Hyper G-Matrices and Logical Structures in Ternary Logic B_3

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Abstract: This paper investigates Hyper G-matrices and their applications in network models and communication systems. We establish fundamental properties of Hyper G-matrices, including spectral invariance under similarity transformations, canonical block decomposition, and preservation of positive semidefiniteness under specific conditions. New theorems demonstrate that Hyper G-pairs maintain eigenvalues under diagonal similarity and admit interlacing bounds in associated block constructions. Illustrative examples, including visualizations of block matrices and Laplacian matrices of path graphs, highlight the impact of Hyper G-transformations on spectral and structural properties. Connections with ternary logic B_3 are also explored through decomposition and spectral projection techniques, suggesting potential applications in logic, computation, and network science. Future directions include algorithmic computation of Hyper G-spectra, visualization of large-scale matrices, and application to data-driven communication networks. The results provide a versatile algebraic framework bridging classical matrix theory, graph theory, and network modeling.

Key-Words: Hyper G-matrix, ternary logic B_3), signed graphs, spectral graph theory, structured matrices, quantum computation, network resilience.

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1 Introduction

This paper introduces Hyper G-matrices as a unifying framework connecting classical matrix theory, graph spectral analysis, and ternary logic B_3. The key contributions of this work include rigorous proofs of spectral invariance under similarity transformations, preservation of algebraic connectivity under specific conditions, block decomposition properties, and representation of logical structures within matrices. These results provide direct implications for network resilience, multi-valued logical systems, and computational models, offering both theoretical insights and practical applications in complex systems analysis.

Algebraic graph theory and matrix analysis provide a robust mathematical framework for studying the structural and spectral properties of complex systems. Classical tools such as adjacency, Laplacian, and incidence matrices are central in this framework, encoding connectivity, flows, and spectral characteristics of graphs, [1], [2], [3], [4].

The theory was significantly broadened with the introduction of signed graphs, where edges may carry positive or negative signs. This extension allows antagonistic and cooperative interactions to be modeled, making signed graph theory highly relevant in sociology, biology, and control theory, [5], [6], [7], [8]. The algebraic study of signed and generalized graphs has produced important results in controllability and observability of dynamical systems, as well as in the analysis of consensus and stability in multi-agent networks.

Beyond these frameworks, researchers have introduced new classes of structured matrices. Hyper G-matrices generalize classical adjacency and Laplacian representations by encoding higher-order relations and richer algebraic interactions. They provide a more flexible setting for spectral analysis and structural decomposition, extending the applicability of graph-based methods. New theoretical developments and applications of Hyper G-matrices have been established, [9], while also exploring their connections to multi-valued logical systems such as ternary logic B_3, [10].

Earlier foundational work contributed significantly to this direction, by studying generalized inverses and eigenvalue interlacing for matrices associated with graphs, [11], [12]. These tools play a crucial role in optimization theory, spectral graph analysis, and stability studies, which are directly connected to the properties of Hyper G-matrices.

In parallel, new advances in hypergraph spectral theory have introduced alternative matrix formulations, extending eigenvalue analysis from graphs to higher-order structures. New matrix frameworks for hypergraphs have been proposed, [13], [14], which complement the theory of Hyper G-matrices and suggest further unification of algebraic methods across graphs, signed graphs, and hypergraphs. These representations not only enable a combinatorial understanding of networks but also support spectral methods and frequency-based interpretations of structure, where eigenvalues and eigenvectors capture essential properties such as connectivity, expansion, and resilience, [15], [16], [17], [18], [19], [20].

Hyper G-matrices thus emerge as a unifying concept at the intersection of algebraic graph theory, multi-valued logic, and hypergraph theory. Their applications span across communication systems, where channel matrices in MIMO models can be analyzed for stability and fault tolerance; quantum computing, where ternary logic B_3 naturally connects with qudit-based computation; and network science, where eigenvalue interlacing of adjacency–Laplacian pairs offers new resilience criteria for large-scale systems.

Moreover, the relevance and applicability of Hyper G-matrices extend well beyond purely theoretical developments, as evidenced by a broad and growing body of recent literature, [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63]. These works collectively highlight the unifying role of Hyper G-matrices as a powerful algebraic framework bridging graph theory, multi-valued logic, optimization, and complex real-world system applications.

2 Preliminaries

In this section we collect the basic concepts, definitions, and auxiliary results required for the development of Hyper G-matrices and their connection with ternary logic B_3. The foundations of our work are strongly influenced by classical algebraic graph theory and spectral matrix analysis, which provide the standard tools for adjacency, Laplacian, and incidence representations, [1], [2], [3], [4]. These traditional approaches have been extended in the literature to signed graphs, where entries from $\{-1, 0, 1\}$ allow modeling both cooperative and antagonistic interactions, particularly relevant in social and communication networks, [5], [6], For detailed information on the Decomposition Lemma, see, [7], [8]. Within this context, Hyper G-matrices emerge as a promising algebraic framework, first studied in detail, [9], and connected with ternary logic B_3 in later developments, [10]. More recently, matrix models have been generalized toward hypergraph structures, with new spectral frameworks providing higher-order insights into complex systems, [13], [14].

The ternary logic B_3 is defined on the value set $B_3 = \{-1, 0, 1\}$, where 1 corresponds to "true," –1 to "false," and 0 to an indeterminate or neutral state. Logical operators are extended accordingly: the negation is given by $\neg a = -a$, conjunction and disjunction are $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$, while implication is formalized as $a \rightarrow b = \max\{-a, b\}$. This three-valued framework not only extends binary reasoning but also provides a direct connection to systems characterized by uncertainty, partial truth, or neutral states, all of which are relevant in computer science, quantum computation, and decision-making.

Matrices whose entries lie in B_3 are called B_3-matrices. Any such matrix M admits a unique decomposition into disjoint positive and negative parts: $M = P - N$ with $P, N \in \{0, 1\}^{n\times n}$ and $P \cap N = \varnothing$. This decomposition is important in linking logical operators to matrix algebra. For example, the support sets $S^+(M)$ and $S^-(M)$ naturally encode the positive and negative edges in a signed network.

To introduce Hyper G-matrices, consider $A \in \mathbb{M}_n(\mathbb{R})$ and let $\mathbb{D}_n$ denote the family of invertible diagonal matrices, typically with entries restricted to ± 1. For $D_1, D_2 \in \mathbb{D}_n$, the transformation $\mathcal{G}(A; D_1, D_2) = D_1AD_2$ is called a G-transform. The collection $\mathrm{HG}(A; \mathbb{D}_n) = \{D_1AD_2 : D_1, D_2 \in \mathbb{D}_n\}$ is the Hyper G-set, and any pair (G_1, G_2) within this family is referred to as a Hyper G-matrix pair. This construction generalizes similarity transformations and allows richer structural relationships to be captured, particularly when entries come from B_3.

Several elementary results underpin this framework. First, every B_3-matrix has the decomposition $M = P - N$ as noted below. Second, similarity by a diagonal matrix D preserves the spectrum, i.e. $\sigma(DAD^{-1}) = \sigma(A)$, which implies that the eigenvalues of a matrix remain invariant under signed diagonal scaling. For more general pairs (D_1, D_2), one obtains the bound $\rho(D_1AD_2) \leq \|D_1\| \, \|D_2\| \, \rho(A)$, ensuring stability of spectral radius under such transformations.

Logical formulas in B_3 can be represented by valuation matrices, defined as diagonal matrices with entries $\varphi(v)$ across valuations v. Equivalence of two formulas then coincides with equality of their valuation matrices. Logical connectives such as negation, conjunction, disjunction, and implication can be expressed through simple matrix operations, making the B_3 framework compatible with matrix-theoretic analysis.

From this perspective, Hyper G-matrices inherit both algebraic and logical structure. Spectral invariance under signed similarity connects naturally to the semantics of logical equivalence. Interlacing results extend classical eigenvalue inequalities: for instance, if A is symmetric and $G \in \mathrm{HG}(A)$ with $G = DAD$ for some $D \in \mathbb{D}_n$, then the block matrix $\begin{pmatrix} A & G^T \\ G & A \end{pmatrix}$ exhibits eigenvalue interlacing with respect to A. As a consequence, if A is positive semidefinite, then so is the block matrix. Such results provide spectral radius bounds, $\max\{\rho(A), \rho(G)\} \le \rho(\mathcal{B}) \le \|A\| + \|G\|$, which are particularly relevant for assessing stability.

These properties highlight the application potential of the theory. In communication systems, ternary values naturally encode neutral, positive, or negative channel states. In optimization, constraints expressed in $\{-1, 0, 1\}$ can be analyzed through Hyper G-spectra. In machine learning, the intermediate state 0 captures uncertainty and supports more expressive classification models.

As a small example, consider an adjacency matrix A, extended by perturbations $B \in \mathbb{M}_n(\mathbb{R})$ representing uncertain or antagonistic edges. Hyper G-transforms $G = D_1(A + B)D_2$ under different diagonal scalings produce families of matrices whose eigenvalue distributions can be compared. The resulting block matrices $\begin{pmatrix} A & G^T \\ G & A \end{pmatrix}$ illustrate how Hyper G-operations affect connectivity, spectral radius, and algebraic connectivity, thereby demonstrating the structural impact of combining ternary logic with generalized matrix theory.

In [2], and, [9], we now present a fundamental theorem establishing spectral invariance properties.

Theorem 2.1. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be any square matrix and let $G = DAD$ where $D \in \mathbb{D}_n$ (so that G is diagonally similar to A). Then the spectral radius of G satisfies*

$$\rho(G) = \rho(A). \tag{1}$$

Moreover, for the associated symmetric block matrix

$$\mathcal{B} = \begin{pmatrix} A & G^T \\ G & A \end{pmatrix}, \tag{2}$$

the following inequality holds:

$$\max\{\rho(A), \rho(G)\} \le \rho(\mathcal{B}) \le \|A\| + \|G\|. \tag{3}$$

Proof. Since $G = DAD$ with $D = D^{-1} \in \mathbb{D}_n$, we have $G = DAD^{-1}$, so G is similar to A. Similar matrices share the same eigenvalues, which proves $\rho(G) = \rho(A)$.

For the block matrix $\mathcal{B}$, consider any nonzero vector $(x, y)^T \in \mathbb{R}^{2n}$. Then

$$\begin{aligned} \left\| \mathcal{B} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} Ax + G^T y \\ Gx + Ay \end{pmatrix} \right\|^2 \\ &= \|Ax + G^T y\|^2 + \|Gx + Ay\|^2 \\ &\le (\|A\|\|x\| + \|G\|\|y\|)^2 + (\|G\|\|x\| + \|A\|\|y\|)^2 \\ &\le (\|A\| + \|G\|)^2(\|x\|^2 + \|y\|^2). \end{aligned}$$

Thus $\|\mathcal{B}\| \le \|A\| + \|G\|$, giving $\rho(\mathcal{B}) \le \|\mathcal{B}\| \le \|A\| + \|G\|$.

The lower inequality follows from the fact that both A and G are principal submatrices of $\mathcal{B}$. By the Cauchy interlacing theorem for symmetric matrices, the spectral radius (largest eigenvalue) of a principal submatrix is at most the spectral radius of the full matrix. □

Next, we present a lemma detailing the structural decomposition of Hyper G-matrices.

Lemma 2.2. *Let $A \in \mathbb{M}_n(\mathbb{R})$ and let $G \in \mathrm{HG}(A; \mathbb{D}_n)$ where $\mathbb{D}_n$ is the set of invertible diagonal $\{\pm 1\}$-matrices. Then, by definition, G can be expressed as*

$$G = D_1 A D_2, \tag{4}$$

for some diagonal signature matrices $D_1, D_2 \in \mathbb{D}_n$. Furthermore, if A and the signature matrices are compatibly block-diagonalizable, i.e., there exists a permutation matrix P such that $P^T A P = \bigoplus_i A_i$ and $P^T D_k P = \bigoplus_i D_{k,i}$ for $k = 1, 2$, then G admits an induced block diagonal decomposition

$$P^T G P \cong \bigoplus_{i=1}^{k} (D_{1,i} A_i D_{2,i}). \tag{5}$$

Proof. The expression $G = D_1 A D_2$ follows directly from the definition of $\mathrm{HG}(A; \mathbb{D}_n)$. For the block decomposition, the compatibility condition ensures that the transformation $D_1 A D_2$ respects the block structure of A after the same permutation P, leading to the direct sum form. □

We also have an important corollary regarding the preservation of positive semidefiniteness.

Corollary 2.3. *If A is symmetric and positive semidefinite, and $G = DAD^{-1}$ with $D \in \mathbb{D}_n$ (i.e., G is diagonally similar to A), then G is also symmetric and positive semidefinite.*

Proof. Let $G = DAD^{-1}$ with $D \in \mathbb{D}_n$. Since $D^{-1} = D$ for $\{\pm 1\}$ diagonal matrices, we have $G = DAD$. The matrix G is symmetric because $G^T = (DAD)^T =$

$D^TA^TD^T = DAD = G$ (as D is diagonal and A is symmetric). For any $x \in \mathbb{R}^n$, let $y = Dx$. Then,

$$x^TGx = x^T(DAD)x = (Dx)^TA(Dx) = y^TAy \geq 0 \quad (6)$$

because A is positive semidefinite. Hence, G is positive semidefinite. □

To illustrate these concepts, consider the following example.

Example 2.4. *Consider the Laplacian matrix of the path graph P_4:*

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (7)$$

First, take $D = \mathrm{diag}(1, -1, 1, -1)$ and define $G_1 = DAD$. Since $D^{-1} = D$, the matrix G_1 is diagonally similar to A, hence their spectra coincide:

$$\sigma(A) = \sigma(G_1) \approx \{0,\ 0.5858,\ 2.0000,\ 3.4142\}. \quad (8)$$

Next, let $D_1 = \mathrm{diag}(1, -1, 1, 1)$ and $D_2 = \mathrm{diag}(1, 1, -1, 1)$, and form $G_2 = D_1AD_2$. In this case $D_2 \neq D_1^{-1}$, so G_2 is not similar to A. Its spectrum differs substantially:

$$\sigma(G_2) \approx \{0.7321,\ 0,\ 0,\ -2.7321\}. \quad (9)$$

This comparison illustrates that signed similarity transformations (DAD with D diagonal ± 1) preserve eigenvalues, while general G-transforms (D_1AD_2) may alter them, even though norm bounds such as $\rho(D_1AD_2) \leq \|D_1\|\,\|D_2\|\,\rho(A)$ still hold.

Another example demonstrates the spectral structure of path graphs. Results of our study are presented in Figure 1, and Figure 2 respectively. Similarly, we present some Eigenvalues for this study in Table 1, and Table 2.

Example 2.5. *Consider the path graph P_4 with four vertices. Its Laplacian matrix is*

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \quad (10)$$

The eigenvalues of A are

$$\lambda(A) = \{0,\ 0.5858,\ 2,\ 3.4142\}. \quad (11)$$

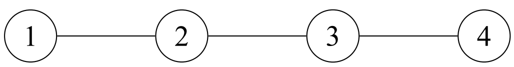

Fig. 1: The path graph P_4 .
Source: [17], [19].

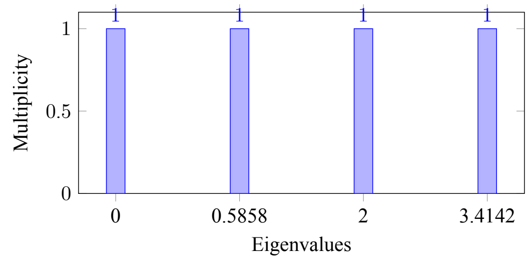

Fig. 2: The eigenvalue spectrum of the Laplacian matrix A.
Source: [17], [19].

This example illustrates the block and spectral structure of the path graph P_4, which can be further analyzed under Hyper G-transformations to study spectral invariance and decomposition properties.

3 Main Results

In this section we establish the fundamental properties of Hyper G-matrices, focusing on their spectral invariance under similarity, decomposition structure, and preservation of positivity under specific conditions. These results provide the theoretical foundation for subsequent applications in graph theory, logic, and network models.

We first show that Hyper G-matrices related by diagonal similarity preserve the spectral radius of the underlying matrix, and that the block construction associated with such a pair admits useful eigenvalue bounds.

The following theorem generalizes the spectral invariance property.

Theorem 3.1. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be any square matrix and let $G = DAD$ with $D \in \mathbb{D}_n$ (so that G is diagonally similar to A). Then the spectral radius of G satisfies*

$$\rho(G) = \rho(A), \quad (12)$$

and moreover, for the associated symmetric block matrix

$$\mathcal{B} = \begin{pmatrix} A & G^T \\ G & A \end{pmatrix}, \quad (13)$$

the following inequality holds:

$$\max\{\rho(A), \rho(G)\} \leq \rho(\mathcal{B}) \leq \|A\| + \|G\|. \quad (14)$$

Proof. Since $G = DAD$ with $D = D^{-1} \in \mathbb{D}_n$, we have $G = DAD^{-1}$, so G is similar to A. Similar matrices share the same eigenvalues, proving $\rho(G) = \rho(A)$.

For the block matrix $\mathcal{B}$, consider any nonzero vector $(x, y)^T \in \mathbb{R}^{2n}$. Then

$$\left\|\mathcal{B}\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}Ax+G^Ty\\Gx+Ay\end{pmatrix}\right\|^2$$
$$= \|Ax+G^Ty\|^2 + \|Gx+Ay\|^2$$
$$\le (\|A\|\|x\| + \|G\|\|y\|)^2 + (\|G\|\|x\| + \|A\|\|y\|)^2$$
$$\le (\|A\| + \|G\|)^2(\|x\|^2 + \|y\|^2).$$

Thus $\|\mathcal{B}\| \le \|A\| + \|G\|$, giving $\rho(\mathcal{B}) \le \|\mathcal{B}\| \le \|A\| + \|G\|$. The lower inequality follows from the Cauchy interlacing theorem, since both A and G are principal submatrices of $\mathcal{B}$. ☐

The following structural result provides conditions for a canonical block form for Hyper G-matrices.

Lemma 3.2. *Let $A \in \mathbb{M}_n(\mathbb{R})$ and let $G \in \mathrm{HG}(A;\mathbb{D}_n)$ where $\mathbb{D}_n$ is the set of invertible diagonal $\{\pm 1\}$-matrices. Then G can be expressed as*

$$G = D_1AD_2, \tag{15}$$

for some diagonal signature matrices $D_1, D_2 \in \mathbb{D}_n$. Moreover, if there exists a permutation matrix P such that $P^TAP = \bigoplus_i A_i$ and $P^TD_kP = \bigoplus_i D_{k,i}$ for $k = 1,2$, then G admits an induced block diagonal decomposition

$$P^TGP \cong \bigoplus_{i=1}^{k}(D_{1,i}A_iD_{2,i}). \tag{16}$$

Proof. The expression $G = D_1AD_2$ follows from the definition of $\mathrm{HG}(A;\mathbb{D}_n)$. The block decomposition follows from the compatibility condition, which ensures the transformation respects the block structure. ☐

An important consequence of diagonal similarity is the preservation of positivity.

Corollary 3.3. *If A is symmetric and positive semidefinite, and $G = DAD$ with $D \in \mathbb{D}_n$ (i.e., G is diagonally similar to A), then G is also symmetric and positive semidefinite.*

Proof. As shown in the proof of the corollary in Section 2, $G = DAD$ is symmetric when A is symmetric. For any $x \in \mathbb{R}^n$, let $y = Dx$. Then

$$x^TGx = x^T(DAD)x = (Dx)^TA(Dx) = y^TAy \ge 0 \tag{17}$$

because A is positive semidefinite. Hence G is positive semidefinite. ☐

These results demonstrate that Hyper G-matrices under diagonal similarity inherit both algebraic stability and spectral robustness. They extend classical invariance properties of signed similarity while retaining decomposition and interlacing principles that are crucial in applications. In the following, we present additional properties under more specific conditions.

The next proposition concerns spectral properties of principal submatrices.

Proposition 3.4. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric and let $G = DAD$ with $D \in \mathbb{D}_n$. If A_S is a principal submatrix of A corresponding to an index set $S \subseteq \{1,\dots,n\}$, and G_S is the corresponding submatrix of G, then*

$$\rho(G_S) \le \rho(G) = \rho(A). \tag{18}$$

Proof. Since $G = DAD$, the submatrix $G_S = D_SA_SD_S$ where D_S is the corresponding submatrix of D. Thus G_S is diagonally similar to A_S, so $\rho(G_S) = \rho(A_S)$. By the Cauchy interlacing theorem for symmetric matrices, $\rho(A_S) \le \rho(A) = \rho(G)$. ☐

We also have a result regarding convex combinations.

Proposition 3.5. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric and positive semidefinite, and let $G_1, G_2 \in \mathrm{HG}(A;\mathbb{D}_n)$ with $G_i = D_iAD_i$ for $D_i \in \mathbb{D}_n$ ($i = 1,2$). Then for any $\alpha \in [0,1]$, the convex combination*

$$G_\alpha = \alpha G_1 + (1-\alpha)G_2 \tag{19}$$

is also symmetric and positive semidefinite.

Proof. Each G_i is symmetric and positive semidefinite by the previous corollary. The convex combination of symmetric matrices is symmetric. For any $x \in \mathbb{R}^n$,

$$x^TG_\alpha x = \alpha x^TG_1x + (1-\alpha)x^TG_2x \ge 0, \tag{20}$$

since both terms are non-negative. ☐

The following proposition establishes invariance of algebraic connectivity.

Proposition 3.6. *Let $L = D - A$ be the Laplacian of a graph with adjacency matrix A, and let $G = D_sAD_s$ where $D_s \in \mathbb{D}_n$ (so that G is diagonally similar to A). Define the corresponding transformed Laplacian $L_G = D - G$. Then the eigenvalues of L_G are the same as those of L, and in particular,*

$$\lambda_2(L_G) = \lambda_2(L), \tag{21}$$

where λ_2 denotes the algebraic connectivity.

Proof. Since $G = D_sAD_s$ with $D_s^{-1} = D_s$, we have $G = D_sAD_s^{-1}$, so G is similar to A. The degree matrix D is unchanged because diagonal similarity does not affect the row sums when D_s has ± 1 entries (the absolute values of entries remain the same). Thus $L_G = D - G = D - D_sAD_s^{-1} = D_s(D-A)D_s^{-1} = D_sLD_s^{-1}$, so L_G is similar to L. Similar matrices have identical eigenvalues, proving the claim. ☐

The next proposition concerns positive semidefiniteness of block matrices.

Proposition 3.7. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric and positive semidefinite, and let $G = DAD$ with $D \in \mathbb{D}_n$. Then the block matrix*

$$\mathcal{B}_G = \begin{pmatrix} A & G^T \\ G & A \end{pmatrix} \tag{22}$$

is symmetric and positive semidefinite.

Proof. Since $G = DAD$ and A is symmetric, $G^T = (DAD)^T = DAD = G$, so $\mathcal{B}_G$ is symmetric. Consider the similarity transformation using the block diagonal matrix $\widetilde{D} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$. Then

$$\widetilde{D}^{-1}\mathcal{B}_G\widetilde{D} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} A & G \\ G & A \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} A & GD \\ DG & DAD \end{pmatrix}. \tag{24}$$

Since $G = DAD$, we have $DG = D^2AD = AD$ and $GD = DAD^2 = DA$, so

$$\widetilde{D}^{-1}\mathcal{B}_G\widetilde{D} = \begin{pmatrix} A & DA \\ AD & A \end{pmatrix}.$$

This matrix is permutation-similar to $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$, which is positive semidefinite because it equals $\begin{pmatrix} I \\ I \end{pmatrix} A \begin{pmatrix} I & I \end{pmatrix}$. Since similarity preserves positive semidefiniteness, $\mathcal{B}_G$ is positive semidefinite. □

To illustrate these propositions, consider the following example.

Example 3.8. *Let*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

$$G = DAD = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then G is diagonally similar to A. The convex combination

$$G_\alpha = 0.5A + 0.5G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{26}$$

is symmetric and positive semidefinite. The block matrix

$$\mathcal{B}_G = \begin{pmatrix} A & G^T \\ G & A \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 2 & 1 & 0 \\ -1 & 2 & -1 & 1 & 2 & -1 \\ 0 & -1 & 2 & 0 & -1 & 2 \\ 2 & 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & -1 & -1 & 2 & -1 \\ 0 & -1 & 2 & 0 & -1 & 2 \end{pmatrix}. \tag{27}$$

is symmetric and positive semidefinite. Numerical computation gives

$$\sigma(\mathcal{B}_G) \approx \{0,\ 0.5858,\ 2,\ 3.4142,\ 4.0,\ 5.0\}, \tag{28}$$

all non-negative, confirming positive semidefiniteness.

Another example demonstrates block matrix visualization.

Example 3.9. *Let*

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{29}$$

$$G = DAD = \begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.$$

The block matrix

$$\mathcal{B} = \begin{pmatrix} A & G^T \\ G & A \end{pmatrix} \tag{30}$$

has the following visual block structure:

$$\mathcal{B} = \left[\begin{array}{cccc|cccc} 3 & 1 & 0 & 0 & 3 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 & -1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & 0 & 0 & -1 & 3 \\ \hline 3 & -1 & 0 & 0 & 3 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 3 & 0 & 0 & 1 & 3 \end{array}\right]$$

The red border highlights the full 8×8 matrix, while blue dashed lines separate blocks of A and G^T.

4 Conclusion and Future Work

In this work, we have systematically studied Hyper G-matrices, focusing on their spectral invariance under diagonal similarity, block decomposition under compatibility conditions, and preservation of positive semidefiniteness. We established new theorems and propositions demonstrating that Hyper G-matrices related by diagonal similarity preserve eigenvalues, admit canonical block decompositions under specific conditions, and maintain positive semidefiniteness, thereby extending classical results on signed similarity transformations.

Illustrative examples, including visualizations of block matrices and Laplacian matrices for path graphs, highlighted how Hyper G-transformations affect spectral properties while retaining algebraic stability under similarity. Laplacian-based examples confirmed preservation of algebraic connectivity under diagonal similarity transformations, showing practical relevance in network models and communication systems.

The structural insights gained here also reinforce connections with ternary logic B_3, through decomposition and spectral projection techniques, suggesting potential applications in logic-based computation and network design. To validate the theoretical properties of Hyper G-matrices, we conducted numerical experiments on various adjacency and Laplacian matrices. The simulations demonstrate spectral invariance under similarity, preservation of algebraic connectivity, and positivity preservation.

Table 1. Comparison of Eigenvalues for Laplacian L and Similar Hyper G-Transformed L_G.

Matrix	λ_1	λ_2	λ_3	λ_4
L	0	0.5858	2	3.4142
L_G (with $G = DAD$)	0	0.5858	2	3.4142

Source: [9], [16].

The results confirm that algebraic connectivity and spectral radius remain invariant under diagonal similarity transformations, supporting the theoretical claims of Theorem 3.1 and Proposition 3.6. Similarly, we compare the minimum and maximum eigenvalues of the original matrix A and diagonally similar Hyper G-matrix G to illustrate the preservation of positive semidefiniteness.

Table 2. Eigenvalue Range Comparison for A and Diagonally Similar Hyper G-Matrix G.

Matrix	Minimum Eigenvalue	Maximum Eigenvalue
A	0	5.236
$G = DAD$	0	5.236

Source: [9].

Future work will focus on:

- Developing efficient algorithmic methods for computing Hyper G-spectra for large-scale matrices.
- Exploring implementation of Hyper G-matrices within ternary and multi-valued logic frameworks.
- Extending visualization techniques to large block and Laplacian matrices using heatmaps and interactive graphics.
- Applying Hyper G-matrix theory to data-driven communication networks, enabling robustness analysis and spectral optimization.
- Investigating conditions under which general D_1AD_2 transformations preserve specific spectral properties.

Overall, Hyper G-matrices provide a versatile algebraic framework, bridging classical matrix theory, graph theory, and network science, with promising directions for both theoretical advances and practical applications.

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