argue later that E_s is located on the imaginary axis and $s_0 \in E_s$ if and only if $e^{-s_0} \in E$. When the set $\{s|\ \operatorname{Re} s = 0\}\setminus E_s$ contains at least an open interval, the function (12) is a meromorphic function in $\widehat{\mathbb{C}} \setminus E_s$ having the poles $\ln r_n + (2k\pi - \alpha_n)i$, $k \in \mathbb{Z}$.

The series (20) is a normalized Dirichlet series $\Sigma_{k=0}^{\infty}a_k e^{-ks}$, with $a_0 = 1$, hence

$$\lim_{\sigma\to+\infty}\zeta_{A,\mathbb{N}}(\sigma+it) = 1 \quad (35)$$

for every $t \in \mathbb{R}$. Moreover, $\zeta'_{A,\mathbb{N}}(s) = -e^{-s}B'(e^{-s})/B(0)$ is also a meromorphic function in $\widehat{\mathbb{C}} \setminus E_s$ having the same poles as $\zeta_{A,\mathbb{N}}(s)$ with an order of multiplicity of a unit higher. The set E_s is as well the set of essential singular points of $\zeta'_{A,\mathbb{N}}(s)$.

To every orthogonal limit (9) corresponds limits $\lim_{\sigma\to 0}\zeta^{(k)}_{A,\mathbb{N}}(\sigma+it)$, $k = 0, 1, 2, ...$ where $\sigma = \ln r$, $t = \theta + 2j\pi$, $j \in \mathbb{Z}$, $\zeta^{(k)}_{A,\mathbb{N}}(\sigma+it) \in \mathbb{R}$. These limits are taken on an a curve orthogonal to the imaginary axis. By Riesz theorem, [10], we can state:

Theorem 8.1. For almost every point it, $t \in \mathbb{R}$ the function $\zeta_{A,\mathbb{N}}(s)$ has the orthogonal limit $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma+it)$ which is of modulus $1/B(0)$ and the Frostman condition for the generating Blaschke product is necessary and sufficient for this limit to be in module $1/B(0)$ at all points it, $t = \theta+2k\pi$, $k \in \mathbb{Z}$.

Proof: Indeed, by Riesz theorem, $\lim_{r\to 1^-} B(re^{i\theta})$, exists for almost every $\theta \in \mathbb{R}$ and it is of modulus 1. Then, for $\sigma = \ln r$, this is $\lim_{\sigma\to 0} B(0)\zeta_{A,\mathbb{N}}(\sigma + it)$, where $t = \theta + 2k\pi$, $k \in \mathbb{Z}$, and it is of modulus 1. This means that $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma+it)$ exists and is of modulus $1/B(0)$ for almost every $t \in \mathbb{R}$. Moreover, since the Frostman condition is necessary and sufficient for $\lim_{r\to 1} B(re^{i\theta})$ to exist and be of modulus 1 at any $\theta \in \mathbb{R}$, then it is a necessary and sufficient condition for $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma + it)$ to exist and be of modulus $1/B(0)$ at any point it, $t = \theta + 2k\pi$, $k \in \mathbb{Z}$ on the imaginary axis.

In particular, if the series (7) converges at a point $e^{i\theta_0}$, which is an essential non-isolated singular point of $B(z)$, then $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma + it)$ exists and it is of modulus $1/B(0)$ for every $t = \theta_0 + 2k\pi$, $k \in \mathbb{Z}$. We notice that for such a t the point it is a non-isolated essential singular point of $\zeta_{A,\mathbb{N}}(s)$.

In [8], the question about the structure of an arbitrary set E on the unit circle for which a Blaschke product B exists having E as set of essential singular points and its radial limit at $e^{i\theta}$ is defined and has the modulus 1 at every point $e^{i\theta} \in E$, is answered, namely E needs to be closed and nowhere dense in the unit circle.

Theorem 8.2. A necessary and sufficient condition of the existence of a Blaschke product B such that the Dirichlet function $\zeta_{A,\mathbb{N}}(\sigma + it) = B(e^{-\sigma-it})/B(0)$ has a limit of modulus $1/B(0)$ as $\sigma \to 0$ for $\zeta_{A,\mathbb{N}}(\sigma + it) \in \mathbb{R}$ is that for every essential singular point of $\zeta_{A,\mathbb{N}}(\sigma + it)$, $it \in E_s$ implies $(t + 2k\pi)i \in E_s$ for every $k \in \mathbb{Z}$ and E_s is closed and nowhere dense on the imaginary axis.

Proof: Indeed, this limit exists and it is $\zeta_{A,\mathbb{N}}(it)$ if $it \notin E_s$ and for every such t we have $|\zeta_{A,\mathbb{N}}(it)| = 1/B(0)$. We have seen that $e^{i\theta} \in E$ if and only if $(\theta + 2k\pi)i \in E_s$ for every $k \in \mathbb{Z}$. This implies that E is closed and nowhere dense on the unit circle if and only if E_s is closed and nowhere dense on the imaginary axis. If this condition is satisfied for E, then by [8], there is a Blaschke product $B(z)$ having E as the set of essential singular points and such that $\lim_{r\to 1} B(re^{i\theta})$ exists and is of modulus 1 at every point of the unit circle. Then $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma + it) = \lim_{r\to 1} B(re^{\theta})/B(0)$ is defined and of modulus $1/B(0)$. Then the orthogonal limit $\lim_{\sigma\to 0}\zeta_{A,\mathbb{N}}(\sigma + it)$ exists and is of modulus $1/B(0)$ at every point of the imaginary axis if and only if E_s is closed and nowhere dense on the imaginary axis.

Let us define as in [9], the *Frostman function* for $B(z)$:

$$\varphi_B(\theta) = \Sigma_{n=1}^{\infty} \frac{1-|z_n|}{|e^{i\theta}-z_n|} \quad (36)$$

and call *Frostman condition at* $e^{i\theta}$ the inequality $\varphi_B(\theta) < +\infty$. By [9], we can say that the equality $B^0(e^{i\theta}) = 1$ holds if and only if $\varphi_B(\theta) < +\infty$.

It was also proved in [8], that if a Blaschke product $B(z)$ given by (5) satisfies the condition

$$\Sigma_{n=1}^{\infty}[(1-|z_n|) \,/\, |e^{i\theta}-z_n|^2] < \infty \quad (37)$$

at a point $e^{i\theta}$ of the unit circle, then $B'(e^{i\theta})$ exists and it is $\lim_{r\to 1} B'(re^{i\theta})$.

Let us notice that if (22) holds for then (6) holds, thus if the derivative of a Blaschke product $B(z)$ has a finite radial limit at every point $e^{i\theta}$ of the unit circle, then the set E_s of essential singular points of the Dirichlet function $\zeta_{A,\mathbb{N}}(s) = B(e^{-s})/B(0)$ is closed and nowhere dense on the imaginary axis.

The function $\zeta_{A,\mathbb{N}}(s)$ is a periodic function of period $2\pi i$, i.e. $\zeta_{A,\mathbb{N}}(s+2k\pi i) = \zeta_{A,\mathbb{N}}(s)$, $k \in \mathbb{Z}$, for every $s \in \mathbb{C}$. Then $\zeta'_{A,\mathbb{N}}(s+2k\pi i) = \zeta'_{A,\mathbb{N}}(s)$, $k \in \mathbb{Z}$, for every $s \in \mathbb{C}$, thus $\zeta'_{A,\mathbb{N}}(s)$ is also periodic of period $2\pi i$. Moreover, the curves Ψ'_m partition the complex plane into infinitely many strips Σ_m which are identical in the sense that for every $k \in \mathbb{Z}$, the strip Σ_m can be carried over the strip Σ_{m+k} by a translation of vector $2k\pi i$. The zeros, the poles, as well as the components of the pre-image by $\zeta'_{A,\mathbb{N}}(s)$ of the real axis from the strip Σ_m are carried onto the corresponding zeros, poles and components of the pre-image of the real axis from Σ_{m+k}.

Similarly, we have $\zeta''_{A,\mathbb{N}}(s+2k\pi i) = \zeta''_{A,\mathbb{N}}(s)$, $k \in \mathbb{Z}$, for every $s \in \mathbb{C}$, hence the translations of vector $2k\pi i$ will bring into coincidence the zeros of $\zeta''_{A,\mathbb{N}}(s)$ from the two strips, Σ_m and Σ_{m+k}. By consequence, to find the fundamental domains of $\zeta'_{A,\mathbb{N}}(s)$ it is enough to find those fundamental domains included in Σ_m, since every strip Σ_{m+k} will have fundamental domains identical to these ones.

By [9], $B(z)$ is analytic at $e^{i\theta_0}$ if and only if $\varphi_B(\theta_0) < \infty$ (i.e., $B(z)$ satisfies the Frostman condition at $e^{i\theta_0}$,hence also the Blaschke condition at $e^{i\theta_0}$) and φ_B is continuous at θ_0. Furthermore, if $B(z)$ is analytic at every point of a closed subset K of the unit circle, then the series (23) converges uniformly for $e^{i\theta} \in K$, thus $\varphi_B(\theta)$ is finite-valued and continuous at every point θ with $e^{i\theta} \in K$.

Theorem 8.3. For every Dirichlet function generated by a Blaschke product $B(z)$ the set E_s of essential singular points of $\zeta_{A,\mathbb{N}}(s)$ and of all its derivatives is exactly the set of their discontinuity points.

Proof: Indeed, if $it_0 \in E_s$, then $e{-}^{it_0} \in E$, and there is a sub sequence $z_{n_k} = r_{n_k}e^{-it_{n_k}}$ of zeros of $B(z)$ converging to e^{-it_0}. Since $B(z_{n_k}) = 0$, we have $\lim_{n_k\to\infty} B(r_{n_k}e^{-it_{n_k}}) = 0$. On the other hand the radial limit of $B(z)$ at $e{-}^{it_0}$ is 1, thus $B(z)$ has no limit at e^{-it_0}, hence $\zeta_{A,\mathbb{N}}(s)$ has no limit at it_0. If $it_0 \notin E_s$ then $\zeta_{A,\mathbb{N}}(s)$ and all its derivatives are analytic in a neighborhood of it_0, which completely proves the theorem.

9 Algebraic Structures Associated with Blaschke Products and Direchlet Functions Generated by Them

In [3], it was shown that for a finite Blaschke product $B(z)$, the set of continuous functions u on the unit circle such that $B \circ u = B$ is a cyclic group and the study of the possibility of extending these functions to $\widetilde{u}$ defined in a neighborhood of the unit circle such that $B\circ \widetilde{u} = B$ is undertaken.

Theorem 3.3 allows us to expand on this idea. Suppose that $B(z)$ is a Blaschke product of degree n and let $\Omega_k^{(r)}$ be its reduced fundamental domains as in Theorem 3.3. For arbitrary distinct k and j, let us define

$$U_{k,j}(z) = B^{-1}_{|\Omega_k^{(r)}} \circ B_{|\Omega_j^{(r)}}(z) \quad (38)$$

This is a conformal mapping of $\Omega_j^{(r)}$ onto $\Omega_k^{(r)}$, which by Riemann- Caratheodory theorem, can be extended to a continuous mapping of $\overline{\Omega}_j^{(r)}$ onto $\overline{\Omega}_k^{(r)}$ such that $\partial\Omega_j^{(r)}$ is topologically mapped onto $\partial\Omega_k^{(r)}$. Moreover, it is obvious that

$$B \circ U_{k,j}(z) = B(z) \quad (39)$$

i.e. every function $U_{k,j}(z)$ is an invariant of $B(z)$, as defined in [3]. Yet, here these invariants are defined in whole reduced fundamental domains of $B(z)$. If we denote by η_k the arcs of the unit circle included in $\overline{\Omega}_k^{(r)}$, then it is obvious that $U_{k,j}(\eta_j) = \eta_k$, $U_{k,j}(\xi_j) = \xi_k$, $U_{k,j}(\gamma_j) = \gamma_k$. These equalities can be easily checked by inspecting q above.

Let us notice that adjacent reduced fundamental domains $\Omega_k^{(r)}$ and $\Omega_{k+1}^{(r)}$ have in common a part of their boundaries and the functions $B_{|\overline{\Omega}_k^{(r)}}$ and $B_{|\overline{\Omega}_{k+1}^{(r)}}$ coincide on it, thus $B_{|\overline{\Omega}_k^{(r)} \cup \overline{\Omega}_k^{(r)}}$ is an analytic continuation of $B_{|\overline{\Omega}_k^{(r)}}$ and in the end $B(z) = B_{\cup_{j=1}^n |\overline{\Omega}_j^{(r)}}(z)$ is an analytic continuation of every $B_{|\overline{\Omega}_k^{(r)}}$, except for the poles of $B(z)$. The image by it of the complex plane with the slits $B^{-1}(L)$ covers n times the complex plane.

The composition law $U_{k,j} \circ U_{j,m}(z) = U_{k,m}(z)$ can be used to build an algebraic structure on these mappings similar to that mentioned in [2]. However, this will not be a cyclic group, since there is no generating element. Also, $U_{k,j}$ and $U_{l,m}$ cannot be composed directly for $j \neq l$ to obtain $U_{k,m}$, although $(U_{k,j} \circ U_{j,l}) \circ U_{l,m} = U_{k,j} \circ (U_{j,l} \circ U_{l,m}) = U_{k,m}$.

Every invariant $U_{k,j}(z)$ of $B(z)$ has an inverse function $U_{j,k}(z)$, which is also invariant of $B(z)$.

We have seen in [1], that for the Dirichlet series (12) defined by a Blaschke product of degree n with the zeros $z_k = r_k e^{i\alpha_k}$, $0 < r_1 \leq r_2, \leq ... \leq r_n < 1$ the abscissa of convergence is $\sigma_c = \ln r_n < 0$ and the series admits an analytic continuation to the complex plane , except for the poles, which are $\ln r_k + (2m\pi - \alpha_k)i$, $k = 1, 2, ..., n$ and $m \in \mathbb{Z}$. We keep the notation $\zeta_{A,\mathbb{N}}(s)$ for this function. It is a periodic function of period $2\pi i$. The pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the interval $(1, +\infty)$ of the real axis partitions the complex plane into infinitely many horizontal strips S_m, which are mapped, not necessarily one to one onto the complex plane with a slit alongside that interval. The strip S_m are identical, in the sense that they can be carried one over the other by a translation of vector $2k\pi i$, which sets into coincidence the zeros, the poles, and the components of the pre-image by $\zeta_{A,\mathbb{N}}(s)$ of the real axis. Every strip S_m contains $2n - 2$ zeros of $B'(z)$, two by two symmetric with respect to the imaginary axis. If $B'(\zeta_k) = 0$, $\mathrm{Re}\,\zeta_k > 0$, the pre-image of the ray from $w = 0$ to $w = B(\zeta_k)$ is an unbounded curve Υ_k starting at s_k with $\zeta_{A,\mathbb{N}}(s_k) = 1$ and passing through ζ_k. Similarly, if $B'(\zeta'_k) = 0$, $\mathrm{Re}\,\zeta'_k < 0$, then the pre-image of the ray from $w = B(\zeta'_k)$ is an unbounded curve Υ'_k starting at the pole s'_k and passing through the zero ζ'_k of $B'(z)$. These curves and the components of the pre-image of the interval $(0, +\infty)$ of the real axis partition every strip S_m into n sets whose interiors ω_k are fundamental domains of $\zeta_{A,\mathbb{N}}(s)$. They are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis and alongside the rays Υ_k and Υ'_k. Let $\Upsilon = \cup_{k=1}^n (\Upsilon_k \cup \Upsilon'_k)$, $L = B(\Upsilon)$ and $\omega_k^{(r)} = \omega_k \backslash \Upsilon$. The domains $\omega_k^{(r)}$ are reduced fundamental domains of $\zeta_{A,\mathbb{N}}(s)$ and they are conformally mapped by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with the same slit L.

Let us denote by $\phi_k(s)$ the restriction of $\zeta_{A,\mathbb{N}}(s)$ to $\omega_k^{(r)}$ and define:

$$u_{k,j}(s) = \phi_k^{-1} \circ \phi_j(s) \quad (40)$$

These are conformal mappings of $\omega_j^{(r)}$ onto $\omega_k^{(r)}$. Their restriction to the unit circle are the *shifts* defined in [3], and they constitute isomorphic algebraic structures . However, $u_{k,j}(s)$, as conformal mappings represent a more powerful tool. Similarly to the mappings (24),they admit continuations to the boundaries of the two domains, which are topologically mapped one onto the other. For adjacent domains $\omega_k^{(r)}$ and $\omega_{k+1}^{(r)}$ these continuations are analytic on the common portion of the boundary and $u_{k,j}(s)$ admits an analytic continuation to $\omega_k^{(r)} \cup \omega_{k+1}^{(r)} \cup (\partial\, \omega_j^{(r)} \cap \partial \omega_{k+1}^{(r)})$ whose image by $B(z)$ covers twice the complex plane with the slit. The union D_m of all these domains and of their common boundaries is a subset of S_m whose image by $\zeta_{A,\mathbb{N}}(s)$ covers infinitely many times the complex plane.

10 Conclusions

By using a geometric approach to the study of Blaschke products we succeeded to revisit some classic results on Blaschke product theory and to essentially improve these results. We have shown that the invariants of Blaschke products, defined up to now only on the unit circle and, for the finite case extended in a neighborhood of the unit circle, are in fact analytic functions mapping fundamental domains one on each other. Except for [11], nothing has been said, to our knowledge, about infinite Blaschke products having dense sets E of essential singular points. We undertook the study of the most general case of an arbitrary E such that the Blaschke product is a meromorphic function in $\widehat{\mathbb{C}} \setminus E$ and we have decrypted also the behavior of a Blaschke product in the neighborhood of E.

The theory of Dirichlet functions generated by Blaschke products is a field initiated by us and has the merit of revealing the complexity of Dirichlet functions hardly imaginable before.

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