On n^{th} Roots of Unity and k^{th} Order Involutive Matrices

HASAN KELEŞ[1], FRANK J. HALL[2]
[1]Department of Mathematics
Karadeniz Technical University
Campus of Kanuni, Ortahisar, 61080 Trabzon
TÜRKÍYE

[2]Department of Mathematics and Statistics
Georgia State University, Georgia Atlanta
USA

Abstract: The n^{th} roots of unity play a central role in algebra and complex analysis, with important applications in number theory, Fourier analysis, and quantum mechanics. This paper investigates the solutions of the equation $x^n = 1$ in both real and complex domains, emphasizing their algebraic structure and geometric representation.
We further introduce k^{th} order involutive matrices as a generalization of classical involutive matrices and analyze their fundamental spectral properties, including eigenvalue distributions an^{th} roots of unity and the eigenvalues of k^{th} order involutive matrices is established, providing new insight into their cyclic behavior. These results underscore the relevance of higher-order involutive matrices in linear algebra, quantum mechanics, and computational mathematics.

Key-Words: Roots of unity, involutive matrices, k^{th} order involutive, eigenvalues, linear algebra, spectral analysis, Fourier analysis, quantum mechanics.
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1 Introduction

The equation

$$x^n = 1 \tag{1}$$

represents one of the most fundamental problems in algebra: finding all numbers which, when raised to the n-th power, yield unity. Solutions to this equation, called n^{th} roots of unity, play a significant role in diverse areas of mathematics.

In number theory, n^{th} roots of unity underpin cyclotomic polynomials and integer factorization problems, [1], [2], [3]. In Fourier analysis and signal processing, they form the basis of the discrete Fourier transform (DFT), which is widely used in digital communications, image processing, and data compression, [4], [5]. In quantum mechanics, n-th roots of unity appear naturally in the description of quantum gates, unitary operators, and periodic systems, [6], [7]. Additionally, recent work, [8] studies generalized G-matrices and involutive matrices with applications in higher-order matrix theory, highlighting the relevance of n^{th} roots of unity in constructing special matrix classes. Recent work on Hyper G-matrices, [9] provides important insights into the diagonal equivalence relations and matrix division operations that form the foundation for the properties examined in this section. Similarly, [10] investigates structural properties of involutive matrices within triangular matrix rings, providing further insight into the algebraic behavior of these matrices.

In linear algebra, the notion of involutory matrices, i.e., matrices A satisfying

$$A^2 = I, \tag{2}$$

appears naturally in various contexts including reflection operators, symmetric transformations, and certain classes of linear systems, [11], [12], [13]. Extending this idea, we explore k^{th} *order involutive matrices* as matrices A satisfying

$$A^k = I, \tag{3}$$

for integer $k \geq 2$. For a rigorous treatment of the complex roots of unity and their properties in $\mathbb{C}$, we refer the reader to, [14]. The set of complex matrices of order n is denoted by $\mathbb{M}_n(\mathbb{C})$. Of course, this includes all $n \times n$ real matrices. These matrices generalize classical involutive matrices and have eigenvalues that are precisely k^{th} roots of unity. Such matrices are relevant in group representations, periodic linear transformations, quantum mechanics, and cryptography, [14], [15]. In the context of cryptographic protocols and algebraic structures, [16] proposed a double threshold condition for multisignature schemes in blockchain and a key exchange protocol based on non-commutative groups with a semidirect product structure. Understanding the structural properties of k^{th} order involutive matrices allows for deeper insights into matrix exponentiation, spectral analysis, and applications in numerical linear algebra, [17].

[18] contributes to the structural and factorisation properties of matrices, complementary aspects of the theory of special matrices. Additional studies on matrix congruence and involutive equivalence relations have been developed, [19], providing important classification results. The algebraic structure of matrices generated by roots of unity has also been examined, [20], especially in the context of unitary representations and spectral decompositions.

Furthermore, eigenvalue distributions and trace identities associated with structured matrices play an important role in matrix theory and graph theory. Inequalities involving eigenvalues of strongly regular graphs have been investigated, [21], while singular linear systems and their eigenstructure behavior have been studied, [22], offering complementary analytical tools relevant to the spectral framework considered in this work.

This paper is structured as follows. Section 2 examines the solutions of (1), which are straightforward but provide insight into the symmetry and parity considerations of the equation. This section explores k^{th} order involutive matrices, their definitions, properties, eigenvalues, and examples.

In Section 3, we study k^{th} order involutive matrices. Since $A^k = I_n$, all eigenvalues of A are kth roots of unity. If $k_1, k_2, \ldots, k_m$ denote their multiplicities, then

$$k_1 + k_2 + \cdots + k_m = n, \tag{4}$$

which forms the basis for their classification.

In conclusion, to summarize, this work investigates k^{th} order involutive matrices, emphasizing their spectral properties and structural behavior. We show that such matrices have eigenvalues that are k^{th} roots of unity and are diagonalizable over $\mathbb{C}$, while also revealing interesting commutativity phenomena and potential applications in algebra, quantum computing, and graph theory.

Recent studies have further expanded the theory of roots of unity and their matrix applications. [23] investigated the composition series of solvable multiplicative abelian groups over regular even nth roots of unity, providing new algebraic insights. [24] examined maximal determinants of matrices constructed over roots of unity, revealing important extremal properties. [25] explored applications of nth powers of 2×2 matrices to second-order linear recurrences and birth-death processes. [26] provided a classical algebraic perspective on roots of unity in the context of advanced modern algebra. [27] developed an algorithm for computing nth roots of unity in bit-reversed order, with applications to signal processing. [28], [**?**], [30] conducted extensive research on associative ternary algebras and ternary Lie algebras at cube roots of unity. [31] analyzed Hermitean matrices of roots of unity and their characteristic polynomials. [32] established combinatorial and Gaussian foundations for rational nth root approximations. [33] studied complex equiangular Parseval frames and Seidel matrices containing pth roots of unity. [34] proposed improved security distributed matrix multiplication via roots of unity and grid partitioning. [35], [36] investigated extended T-systems, Q matrices, and T-Q relations for sl(2) models at roots of unity. [37] examined hidden twisted sectors and exponential degeneracy in root-of-unity XXZ Heisenberg chains. [38] studied ternary associators, ternary commutators, and ternary Lie algebras at cube roots of unity. [39] analyzed linear combinations of tripotent, idempotent, and involutive matrices. Finally, [40] investigated Hermite-Birkhoff interpolation in the nth roots of unity, providing fundamental results in approximation theory.

2 Basic Results

In this section, we introduce fundamental definitions, theorems, lemmas, and properties that will play a crucial role in the subsequent analysis of the equation $x^n = 1$ (see (1)) and k^{th} order involutive matrices (see (3)). These results are primarily based on [8], [10] together with classical references in matrix theory [2], [11].

We begin with a well-known concept in the literature, which serves as a foundation for this study. A complex number $x \in \mathbb{C}$ is called an n^{th} root of unity if

$$x^n = 1 \tag{5}$$

In a similar way, many references in linear algebra adopt and further develop the classical definition of involutive matrices.

Definition 2.1. [15] A square matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called involutory if

$$A^2 = I_n \tag{6}$$

where I_n is the identity matrix of order n. In [15], the term involution is also used in this case.

Proposition 2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a nonsingular matrix. Then*

$$A^2 = I_n \iff A^{-1} = A \tag{7}$$

After the definition of matrix division, we focused on the factors of matrices and extended the definition of involutive matrices, which was originally defined only for the second order as in (6).

Definition 2.3 ([8])**.** A square matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called k^{th} order involutive matrix if

$$A^k = I_n \tag{8}$$

for an integer $k \geq 2$. The smallest positive integer k that satisfies this condition is called the involutive order.

The set of k^{th} order involutive matrices is denoted by

$$I^k_{\mathbb{M}_n(\mathbb{C})} = \{A | A^k = I_n, A \in \mathbb{M}_n(\mathbb{C})\}.$$

We next mention some elementary results:

Lemma 2.4 ([16, pp. 2278–2279])**.** *If A is a k^{th} degree involutive matrix (i.e. $A^k = I$ as in (8)), then every eigenvalue λ of A satisfies*

$$\lambda^k = 1 \tag{9}$$

In general, as shown in Lemma 2.4, the eigenvalues of A^k are the k^{th} power of the eigenvalues of A (see [15, Theorem. 1.3.12]) polynomial functions of matrices.

Proposition 2.5. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be diagonal and invertible matrices satisfying $A^s = I_n$ and $B^k = I_n$ for some positive integers s, k, where $m = \operatorname{lcm}(s, k)$. Then,*

$$(AB)^m = I_n, \quad \textit{for some } m \in \mathbb{N} \tag{10}$$

Proof. Since A and B are diagonal, $AB = BA$. Then we can write

$$(AB)^m = A^m B^m.$$

If $A^s = I_n$ and $B^k = I_n$, let $m = \operatorname{lcm}(s, k)$. Then

$$A^m = I_n, \quad B^m = I_n \Rightarrow (AB)^m = A^m B^m = I_n.$$

□

Example 2.6. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^2 = I_3,$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad B^3 = I_3,$$

where

$$\omega = e^{2\pi i/3}.$$

Since A and B are diagonal, they commute:

$$AB = BA. \tag{11}$$

Furthermore, as shown in Proposition 2.5 (see (10)),

$$(AB)^6 = A^6 B^6 = (A^2)^3 (B^3)^2 = I_3.$$

Theorem 2.7. *Every k^{th} order involutive matrix (see Definition 2.3 and (8)) with distinct eigenvalues is diagonalizable over $\mathbb{C}$.*

In fact, as stated in Theorem 2.7, every matrix with distinct eigenvalues is diagonalizable.

Lemma 2.8 ([16])**.** *Let T be an upper triangular k^{th} order involutive matrix satisfying (8). Then the diagonal entries of T are k^{th} roots of unity as in (9).*

It is well known that the k^{th} power of an upper triangular matrix T is an upper triangular matrix, with diagonal entries that are the k^{th} power of the diagonal entries of T.

Theorem 2.9. *Let $A, B \in I^k_{\mathbb{M}_n(\mathbb{C})}$ be nonsingular matrices such that $A^k = I_n$ and $B^k = I_n$ as in (8). If $AB = BA$, then $AB \in I^k_{\mathbb{M}_n(\mathbb{C})}$.*

Proof. Since A and B commute, we claim that

$$(AB)^m = A^m B^m \quad \text{for all } m \in \mathbb{Z}^+.$$

We prove this by induction on m.

Base step: For $m = 1$, clearly $(AB)^1 = AB = A^1B^1$.

Inductive step: Assume $(AB)^m = A^mB^m$ holds for some $m \geq 1$. Then

$$(AB)^{m+1} = (AB)^m(AB) = (A^mB^m)(AB).$$

Since A and B commute, $B^mA = AB^m$. Hence

$$(A^mB^m)(AB) = A^m(B^mA)B = A^m(AB^m)B = A^{m+1}B^{m+1}.$$

Thus, by induction, $(AB)^m = A^mB^m$ for all $m \in \mathbb{Z}^+$.

Finally, for $m = k$ we obtain

$$(AB)^k = A^kB^k = I_nI_n = I_n.$$

This completes the proof. □

Example 2.10. Given the nonsingular matrices:

$$A = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we have

$$A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^4 = (A^2)^2 = I_3,$$

$$B^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B^4 = (B^2)^2 = I_3,$$

$$AB = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$BA = \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with

$$(AB)^4 = \begin{bmatrix} (1)^4 & 0 & 0 \\ 0 & (1)^4 & 0 \\ 0 & 0 & 1^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

The converse of Theorem 2.9 is generally not true. That is, even if $AB \in I^k_{\mathbb{M}_n(\mathbb{C})}$, it doesn't necessarily mean that $AB = BA$.

Example 2.11. Let our matrices be:

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We first compute the square of each matrix to check their order.

$$A^2 = A \cdot A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2.$$

We get:

$$A^4 = A^2 \cdot A^2 = I_2.$$

Similarly for matrix B:

$$B^2 = B \cdot B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2,$$

we have:

$$B^4 = (-I_2)(-I_2) = I_2.$$

And with

$$(AB)^2 = (AB) \cdot (AB) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$
$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2,$$

we have

$$(AB)^4 = (AB)^2 \cdot (AB)^2 = I_2.$$

But for the following operations, $AB \neq BA$.

$$AB = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

Theorem 2.12. *If S is a nonsingular matrix in $\mathbb{M}_n(\mathbb{C})$ and $A \in I^k_{\mathbb{M}_n(\mathbb{C})}$ satisfying* (8)*, then $S^{-1}AS$ is a k^{th} order involutive matrix which has the same eigenvalues as A.*

The proof is straightforward and makes use of a collapsing matrix product.

In this way, as shown in Theorem 2.12, we may go from a sparse k^{th} order involutive matrix to one which is dense.

Theorem 2.13. *If $A \in I^k_{\mathbb{M}_n(\mathbb{C})}$ as in* (8)*, s is any positive integer, and*

$$\alpha = e^{\frac{2\pi i}{sk}},$$

then αA is an sk-order involutive matrix. Furthermore, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A satisfying (9)*, then $\alpha\lambda_1, \ldots, \alpha\lambda_n$ are the eigenvalues of αA.*

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The proof is easy and makes use of the fact that $(\alpha A)^{sk} = e^{2\pi i} A^{sk} = 1 I_n$. Because of this result (Theorem 2.13), we can rotate the eigenvalues of an involutive matrix in obtaining a new involutive matrix.

In Problem 28 on page 199 of [15], it is indicated that every involutory matrix (see Definition 2.1 and (6)) is diagonalizable. We now consider an arbitrary k^{th} order involutive $n \times n$ matrix A. Let $A = SJS^{-1}$, where S is nonsingular and J is the Jordan form of A. Recall that J is unique up to an ordering of the diagonal Jordan blocks. Now,

$$A^k = I \quad \textit{iff} \quad J^k = I \quad \textit{iff} \quad [J_p(\lambda)]^k = I \qquad (12)$$

for an arbitrary Jordan block $J_p(\lambda)$ of J. It can be observed that by writing $J_p(\lambda)$ as $\lambda I + N$ where N is nilpotent, and expanding $(\lambda I + N)^k$, we have

$$[J_p(\lambda)]^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{p-1}\lambda^{k-p+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{p-2}\lambda^{k-p+2} \\ 0 & 0 & \lambda^k & \cdots & \binom{k}{p-3}\lambda^{k-p+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \cdots & \lambda^k \end{bmatrix} \qquad (13)$$

We have $\lambda \neq 0$ and $k > 1$. For (12) to hold, we must have $p = 1$, that is, all Jordan blocks are 1×1! We thus have an important result, stated in Theorem 2.14.

Theorem 2.14. *Every k^{th} order involutive matrix A satisfying* (8) *is diagonalizable.*

This means that the geometric and algebraic multiplicities of any eigenvalue λ of A are equal, that is, λ is a semisimple eigenvalue (see [15, p. 181]). Of course, we can have repeated eigenvalues.

We can then find the Jordan forms of k^{th} order involutive matrices by enumeration. For given values of n and k, we can systematically arrange any k^{th} roots of unity as diagonal entries of a diagonal matrix J. In fact, we then have classes of k^{th} order involutive matrices of the form

$$\{SJS^{-1} \mid S \in \mathbb{M}_n(\mathbb{C}) \text{ is any nonsingular matrix}\}. \qquad (14)$$

All k^{th} order involutive matrices are obtained by such classes as shown in (14).

We now consider specific structures of matrices.

Example 2.15. Consider a general 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For $A^k = I_2$ as in (8), $k > 2$, the necessary conditions are:

$$a + d = 2\cos\left(\frac{2\pi m}{k}\right),$$
$$ad - bc = 1, \qquad m \in \{1, 2, \ldots, k-1\}.$$

For $k = 3$, choose $m = 1$, then

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Set $a = 0$, then $d = -1$, choose $b = 1$, $c = -1$ to satisfy $ad - bc = 1$. Hence:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A^3 = I_2.$$

Summary of Non-Diagonal Matrices for $k > 2$

- 2×2: $A = \begin{bmatrix} a & b \\ c & 2\cos(2\pi m/k) - a \end{bmatrix}$, $ad - bc = 1$.
- $n \times n$: Combine 2×2 and 1×1 blocks to construct non-diagonal matrices.
- Eigenvalues of each block satisfy $\lambda^k = 1$ as in (9).
- Real matrices require complex eigenvalues to occur in conjugate pairs.

Example 2.16. A 3×3 matrix can be constructed using a 2×2 non-diagonal block (similar to Example 2.15) and a 1×1 diagonal block, as shown in Figure 1:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^3 = I_3.$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}.$$

Figure 1: Block structure of the 3×3 matrix in Example 2.16

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Example 2.17. Construct A using two 2×2 blocks as in Example 2.15, as illustrated in Figure 2:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ i^2 & i^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i^2 & i^2 \end{bmatrix}, \quad A^3 = I_4.$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ i^2 & i^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i^2 & i^2 \end{bmatrix}.$$

Figure 2: Block structure of the 4×4 matrix in Example 2.17

Next, let A be an arbitrary 3×3 matrix. Suppose

$$A^2 = I_3 \quad \text{and} \quad A \neq I_3 \tag{15}$$

And

$$P := \frac{A + I_3}{2} \tag{16}$$

Then P is idempotent, i.e. $P^2 = P$. Conversely, for any projection (idempotent) matrix P we obtain

$$A = 2P - I_3, \tag{17}$$

which automatically satisfies $A^2 = I_3$ as in (15).

Therefore, requiring $A \neq I_3$ is equivalent to $P \neq I_3$. In other words, P is a nontrivial projection (its rank is $0, 1$, or 2).

Since $A^2 = I_3$ from (15), every eigenvalue of A must satisfy $\lambda^2 = 1$, which means that all eigenvalues are either $+1$ or -1. Because A is a 3×3 matrix, it has exactly three eigenvalues; let us assume that k of them are equal to $+1$ and the remaining $3 - k$ are equal to -1. The trace of a matrix is the sum of its eigenvalues, so in this case the trace equals k copies of $+1$ plus $(3 - k)$ copies of -1. This gives the expression

$$\operatorname{tr}(A) = k - (3 - k) = 2k - 3 \tag{18}$$

With

$$\operatorname{tr}(A) = 2 \operatorname{tr}(P) - 3, \qquad \operatorname{tr}(P) = \frac{\operatorname{tr}(A) + 3}{2}.$$

Since k can only take the values $0, 1, 2$, or 3, the possible traces from (18) are $-3, -1, 1$, and 3. The case $k = 3$ corresponds to all eigenvalues being equal to $+1$, which means $A = I_3$, and this is excluded under the condition $A \neq I_3$. Therefore, for a non-identity matrix with $A^2 = I_3$ as in (15), the trace can only take the values $1, -1$, or -3. We also observe the following.

Proposition 2.18. *Let A be a 3×3 matrix. Then $A^2 = I_n$ (see (6)) and $A \neq I_3$ if and only if A is similar to one of the diagonal matrices*

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This characterization generalizes the case $A^2 = I_3$ from (15): the matrix is always diagonalizable, invertible, and can be described in terms of its eigenvalues and corresponding spectral projections [2].

3 Eigenvalue Structure of Matrices Satisfying $A^s = I_n$ with $A \neq I_n$

Let $A = [a_{ij}]_3$ be a 3×3 matrix satisfying $A^3 = I_3$ and $A \neq I_3$. Since $A^3 = I_3$ as in (8), every eigenvalue λ of A must satisfy $\lambda^3 = 1$ as in (9), which means that all eigenvalues are cube roots of unity:

$$\lambda \in \{1, c, c^2\}, \quad \text{where } c = e^{2\pi i/3} \tag{19}$$

Because $A \neq I_3$, at least one eigenvalue must be different from 1, i.e., at least one eigenvalue is c or c^2 from (19). Denoting the number of eigenvalues equal to 1 by k_1, the number equal to c by k_2, and the number equal to c^2 by k_3, we have

$$k_1 + k_2 + k_3 = 3 \tag{20}$$

The trace of A is the sum of its eigenvalues:

$$\operatorname{tr}(A) = k_1 \cdot 1 + k_2 \cdot c + k_3 \cdot c^2 \tag{21}$$

For a 3×3 matrix different from the identity, possible distributions of eigenvalues include $(1, 1, c)$, $(1, 1, c^2)$, $(1, c, c^2)$, (c, c, c^2), and (c, c^2, c^2). In all cases, the necessary condition is that all eigenvalues belong to $\{1, c, c^2\}$ from (19) and at least one eigenvalue is not equal to 1. This ensures that A is invertible, and a non-identity cube root of the identity matrix.

The matrix A is invertible with $A^{-1} = A^2$, and it is diagonalizable because its minimal polynomial divides $x^3 - 1$, which has distinct roots. Also, see Theorem 2.14. In a suitable basis, A is similar to a diagonal matrix whose diagonal entries are some arrangement of 1, c, and c^2 from (19).

Equivalently, A can be expressed using projection (idempotent) as

$$A = P_1 + cP_c + c^2 P_{c^2} \tag{22}$$

where P_1, P_c, and P_{c^2} are idempotents that sum to the identity matrix and have ranks equal to the multiplicities of the corresponding eigenvalues. This fact can be explained as follows. Let $A = SJS^{-1}$, where $J = \mathrm{Diag}(1, c, c^2)$ as in (14). Partition S into its 3 columns x_1, x_2, x_3 and S^{-1} into its 3 rows $(y_1)^T, (y_2)^T, (y_3)^T$. Then, it can be seen that $P_1 = x_1(y_1)^T, P_c = x_2(y_2)^T$, and $P_{c^2} = x_3(y_3)^T$.

A concrete example of a non-diagonal matrix satisfying $A^3 = I_3$ is the cyclic permutation matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{23}$$

which also has eigenvalues $\{1, c, c^2\}$ from (19). Matrices of this type are always invertible, with inverse given by $A^{-1} = A^2$, and they naturally arise in the study of cyclic symmetries and finite group actions in linear algebra.

Example 3.1. Let $A \in I^3_{\mathbb{M}_3(\mathbb{C})}$ be satisfying $A \neq I_3$. In this case, all eigenvalues of A must be cube roots of unity as in (19):

$$\lambda \in \{1, c, c^2\}, \qquad c = e^{2\pi i/3}.$$

Table 1: Eigenvalue distributions for 3×3 matrices satisfying $A^3 = I_3$, $A \neq I_3$

Eigenvalue distribution	Example diagonal matrix	Trace
$(1, 1, c)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$	$2 + c$
$(1, 1, c^2)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	$2 + c^2$
$(1, c, c)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$	$1 + 2c$
$(1, c, c^2)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	0
$(1, c^2, c^2)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	$1 + 2c^2$
(c, c, c^2)	$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	$2c + c^2$
(c, c^2, c^2)	$\begin{bmatrix} c & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	$c + 2c^2$

The table was prepared by the authors.

The possible eigenvalue distributions for such matrices are given in Table 1.

Example 3.2. Let

$$c = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad c^2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

as in (19).

Consider the matrices

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{bmatrix}.$$

Then the matrix

$$A = SDS^{-1} = \begin{bmatrix} c & 0 & -c + c^2 \\ 0 & c & -c + c^2 \\ 0 & 0 & c^2 \end{bmatrix}$$

has some non-real entries and has non-real eigenvalues c, c, c^2 from (19), while S is a real invertible matrix.

We didn't include (c, c, c) and (c^2, c^2, c^2) in Table 1. These are covered by the next result.

Proposition 3.3. *Let $c \in \mathbb{C}$ be any complex scalar and let I_n denote the $n \times n$ identity matrix. Define the scalar matrix*

$$J = cI_n.$$

Then, for any nonsingular matrix $S \in \mathbb{M}_n(\mathbb{C})$, the similarity transformation satisfies

$$SJS^{-1} = J = cI_n.$$

Moreover, for any positive integer k, it holds that

$$SJ^k S^{-1} = J^k = c^k I_n.$$

In other words, scalar matrices are invariant under similarity transformations by invertible matrices.

Example 3.4. Let $c = 2 + i$. Then

$$J = cI_4 = (2+i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the nonsingular matrix

$$S = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \det(S) = 1 \neq 0.$$

Then, as shown in Proposition 3.3, the similarity transformation yields

$$SJ^5 S^{-1} = J^5 = c^5 I_4,$$

where

$$c^5 = (2+i)^5 = -46 - 9i.$$

Explicitly,

$$J^5 = (-46-9i)I_4 = \begin{bmatrix} -46-9i & 0 & 0 & 0 \\ 0 & -46-9i & 0 & 0 \\ 0 & 0 & -46-9i & 0 \\ 0 & 0 & 0 & -46-9i \end{bmatrix}$$

Next, consider the following table.

Table 2: General eigenvalue distributions for $A^s = I_n$, $A \neq I_n$

Eigenvalue Distribution	**Example Diagonal Matrix**	**Trace**
$\underbrace{1, \ldots, 1}_{k_0}, \underbrace{\zeta_1, \ldots, \zeta_1}_{k_1}, \ldots, \underbrace{\zeta_{s-1}, \ldots, \zeta_{s-1}}_{k_{s-1}}$	$\mathrm{diag}(\underbrace{1, \ldots, 1}_{k_0}, \underbrace{\zeta_1, \ldots, \zeta_1}_{k_1}, \ldots, \underbrace{\zeta_{s-1}, \ldots, \zeta_{s-1}}_{k_{s-1}})$	$\sum_{j=0}^{s-1} k_j \zeta_j$

This table was prepared by the authors.

In Table 2, general eigenvalue distributions for $A^s = I_n$, $A \neq I_n$, where $\zeta_j = e^{2\pi ij/s}$, multiplicities satisfy $\sum_{j=0}^{s-1} k_j = n$ as in (**??**), and at least one k_j with $j \neq 0$ is positive.

The examples in Table 1 correspond to the special and limited eigenvalue distributions for $n = 3$. This table serves as a small and concrete instance of the spectral structure governed by the same fundamental principles generalized to arbitrary $n \times n$ matrices satisfying the s^{th} root condition.

For general n, constructing a similar table would be significantly larger and more complex, but the underlying structure remains the same: the eigenvalues are chosen from the set of s^{th} roots of unity, and their multiplicities sum up to n as in (**??**).

In a general statement, let A be an $n \times n$ matrix satisfying $A^s = I_n$ as in (8) and $A \neq I_n$, where $s \geq 2$ is a positive integer. In this general case, all eigenvalues of A must be s-th roots of unity as in (9):

$$\lambda \in \{\zeta_0, \zeta_1, \ldots, \zeta_{s-1}\},\ \zeta_k = e^{2\pi ik/s},\ k = 0, 1, \ldots, s-1 \tag{24}$$

Since $A \neq I_n$, at least one eigenvalue is different from 1, that is, at least one eigenvalue is some ζ_k from (24) with $k \neq 0$. All eigenvalues are nonzero, so A is invertible with $A^{-1} = A^{s-1}$, and A is diagonalizable and similar to a diagonal matrix whose entries are some arrangement of the s^{th} roots of unity from (24). In other words, in a suitable basis,

$$A \sim \mathrm{diag}(\underbrace{\zeta_0, \ldots}_{k_0}, \underbrace{\zeta_1, \ldots}_{k_1}, \ldots, \underbrace{\zeta_{s-1}, \ldots}_{k_{s-1}}) \tag{25}$$

where $k_0 + k_1 + \cdots + k_{s-1} = n$ as in (**??**) and each k_j equals the multiplicity of ζ_j from (24). Equivalently, A can be expressed in terms of projections (idempotents) as

$$A = \sum_{k=0}^{s-1} \zeta_k P_k \tag{26}$$

where $P_0, P_1, \ldots, P_{s-1}$ are projections (idempotents) summing to the identity matrix,

$$P_0 + P_1 + \cdots + P_{s-1} = I_n \tag{27}$$

and the rank of each P_k equals the multiplicity of the corresponding eigenvalue ζ_k from (24). The previously discussed cases $A^2 = I_3$ (see (15)) and $A^3 = I_3$ are simply particular examples of this general structure for $s = 2$ and $s = 3$, respectively.

Proposition 3.5. *For some integer $k \geq 2$, let $A \in I^k_{\mathbb{M}_n(\mathbb{C})}$ as in (8). There exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ such that*

$$A = SJS^{-1},$$

where

$$J = \operatorname{diag}(\zeta_0 I_{k_0}, \zeta_1 I_{k_1}, \dots, \zeta_{k-1} I_{k_{k-1}}),$$

with $\zeta_j = e^{2\pi ij/k}$ as in (24) denoting the distinct k^{th} roots of unity, and the multiplicities $k_0, k_1, \dots, k_{k-1}$ satisfying

$$k_0 + k_1 + \cdots + k_{k-1} = n \tag{28}$$

Consequently, A can be expressed as

$$A = \sum_{j=0}^{k-1} \zeta_j P_j,$$

where $P_j \in \mathbb{M}_n(\mathbb{C})$ are mutually orthogonal projections satisfying

$$P_j P_\ell = 0 \quad \text{for } j \neq \ell, \quad \text{and} \quad \sum_{j=0}^{k-1} P_j = I_n$$

as in (27).

Example 3.6. Consider the complex scalar

$$c = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

which is a primitive 3rd root of unity as in (19).

Define the diagonal matrix

$$J = \operatorname{diag}(1, 1, c, c^2) \in \mathbb{M}_4(\mathbb{C}),$$

where $c^2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ as in (19).

Let $S \in \mathbb{M}_4(\mathbb{C})$ be an invertible matrix. Then, as shown in Proposition 3.5, the matrix

$$A = SJS^{-1}$$

satisfies

$$A^3 = SJ^3 S^{-1} = SI_4 S^{-1} = I_4,$$

and $A \neq I_4$ since $J \neq I_4$.

Concretely, for example, take

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \end{bmatrix}.$$

Then,

$$A = SJS^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & c^2 - c \\ 0 & 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -i\sqrt{3} \\ 0 & 0 & 0 & c^2 \end{bmatrix}.$$

is a 4×4 matrix with eigenvalues $\{1, 1, c, c^2\}$ from (19), satisfying $A^3 = I_4$ but $A \neq I_4$.

4 Examples of k^{th} Order Involutive Matrices

In this section, we present concrete examples of 4x4 matrices that satisfy $A^4 = I_4$ as in (8), demonstrating the practical relevance of k^{th} order involutive matrices across various fields.

In cryptography, matrices whose powers yield the identity are valuable for constructing efficient cryptographic primitives.

Example 4.1. A cyclic shift matrix used in cryptographic protocols:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This matrix satisfies $A_1^4 = I_4$ as in (8). Its eigenvalues are the fourth roots of unity from (24): $\lambda \in \{1, i, -1, -i\}$. In cryptography, such matrices provide efficient diffusion in block ciphers.

In cryptography, MDS matrices ensure optimal diffusion properties for block cipher security.

Example 4.2. An MDS (Maximum Distance Separable) matrix over GF(2^8) used in block ciphers:

$$A_2 = \begin{bmatrix} \alpha & \alpha+1 & 1 & 1 \\ 1 & \alpha & \alpha+1 & 1 \\ 1 & 1 & \alpha & \alpha+1 \\ \alpha+1 & 1 & 1 & \alpha \end{bmatrix}_{\mathrm{GF}(2^8)}$$

where α is a primitive element satisfying $\alpha^4 = 1$ as in (24). This matrix satisfies $A_2^4 = I_4$ and provides optimal diffusion for cryptographic security.

In quantum computing, the Quantum Fourier Transform is fundamental for many quantum algorithms.

Example 4.3. The Quantum Fourier Transform matrix for two qubits:

$$Q = \frac{1}{16}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

This unitary matrix satisfies $Q^4 = I_4$. It's used in quantum algorithms including Shor's factoring algorithm and quantum phase estimation.

Tensor products of Pauli matrices generate important quantum gates for error correction.

Example 4.4. The tensor product $X \otimes Z$ of Pauli matrices:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

This matrix satisfies $P^4 = I_4$ and represents a quantum gate used in stabilizer formalism for quantum error correction.

Controlled-phase gates implement conditional phase shifts in quantum circuits.

Example 4.5. A controlled-phase gate matrix with phase $\pi/2$:

$$CS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

where $i = e^{i\pi/2}$ as in (24). This matrix satisfies $(CS)^4 = I_4$ and implements a conditional phase shift in quantum circuits.

In signal processing, the Discrete Fourier Transform matrix is central to frequency analysis.

Example 4.6. The 4-point Discrete Fourier Transform matrix:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

This matrix satisfies $F_4^4 = 16I_4$. The normalized version $\frac{1}{2}F_4$ satisfies $(\frac{1}{2}F_4)^4 = I_4$, with all eigenvalues being fourth roots of unity as in (24).

Cyclic shift matrices implement circular convolution operations in signal processing.

Example 4.7. A cyclic shift matrix used in digital signal processing:

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This matrix satisfies $S^4 = I_4$ as in (8) and its eigenvalues are the fourth roots of unity: $\lambda \in \{1, i, -1, -i\}$ from (24). In signal processing, cyclic shift matrices implement circular convolution operations and are fundamental in filter design.

In coding theory, circulant matrices generate structured cyclic codes.

Example 4.8. A circulant matrix over GF(2) satisfying $C^3 = I_4$:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\mathrm{GF}(2)}$$

This matrix satisfies $C^3 = I_4$ as in (8). It consists of a 3-cycle permutation block and a 1×1 identity block.

Example 4.9. For the cyclic shift matrix A_1 from Example 4.1:

$$\text{Eigenvalues: } \lambda = e^{2\pi ik/4}, \; k = 0, 1, 2, 3 \tag{29}$$

$$\text{Trace: } \mathrm{tr}(A_1) = 0 = 1 + i + (-1) + (-i) \tag{30}$$

$$\text{Determinant: } \det(A_1) = (-1)^3 = -1 \tag{31}$$

$$\text{Check: } A_1^4 = I_4 \text{ (verified by computation)} \tag{32}$$

Table 3: Summary of 4×4 k^{th} order involutive matrices examples

Example	Matrix Type	Application Field	Key Property
A_1	Cyclic shift matrix	Cryptography	Eigenvalues: $\{1, i, -1, -i\}$; Provides diffusion
A_2	MDS matrix over GF(2^8)	Cryptography	Optimal diffusion; $\alpha^4 = 1$
Q	Quantum Fourier Transform	Quantum Computing	Unitary; $Q^4 = I_4$; Used in Shor's algorithm
$P = X \otimes Z$	Pauli tensor product	Quantum Computing	$P^4 = I_4$; Quantum error correction
CS	Controlled-phase gate	Quantum Computing	Phase $\pi/2$; Conditional phase shift
F_4	DFT matrix	Signal Processing	$F_4^4 = 16I_4$; Normalized version is involutive
S	Cyclic shift matrix	Signal Processing	Eigenvalues: $\{1, i, -1, -i\}$; Circular convolution
C	Circulant over GF(2)	Coding Theory	$C^3 = I_4$; 3-cycle permutation block

This table was prepared by the authors.

Table 3 provides a comprehensive summary of all 4×4 k^{th} order involutive matrices presented in this section, including their types, application fields, and key properties.

5 Conclusion and Future Work

This paper explored the strong connection between n^{th} roots of unity and the properties of k^{th} order involutive matrices. We showed that the eigenvalues of a matrix A satisfying the equation $A^k = I_n$ are precisely the k^{th} roots of unity, establishing a direct link between these two algebraic concepts. We also proved that all such matrices are diagonalizable over the complex number field.

Our analysis, supported by various examples, confirmed that while diagonal matrices always commute, non-diagonal k^{th} order involutive matrices can, and often do, exhibit non-commutative behavior, even when their product is also k^{th} order involutive. This finding highlights the rich and complex structure of these matrices.

Future work could focus on extending these results to non-diagonalizable cases by considering matrices over fields of prime characteristic, where the failure of complete eigenvalue decomposition—arising from repeated roots of the minimal polynomial—can naturally lead to non-diagonalizable k^{th} order involutive matrices. It would also be valuable to explore the application of these matrices in quantum computing, particularly in the construction of new quantum gates and operators with specific periodic properties. Additionally, investigating connections to graph theory and combinatorics could reveal novel applications for k^{th} order involutive matrices.

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