

Derivation of Analytical Solution to ARL Integral Equation of the Extended EWMA Control Chart for Time Series ARMAX Process

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Abstract: - Control charts are effective tools in statistical process control that are used for monitoring process quality and detecting unusual changes. The extended exponentially weighted moving average (extended EWMA) control chart is one of a traditional EWMA-type improvement which showed a greater performance in detecting changes in autocorrelated process. Average run length (ARL) is a basic performance metric for measuring the efficiency of statistical control charts. ARL can be expressed as a Fredholm integral equation of the second kind and approximated numerically. This study aims to propose an analytical solution to the ARL integral equation of the extended EWMA control chart for an autoregressive moving average model with exogenous variables (ARMAX) and an exponential white noise. The analytical solution of the ARL integral equation was derived and compared with the accuracy to numerical integration method. The simulation results indicated that the analytical expression provides accurate ARL values and significantly decreases computation time to less than 0.001 seconds. The comparisons on the ARL and other overall performance criteria for the extended EWMA, the EWMA, and the cumulative sum (CUSUM) procedures illustrated that the extended EWMA control chart outperforms the others in detecting the process mean changes.

Key-Words: - Average Run Length, Analytical Solution, Extended Exponentially Weighted Moving Average Control Chart, Autoregressive Moving Average Model, Exogenous Variables, Numerical Integration.

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1 Introduction

Control charts are valuable statistical methods that were initially employed in manufacturing industries and then applied across many areas to determine whether a process is in a state of control or not. Indeed, these graphics tools monitor and quickly detect the change or the lack of stability, thus the practitioners can take proper action to move an out-of-control state to the in-control process. Exponentially weighted moving average (EWMA) [1] and cumulative sum (CUSUM) [2] procedures are conventional memory control charts designed to use historical data with past information, which making them more sensitive in detecting small shifts in process parameter than the memoryless types. There are many developments based on EWMA-type procedure, including adaptive EWMA [3], modified EWMA [4], and the extended EWMA [5] control charts, which show greater performance than the traditional charts.

An independence assumption of observations is violated in several continuous processes, such as chemical, environmental, and financial process which can imply that the consecutive observations are autocorrelated, which are dependent on the same series, [6]. Autoregressive integrated moving average (ARIMA) methodology [7] is used to describe the behavior of time series and forecast future values. The stationary ARMA model with exogenous variables or ARMAX, has been applied in various processes and has been able to increase the accuracy of the predictive model, [8], [9], [10], [11]. Generally, residuals produce white noise. However, exponential white noise may be present in certain datasets, [12], [13], prompting this research to focus on the investigation of exponential white noise.

Average run length (ARL) is a basic performance measure of statistical control charts. The ARL can be divided into in-control ARL (ARL_{IC}) and out-of-control ARL (ARL_{OOC}). The

ARL_{IC} is the expected number of observations taken from a stable process until a control chart signals an out-of-control state. A large ARL_{IC} is required to avoid incorrect signals when there is no unacceptable change in the process. On the other hand, the ARL_{OOC} is the average number of samples taken from an out-of-control process before a control chart signals. Therefore, the ARL_{OOC} is desirable to be as small as possible.

The ARL value is an important factor in studying and comparing the detection sensitivity of control charts. Several procedures have been applied to approximate the ARL, such as Monte Carlo simulation, Markov Chain approach, and Integral equation, [14], [15]. In general, the ARL integral equation is in the form of a Fredholm integral equation of the second kind, and hard to find an exact solution and numerical simulation is time-consuming. The analytical solution to ARL integral equation has been studied under conditions of autocorrelation and an exponential white noise process, [16], [17], [18], [19], [20], [21].

This study aims to derive the analytical solution to the ARL integral equation of the extended EWMA chart for the ARMAX process when the white noise is an exponential distribution and ensure the accuracy with the numerical integration method. The comparison of the detection sensitivity of the extended EWMA, the EWMA, and the CUSUM control charts was conducted. The financial dataset was applied to evaluate the analytical solution and illustrate the sensitivity of competing control charts.

2 Preliminaries

This section provides the definitions and brief descriptions of time series models, control charts, and their characteristic, which are known as average run length.

2.1 ARMAX Process

The ARMAX(p, q, r) is the combination of the autoregressive (AR) component order p , the moving average (MA) term order q , and the number r of exogenous variables. The model has the following structure;

$$\xi_t = \eta + \sum_{i=1}^p \phi_i \xi_{t-i} + \varepsilon_t - \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \sum_{k=1}^r c_k X_{kt} \quad (1)$$

where η is a constant, ϕ_i is an autoregressive coefficient parameter, θ_j is a moving average coefficient, X_{kt} is an exogenous variable, c_k is a coefficient parameter of X_{kt} and ε_t is an

exponential white noise, $\varepsilon_t \sim \text{Exp}(\alpha), \alpha > 0$ then

$$f(y) = \frac{1}{\alpha} e^{-\frac{y}{\alpha}}.$$

2.2 Control Charts

The main control chart studied in this research is the extended EWMA control chart, moreover we will compare the sensitivity with other traditional procedures.

2.2.1 Extended EWMA Control Chart

The extended EWMA statistic can be expressed as the following recursive equation,

$$E_t = (1 - \lambda_1 + \lambda_2) E_{t-1} + \lambda_1 \xi_t - \lambda_2 \xi_{t-1}, t = 1, 2, 3, \dots \quad (2)$$

where ξ_t is the sequence of observed data, $0 < \lambda_1 \leq 1$ and $0 \leq \lambda_2 < \lambda_1$ are the smoothing constants. The lower control limit (LCL) and the upper control limit (UCL) of the extended EWMA control chart are given as

$$\begin{aligned} \text{UCL} &= b = \mu + L\sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}}, \\ \text{LCL} &= a = \mu - L\sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}} \end{aligned}$$

where μ is a target mean, σ is a process standard deviation and L is an appropriate control limit width. The stopping time of the extended EWMA chart is $\tau_b = \inf\{t > 0 : E_t < a \cup E_t > b\}$.

2.2.2 Classical EWMA Control Chart

The extended EWMA control chart can convert back to the classical EWMA control chart when $\lambda_2 > 0$, then, the EWMA statistic can be written as,

$$Z_t = (1 - \lambda) Z_{t-1} + \lambda \xi_t, t = 1, 2, 3, \dots \quad (3)$$

where $\lambda \in (0, 1)$ is a smoothing constant. The LCL and the UCL of the EWMA control chart are,

$$\text{UCL} = b' = \mu + L_1 \sigma \sqrt{\frac{\lambda}{2 - \lambda}}, \text{LCL} = a' = \mu - L_1 \sigma \sqrt{\frac{\lambda}{2 - \lambda}}$$

where L_1 is a suitable control limit width.

The stopping time of the EWMA control chart is $\tau_{b'} = \inf\{t > 0 : Z_t < a' \cup Z_t > b'\}$.

2.2.3 CUSUM Control Chart

The CUSUM statistic can be defined as,

$$W_t = \max \{0, W_{t-1} + \xi_t - \psi\}, t = 1, 2, 3, \dots \quad (4)$$

where ψ is the reference constant.

The stopping time of this control chart is $\tau_h = \inf\{t > 0 : W_t > h\}$ where h is the UCL.

2.3 Average Run Length

Average run length (ARL) is the common characteristic to assess the capacity of control chart. It can be described related to the change point in distribution. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t$ be a sequence of independent random variables with a probability density function $f(y, \alpha)$ where α is a parameter. The stopping time (τ) is considered as the signals. The in-control condition has no change in the process parameter, $\alpha = \alpha_0$, thus the ARL_{IC} can be mathematically defined as,

$$ARL_{IC} = \tilde{E}_{\infty}(\tau)$$

where $\tilde{E}_{\infty}(\cdot)$ is the expectation of the stopping time under the assumption that the observations have the distribution $F(y, \alpha_0)$.

The parameter is $\alpha = \alpha_1 \neq \alpha_0$, when the process is out-of-control then the ARL_{OOC} can be defined as,

$$ARL_{OOC} = \tilde{E}_T(\tau - T + 1 | \tau \geq T)$$

where $\tilde{E}_T(\cdot)$ is the expectation of the stopping time under the assumption that the change-point exists at the time T which is often determined at $T = 1$.

2.4 Integral Equation Approach

The main purpose of this research is to derive the analytic expression to the ARL integral equation of the extended EWMA control chart for ARMAX process with exponential white noise. First of all, we suppose that the initial process is in-control, $a \leq E_0 = s \leq b$, and then, let $A(s)$ be the ARL of starting value, s defined as

$$A(s) = \tilde{E}_{\infty}(\tau_b) \quad (5)$$

where τ_b is the stopping time of the extended EWMA control chart.

If ε_1 makes E_1 falls out of control limits, then an out-of-control signal is given, therefore, the run length is equal to 1. On the other hand, if ε_1 gives

E_1 in an in-control state, then $a \leq E_1 \leq b$ or

$$\frac{a - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi_1 - \lambda_2)\xi_0}{\lambda_1} - \eta - \sum_{i=2}^p \phi_i \xi_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} - \sum_{k=1}^r c_k X_{kt} < \varepsilon_1 < \frac{b - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi_1 - \lambda_2)\xi_0}{\lambda_1} - \eta - \sum_{i=2}^p \phi_i \xi_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} - \sum_{k=1}^r c_k X_{kt}$$

Then, we follow the method given in [14], [15]. The $A(s)$ can be expressed as,

$$A(s) = 1 + \int \frac{\frac{b - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi_1 - \lambda_2)\xi_0}{\lambda_1} - \eta - \sum_{i=2}^p \phi_i \xi_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} - \sum_{k=1}^r c_k X_{kt}}{\frac{a - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi_1 - \lambda_2)\xi_0}{\lambda_1} - \eta - \sum_{i=2}^p \phi_i \xi_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} - \sum_{k=1}^r c_k X_{kt}} A(E_1) f(\varepsilon_1) d\varepsilon_1 \quad (6)$$

After changing variable, the $A(s)$ satisfies a Fredholm integral equation of a second kind as,

$$A(s) = 1 + \frac{1}{\lambda_1} \int_a^b A(v) f \left(\frac{v - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi_1 - \lambda_2)\xi_0}{\lambda_1} - \eta - \sum_{i=2}^p \phi_i \xi_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} - \sum_{k=1}^r c_k X_{kt} \right) dv \quad (7)$$

2.5 Functional Analysis

The definition and theorem used for the proof the existence and uniqueness of the solution are given.

Definition 1. A contraction

Let (U, d) be a metric space. An operator $\Omega: U \rightarrow U$ is a contraction if there is a constant $\nu \in [0, 1)$ such that $d(\Omega(u_1), \Omega(u_2)) \leq \nu d(u_1, u_2)$ for all u_1, u_2 in U .

Theorem 1. Banach's fixed point theorem

Let (U, d) be a complete metric space and $\Omega: U \rightarrow U$ be a contraction on U . Then Ω has a unique fixed point such that $\Omega(u) = u, u \in U$.

3 The Proposed Analytical Approach

In this study, white noise process is assumed to be exponentially distributed and $f(y) = \frac{1}{\alpha} e^{-\frac{y}{\alpha}}$. Thus, the $A(s)$ in equation (7) can be described as;

$$A(s) = 1 + \frac{1}{\lambda_1} \int_a^b A(v) \frac{1}{\alpha} e^{-\frac{v}{\alpha}} dv \quad (8)$$

The existence and uniqueness of the analytical solution to the ARL integral equation (8) was proved in 3.1. After that, the proposed exact solution was present in 3.2

3.1 Existence and Uniqueness

Theorem 2. The ARL integral equation, $A(s)$ of extended EWMA control chart for ARMAX(p, q, r) with exponential white noise has a unique solution.

Proof. Let $A(s)$ is in the set of all continuous function $C[a, b]$. Consequently, the space $(C[a, b], \|\cdot\|_\infty)$ is complete with a norm given by $\|A\|_\infty = \sup |A|$. The first condition of theorem 1 is hold.

Let $\Omega: C[a, b] \rightarrow C[a, b]$ be the operator defined as,

$$\Omega(A(s)) = A(s) = 1 + \frac{1}{\alpha \lambda_1} \int_a^b A(s) k(s, z) dz \quad (9)$$

where

$$k(s, z) = e^{-\frac{z + (1-\lambda_1 + \lambda_2)s + (\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1} + \frac{\eta - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha}}$$

is a kernel function. Let A_1 and A_2 are continuous functions in $C[a, b]$. Consider

$$\begin{aligned} \|\Omega(A_1) - \Omega(A_2)\|_\infty &= \sup_{s \in [a, b]} |A_1(s) - A_2(s)| \\ &= \left| \frac{1}{\alpha \lambda_1} \sup_{s \in [a, b]} \int_a^b k(s, z) (A_1(z) - A_2(z)) dz \right| \\ &\leq \frac{1}{\alpha \lambda_1} \sup_{s \in [a, b]} \int_a^b |k(s, z)| |A_1(z) - A_2(z)| dz \\ &\leq \|A_1 - A_2\|_\infty \frac{1}{\alpha \lambda_1} \sup_{s \in [a, b]} \int_a^b |k(s, z)| dz \\ &= \nu \|A_1 - A_2\|_\infty \end{aligned}$$

where $\nu = \frac{1}{\alpha \lambda_1} \sup_{s \in [a, b]} \int_a^b |k(s, z)| dz < 1$ is a positive constant. Therefore, Ω is a contraction and the second condition of Theorem 1 is hold. We can imply that the ARL integral equation has one solution.

3.2 Analytical Solution

Theorem 3. The derived analytical solution to the ARL integral equation of the extended EWMA control chart for ARMAX(p, q, r) process with exponential white noise can be written as,

$$A(s) = 1 - \frac{(\lambda_1 - \lambda_2) e^{-\frac{(1-\lambda_1 + \lambda_2)s}{\alpha \lambda_1}} (e^{\frac{-b}{\alpha \lambda_1}} - e^{\frac{-a}{\alpha \lambda_1}})}{(\lambda_1 - \lambda_2) e^{-\frac{(\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1}} + \frac{\mu - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha} + (e^{\frac{-(\lambda_1 - \lambda_2)b}{\alpha \lambda_1}} - e^{\frac{-(\lambda_1 - \lambda_2)a}{\alpha \lambda_1}})} \quad (10)$$

Proof. the ARL integral equation, $A(s)$ in (8) can be rearranged as,

$$A(s) = 1 + \frac{e^{-\frac{(1-\lambda_1 + \lambda_2)s + (\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1} + \frac{\eta - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha}}}{\alpha \lambda_1} \int_a^b A(v) e^{-\frac{v}{\alpha \lambda_1}} dv \quad (11)$$

where $D(s) = e^{-\frac{(1-\lambda_1 + \lambda_2)s + (\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1} + \frac{\eta - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha}}$ and

$G = \int_a^b A(v) e^{-\frac{v}{\alpha \lambda_1}} dv$. Consider $A(v)$ in the form of (11), then

$$\begin{aligned} G &= \int_a^b \left(1 + \frac{D(v)}{\alpha \lambda_1} G \right) e^{-\frac{v}{\alpha \lambda_1}} dv = \int_a^b e^{-\frac{v}{\alpha \lambda_1}} dv + \frac{G}{\alpha \lambda_1} \int_a^b D(v) e^{-\frac{v}{\alpha \lambda_1}} dv \\ G + \frac{G}{(\lambda_1 - \lambda_2)} (e^{-\frac{(\lambda_1 - \lambda_2)b}{\alpha \lambda_1}} - e^{-\frac{(\lambda_1 - \lambda_2)a}{\alpha \lambda_1}}) &= -\alpha \lambda_1 (e^{\frac{-b}{\alpha \lambda_1}} - e^{\frac{-a}{\alpha \lambda_1}}) \\ G &= \frac{-\alpha \lambda_1 (\lambda_1 - \lambda_2) (e^{\frac{-b}{\alpha \lambda_1}} - e^{\frac{-a}{\alpha \lambda_1}})}{(\lambda_1 - \lambda_2) + e^{-\frac{(\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1}} + \frac{\mu - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha} (e^{\frac{-(\lambda_1 - \lambda_2)b}{\alpha \lambda_1}} - e^{\frac{-(\lambda_1 - \lambda_2)a}{\alpha \lambda_1}})} \quad (12) \end{aligned}$$

Substituting G from (12) into (11), we obtain the analytical solution as:

$$A(s) = 1 - \frac{(\lambda_1 - \lambda_2) e^{-\frac{(1-\lambda_1 + \lambda_2)s}{\alpha \lambda_1}} (e^{\frac{-b}{\alpha \lambda_1}} - e^{\frac{-a}{\alpha \lambda_1}})}{(\lambda_1 - \lambda_2) e^{-\frac{(\lambda_1\phi - \lambda_2)\xi_0}{\alpha \lambda_1}} + \frac{\mu - \theta_1 \varepsilon_0 + \sum_{i=2}^p \phi_i \varepsilon_{1-i} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha} + (e^{\frac{-(\lambda_1 - \lambda_2)b}{\alpha \lambda_1}} - e^{\frac{-(\lambda_1 - \lambda_2)a}{\alpha \lambda_1}})}$$

Note that, for in-control process ($\alpha = \alpha_0$), the ARL_{IC} of extended EWMA control chart for ARMAX model is

$$ARL_{IC} = 1 - \frac{(\lambda_1 - \lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)s}{\alpha_0}} (e^{\frac{-b}{\alpha_0\lambda_1}} - e^{\frac{-a}{\alpha_0\lambda_2}})}{(\lambda_1 - \lambda_2)e^{\frac{-(\lambda_1\phi - \lambda_2)\xi_0}{\alpha_0\lambda_1}} + \frac{\mu - \theta_1\varepsilon_0 + \sum_{i=2}^p \phi_i \xi_{i-1} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha_0} + (e^{\frac{-(\lambda_1 - \lambda_2)b}{\alpha_0\lambda_1}} - e^{\frac{-(\lambda_1 - \lambda_2)a}{\alpha_0\lambda_2}})}$$

and, the ARL_{OOC} ($\alpha = \alpha_1 = (1 + \delta)\alpha_0$) is

$$ARL_{OOC} = 1 - \frac{(\lambda_1 - \lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)s}{\alpha_1}} (e^{\frac{-b}{\alpha_1\lambda_1}} - e^{\frac{-a}{\alpha_1\lambda_2}})}{(\lambda_1 - \lambda_2)e^{\frac{-(\lambda_1\phi - \lambda_2)\xi_0}{\alpha_1\lambda_1}} + \frac{\mu - \theta_1\varepsilon_0 + \sum_{i=2}^p \phi_i \xi_{i-1} - \sum_{j=2}^q \theta_j \varepsilon_{1-j} + \sum_{k=1}^r c_k X_{k1}}{\alpha_1} + (e^{\frac{-(\lambda_1 - \lambda_2)b}{\alpha_1\lambda_1}} - e^{\frac{-(\lambda_1 - \lambda_2)a}{\alpha_1\lambda_2}})}$$

where δ is a shift size.

4 Numerical Integration

The numerical integration method for approximating the solution of ARL integral equation which described as a Fredholm integral equation of a second kind is the use of an integration rule to solve the system of linear equations. The quadrature rule is determined by the set of points, $\{x_j, j = 0, 1, \dots, m\}$ obtained from the division of an interval $[a, b]$ into m sections and the set of weights, $\{w_j, j = 0, 1, \dots, m\}$. By following the procedure in [21], the numerical estimation of $A(s)$ is:

$$\hat{A}(s) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \hat{A}(x_j) f \left(\frac{x_j - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi - \lambda_2)\xi_0 - \eta}{\lambda_1} + \theta_1\varepsilon_0 - \sum_{i=2}^p \phi_i \xi_{i-1} + \sum_{j=2}^q \theta_j \varepsilon_{1-j} - \sum_{k=1}^r c_k X_{k1} \right) \quad (15)$$

Composite Boole's rule was applied in this work. The points of the quadrature rule were determined at $x_j = a + dj$ where $d = (b - a) / m$ was equal range and

$$g_j = \frac{x_j - (1 - \lambda_1 + \lambda_2)s - (\lambda_1\phi - \lambda_2)\xi_0 - \eta + \theta_1\varepsilon_0 - \sum_{i=2}^p \phi_i \xi_{i-1} + \sum_{j=2}^q \theta_j \varepsilon_{1-j} - \sum_{k=1}^r c_k X_{k1}}{\lambda_1}$$

Therefore, the $\hat{A}(s)$ was

$$\hat{A}_B(s) = 1 + \frac{1}{\lambda_1} \sum_{j=0}^{m=4n} w_j \ell(x_j) f(g_j) \quad (16)$$

where $w_k = \frac{14d}{45}; k = 0, 4n, w_k = \frac{24d}{45}; k = 2, \dots, 4n - 2,$

$$w_k = \frac{64d}{45}; k = 1, \dots, 4n - 3, 4n - 1,$$

$$w_k = \frac{28d}{45}; k = 4, \dots, 4n - 4$$

5 Simulation Study and Applications

Mathematica program in the 64-bit operating system, AMD Ryzen 7 4700U with Radeon Graphics 2.00 GHz processor was used to calculate the analytical solution and numerical approximation.

5.1 Accuracy of the Analytical Solution

The accuracy of The ARL value from the proposed analytical expression (10) and from the numerical integration was assured by the percentage accuracy,

$$\%Acc = 100 - \left| \frac{A(s) - \hat{A}(s)}{A(s)} \right| \times 100 \quad (17)$$

The steps for assessing the ARL values were

Step 1: Set all of constants, parameters and initial values for ARMAX model and extended EWMA control chart

Step 2: Set $ARL_{IC} = 370$, mean of in-control process, $\alpha = \alpha_0 = 1$ and the LCL, $a = 0$.

Step 3: Calculated the UCL, b by (10)

Step 4: Calculate the ARL_{OOC} by the analytical expression (10) and the numerical integration (16) when $\alpha = \alpha_1 = (1 + \delta)\alpha_0$ and number of subintervals $m = 900$.

Step 5: Computed percentage accuracy (%Acc)

Table 1 (Appendix) shows the example of the UCL, b that obtained from equation (10) when the parameters were set.

Table 2 (Appendix) presents the ARL of the extended EWMA control chart for ARMAX(1,q,r) models obtained from the analytical expression and the numerical method when $\lambda_1 = 0.05$, $\lambda_2 = 0.02$, 0.04, $ARL_{IC} = 370$ and shift sizes, $\delta = 0, 0.001, \dots, 1$. The Average Run Length (ARL_{IC}) under controlled conditions is approximately 370, as it is intrinsically linked to the 3-sigma control limits principle of the Shewhart control chart. Assuming a normal data distribution, establishing the control limits at $\pm 3\sigma$ yields a false alarm probability (Type I error) per observation of approximately 0.027. The results indicate that the ARL of ARMAX(1,1,1), ARMAX(1,2,1) and ARMAX(1,2,2) models obtained from two approaches are not significantly

different assured by 100% of percentage accuracy. One benefit of the analytical solution is not time-consuming; the computation time was less than 0.001 seconds, while numerical integration took around 39-41 seconds. Similarly, Table 3 (Appendix) demonstrated the ARL values of ARMAX(2,q,r) models when $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, 0.15. The results ensure that the analytic solution can provide the precise ARL value compared with the numerical method. The main results agree with the previous studies [16], [17], [18], [19], [20] which investigated and concluded that the ARL values obtained from the analytical approach were close to the ARL values obtained from estimation procedures and consumed less than 0.001 seconds of computation.

5.2 Comparison on Performance of Control Charts

The performance of the extended EWMA control chart was compared with the EWMA and CUSUM charts in detecting changes for ARMAX(p,q,r) models. Standard deviation run length (SDRL) and median run length (MRL) were other performance measures of control chart. For in-control process, the ARL, SDRL and the MRL are defined as,

$$ARL_{IC} = \frac{1}{\rho}, \quad SDRL_{IC} = \frac{\sqrt{1-\rho}}{\rho}, \quad MRL_{IC} = \frac{\log(0.5)}{\log(1-\rho)} \quad (18)$$

where ρ is type I error. In this research, ARL_{IC} was equal to 370. $SDRL_{IC}$ and MRL_{IC} can be calculated by (18). whereas, for out-of-control state, the ARL, SDRL and MRL are defined as

$$ARL_{OOC} = \frac{1}{1-g}, \quad SDRL_{OOC} = \frac{\sqrt{g}}{1-g}, \quad MRL_{IC} = \frac{\log(0.5)}{\log(g)} \quad (19)$$

where g is type II error. The lower value $SDRL_{OOC}$ and MRL_{OOC} indicated the better performance of control charts for each magnitude of shift size.

The overall performance metrics are calculated to summarize the comparison of control charts.

The average extra quadratic loss (AEQL) is defined as:

$$AEQL = \frac{1}{\Delta} \sum_{\delta_i=\delta_{\min}}^{\delta_{\max}} (\delta_i^2 \times ARL(\delta_i)) \quad (20)$$

where δ_i is the value of shift at level i , $ARL(\delta_i)$ is the ARL value for the shift level δ_i and Δ is the number of change levels from $\delta_{\min}$ to $\delta_{\max}$. In this study, $\Delta = 11$ from $\delta_{\min} = 0$ to $\delta_{\max} = 1$. The control chart with the lowest value of AEQL is implied to be the most effective chart.

The relative mean index (RMI) is:

$$RMI = \frac{1}{\Delta} \sum_{i=1}^{\Delta} \frac{ARL(\delta_i) - ARL_{\min}(\delta_i)}{ARL_{\max}(\delta_i)} \quad (21)$$

where $ARL_{\min}(\delta_i)$ and $ARL_{\max}(\delta_i)$ is the lowest and the highest ARL among the control charts for the shift level δ_i . The control chart with lowest RMI shows the best performance in detecting changes.

The expected SDRL (ESDRL) and expected MRL (EMRL) are defined as,

$$ESDRL = \frac{1}{\Delta} \sum_{\delta=\delta_{\min}}^{\delta_{\max}} SDRL(\delta), \quad EMRL = \frac{1}{\Delta} \sum_{\delta=\delta_{\min}}^{\delta_{\max}} MRL(\delta) \quad (22)$$

Table 4 (Appendix) demonstrated the ARL, SDRL and MRL to examine the sensitivity of control charts in detecting process mean changes for ARMAX(1,3,1) processes when smoothing parameters of extended EWMA control charts, $\lambda_1 = 0.05$, $\lambda_2 = 0.45\lambda_1, 0.6\lambda_1, 0.9\lambda_1$ were given. The results indicate that the extended EWMA charts with varying values of λ_2 provided lower ARL, SDRL and MRL values than the traditional control charts for all magnitudes of shift. The extended EWMA control chart with $\lambda_2 = 0.9\lambda_1$ showed the lowest overall performance criteria including AEQL, RMI, ESDRL and EMRL. Moreover, the result recommends that the extended EWMA control chart has greater performance if λ_2 is close to λ_1 .

Table 5 (Appendix) presented the ARL values and other performance measures of participated control charts for ARMAX(2,1,1) models when $\lambda_1 = 0.1$. Similarly, Table 6 (Appendix) proposed the study of ARMAX(3,2,2) processes when $\lambda_1 = 0.2$. The results of the tables are equivalent to Table 4 (Appendix) that the ARL, SDRL and MRL values of extended EWMA control charts were lower than the others under various conditions. It was noticed that the ARL values moderately dropped when λ_2 enlarged. The overall performance criteria confirm

that the extended EWMA control chart with $\lambda_2 = 0.9\lambda_1$ has higher performance.

The results agree with the conclusion in [21] that the extended EWMA was more sensitive to detect small process mean changes when the λ_2 raised closely to λ_1 .

5.3 Application

The United Overseas Bank Ltd (UOB) stock price (SGD) observed daily from November 1st, 2024 to February 28th, 2025 was described in ARMAX process when exchange rate (USD/SGD) was the external factor.

Table 7 (Appendix) revealed the fitted ARMAX(p,q,r) models, the estimated coefficient parameters and statistic tests for observed data. The results showed that the most appropriate model to represent the behavior of the UOB stock price time series was ARMAX(1,1,1) model which had significant test and low model fit indices namely, stationary R-square, root mean square error (RMSE) and normalized Bayesian information criterion (Normalized BIC). Subsequently, the predictive model was,

$$\xi_t = 0.951\xi_t + 0.297\varepsilon_{t-1} + 24.814X_t + \varepsilon_t$$

Table 8 (Appendix) demonstrated the statistical test of residuals. Kolmogorov-Smirnov statistic illustrated that the errors were exponential white noise with mean $\alpha_0 = 0.3248$.

Table 9 (Appendix) presents that the ARL, SDRL and MRL of extended EWMA procedure comparing with two classical control charts for ARMAX(1,1,1) process applied to practical application when $\lambda_1 = 0.2$ and $\lambda_2 = 0.3\lambda_1, 0.45\lambda_1, 0.9\lambda_1$ were given. The results had consistency with the previous section that the extended EWMA control chart was more effective in detection ability than the EWMA and CUSUM charts and λ_2 can affect significantly on the performance of extended EWMA control chart.

6 Conclusion

In this study, the derivation of an analytical solution to the ARL integral equation of the extended EWMA control chart for ARMAX(p,q,r) process with exponential white noise, which satisfied a Fredholm integral equation of the second kind, was proposed. The analytical expression provided the correct ARL assured by the estimation procedure. The extended EWMA procedure outperformed the traditional EWMA and the CUSUM control charts

in detecting the small changes in process mean, as verified by the ARL, SDRL, MRL, and the overall performance measures, including AEQL, RMI, ESDRL, and EMRL. The exponential smoothing parameter λ_2 of extended EWMA control chart influenced the sensitivity of this chart. The coefficients of the autoregressive model cannot be interpreted in terms of sensitivity, as, in practice, distinct coefficient values are obtained from separate datasets. Moreover, the practical situation that frequently produced autocorrelated data was demonstrated to evaluate the proposed method and the performance of the relevant control charts.

The analytical approach for ARL integral equation is suitable as an alternative method to numerical approximation in some situations. Further research could employ this procedure to other correlated processes and new adjusted statistical process control charts.

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References:

- [1] W.S. Roberts, Control Chart Tests Based on Geometric Moving Averages, *Technometrics*, Vol.1, No.3, 1959, pp. 239-250. DOI: 10.2307/1266443.
- [2] E.S. Page, Continuous Inspection Schemes, *Biometrika*, Vol.41, No.1-2, 1954, pp. 100-115. <https://doi.org/10.2307/2333009>.
- [3] G. Capizzi and G. Masarotto, An Adaptive Exponentially Weighted Moving Average Control Chart, *Technometrics*, Vol.45, No.3, 2003, 2003, pp. 199-207. <https://doi.org/10.1198/004017003000000023>
- [4] A. K. Patel and J. Divecha, (2011). Modified Exponentially Weighted Moving Average (EWMA) Control Chart for an Analytical Process Data, *Journal of Chemical Engineering and Materials Science*, Vol.21, No.1, 2011, pp. 12-20. DOI:10.5897/JCEMS.9000014.
- [5] M. Neveed, M. Azam, N. Khan and M. Aslam, Design a Control Chart Using Extended EWMA Statistic, *Technologies*, Vol.6, No.4, 2018, pp. 108-122. DOI: 10.3390/technologies6040108.

- [6] D.C. Montgomery, *Introduction to Statistical Quality Control (7th ed.)*, John Wiley & Sons, 2013.
- [7] G.E.P. Box and G.M. Jenkins, *Time Series Analysis Forecasting and Control . 2nd Edition*, Holden-Day, 1976.
- [8] E. Ayenew, G. Biru, A. Mulatu and M. Berhanu, Modelling and Control of Wind Energy Conversion System: Performance Enhancement, *International Journal of Dynamics and Control*, Vol.12, 2024, pp. 891–914 (2024). DOI: 10.1007/s40435-023-01201-w.
- [9] L.E. Simms, N.Y. Ganushkina and S. Dubyagin, Comparing Influences of Solar Wind, ULF Waves, and Substorms on 20 eV–2 MeV Electron Flux (RBSP) Using ARMAX Models, *Journal of Geophysical Research: Space Physics*, Vol.129, No.6, e2024JA032458. DOI: 10.1029/2024JA032458.
- [10] X. Song, L. Ma, R. Li, X. Ma and X. Li, An ARMAX Model for Structural Damage Quantification and Localisation, *Insight - Non-Destructive Testing and Condition Monitoring*, Vol.67, No. 6, 2025, pp. 335-342. DOI: 10.1784/insi.2025.67.6.335.
- [11] S. Ozgen, A. Wu and F. Ruiz, Modeling approaches for data-driven model predictive control of acid gases in waste-to-energy plants, *Waste Management*, Vol.204, 2025, 114902. DOI: 10.1016/j.wasman.2025.114902.
- [12] I.M.S. Pereira, and M. Antonia Amaral-Turkman, Bayesian Prediction in Threshold Autoregressive Models with Exponential White Noise, *Test*, Vol.13, No.1, 2004, pp. 45-64. DOI: 10.1007/BF02603000.
- [13] S. Suparman, A New Estimation Procedure Using a Reversible Jump MCMC Algorithm for AR Models of Exponential White Noise, *International Journal of GEOMATE*, Vol.15, No.49, 2018, pp. 85-91. DOI:10.21660/2018.49.3622.
- [14] S.V. Crowder, A Simple Method for Studying Run length Distribution of Exponentially Weighted Moving Average Control Charts, *Technometrics*, Vol.29, No.4, 1987, pp. 401-407. DOI: 10.1080/00401706.1987.10488267.
- [15] C. Champ and S.E. Rigdon, A Comparison of the Markov Chain and the Integral Equation Approaches for Evaluating the Run Length Distribution of Quality Control Charts, *Communication in Statistics Simulation and Computation*, Vol.20, No.1, 1991, pp. 191–204. DOI: 10.1080/03610919108812948.
- [16] S. Phanyaem, Average Run Length of Cumulative Sum Control Charts for SARMA(1,1)L Models, *Thailand Statistician*, Vol.15, No.2, 2017, pp. 184-195.
- [17] Y. Supharakonsakun, Comparing the effectiveness of statistical control charts for monitoring a change in process mean, *Engineering Letters*, Vol.29 No.3 (3), 2021, pp. 1108-1114.
- [18] W. Peerajit, Accurate Average Run Length Analysis for Detecting Changes in a Long-Memory Fractionally Integrated MAX Process Running on EWMA Control Chart, *WSEAS Transactions on Mathematics*, Vol. 22, 2023, pp. 514 – 530. <https://doi.org/10.37394/23206.2023.22.58>.
- [19] W. Peerajit, Gauss–Legendre Numerical Integrations for Average Run Length Running on EWMA Control Chart with Fractionally Integrated MAX Process, *WSEAS Transactions on Mathematics*, Vol 23, 2024, pp. 579-590. DOI: 10.37394/23206.2024.23.61.
- [20] J. Neammai, S. Sukparungsee and Y. Areepong, Explicit Analytical Form for the Average Run Length of Double-Modified Exponentially Weighted Moving Average Control Charts Through the MA(q) Process and Applications, *Symmetry*, Vol.17, No.2, 2025. DOI: 10.3390/sym17020238.
- [21] T. Muangngam, Y. Areepong and S. Sukparungsee Analytic Solution to ARL Integral Equation of Extended EWMA Control Chart for Seasonal ARX Model, *WSEAS Transactions on Mathematics*, Vol.24, 2025, pp. 462-476. DOI: 10.37394/23206.2025.24.45.

APPENDIX

Table 1. The UCL (b) calculated from ARL analytical solution of extended EWMA control chart for ARMAX(p,q,r) process when $ARL_{IC}=370$

Model	Coefficient parameter						$\lambda_1 = 0.1$		$\lambda_1 = 0.2$	
	ϕ_1	ϕ_2	θ_1	θ_2	c_1	c_2	$\lambda_2 = 0.05$	$\lambda_2 = 0.075$	$\lambda_2 = 0.05$	$\lambda_2 = 0.075$
(1,1,1)	0.1		0.1		0.25		0.0317235	0.0168724	0.123474	0.08866984
(1,2,1)	0.2		0.1	-0.1	0.25		0.0369102	0.0196113	0.1445958	0.103504
(1,2,2)	0.3		0.1	-0.1	0.25	0.3	0.0272711	0.01451682	0.105562	0.0760134
(2,1,1)	0.1	0.1	0.2		-0.25		0.0526037	0.0278653	0.2102636	0.1489966
(2,2,1)	0.2	0.1	0.2	-0.1	-0.25		0.0612608	0.03239728	0.2476858	0.1744958
(2,2,2)	0.3	0.1	0.2	-0.1	-0.25	0.3	0.0582251	0.0308098	0.2344623	0.1655208

Source: created by the authors

Table 2. The ARL of extended EWMA control chart for ARMAX(1,q,r) when $\lambda_1 = 0.05$ and $\lambda_2 = 0.02, 0.04$

ARMAX	δ	$\lambda_2 = 0.02$				%Acc	$\lambda_2 = 0.04$				%Acc
		A(s)	Time	$\hat{A}_b(s)$	Time		A(s)	Time	$\hat{A}_b(s)$	Time	
$b = 0.0272661$						$b = 0.00992794$					
$\phi = 0.2$ $\theta_1 = 0.2$ $c_1 = 0.3$	0.000	370.04572	<0.001	370.04572	36.469	100%	370.00686	<0.001	370.00686	39.031	100%
	0.001	332.83956	<0.001	332.83956	38.922	100%	321.68357	<0.001	321.68357	40.375	100%
	0.004	255.36104	<0.001	255.36104	39.156	100%	230.62587	<0.001	230.62587	40.344	100%
	0.007	206.80188	<0.001	206.80188	39.719	100%	179.29324	<0.001	179.29324	39.938	100%
	0.010	173.52198	<0.001	173.52198	39.187	100%	146.35183	<0.001	146.35183	39.781	100%
	0.040	64.69079	<0.001	64.69079	39.735	100%	49.66242	<0.001	49.66242	40.984	100%
	0.070	38.64537	<0.001	38.64537	40.14	100%	28.79014	<0.001	28.79014	39.875	100%
	0.100	27.06042	<0.001	27.06042	39.594	100%	19.79419	<0.001	19.79419	40.485	100%
	0.400	5.88175	<0.001	5.88175	40.016	100%	4.06995	<0.001	4.06995	40.593	100%
	0.700	3.24071	<0.001	3.24071	38.969	100%	2.29209	<0.001	2.29209	39.735	100%
1.000	2.34154	<0.001	2.34154	39.375	100%	1.72651	<0.001	1.72651	39.797	100%	
$b = 0.0134583$						$b = 0.00492334$					
$\phi_1 = -0.2$ $\theta_1 = 0.2$ $\theta_2 = -0.3$ $c_1 = 0.3$	0.000	370.05348	<0.001	370.05348	39.367	100%	370.01941	<0.001	370.01941	39.138	100%
	0.001	323.95320	<0.001	323.95320	40.141	100%	313.15710	<0.001	313.15710	39.027	100%
	0.004	235.36950	<0.001	235.36950	38.578	100%	213.74702	<0.001	213.74702	41.072	100%
	0.007	184.41608	<0.001	184.41608	39.047	100%	161.73509	<0.001	161.73509	40.261	100%
	0.010	151.32184	<0.001	151.32184	41.012	100%	129.75629	<0.001	129.75629	39.548	100%
	0.040	52.32634	<0.001	52.32634	39.523	100%	41.69907	<0.001	41.69907	40.364	100%
	0.070	30.56232	<0.001	30.56232	40.425	100%	23.78077	<0.001	23.78077	39.587	100%
	0.100	21.12441	<0.001	21.12441	38.956	100%	16.19098	<0.001	16.19098	40.782	100%
	0.400	4.43754	<0.001	4.43754	40.18	100%	3.26921	<0.001	3.26921	39.165	100%
	0.700	2.49069	<0.001	2.49069	39.255	100%	1.89924	<0.001	1.89924	39.593	100%
1.000	1.85669	<0.001	1.85669	39.715	100%	1.48348	<0.001	1.48348	40.518	100%	
$b = 0.047657$						$b = 0.0172405$					
$\phi_1 = 0.4$ $\theta_1 = 0.3$ $\theta_2 = -0.4$ $c_1 = 0.25$ $c_2 = -0.3$	0.000	370.01116	<0.001	370.01116	41.031	100%	370.18587	<0.001	370.18587	40.531	100%
	0.001	340.56710	<0.001	340.56710	40.656	100%	328.47933	<0.001	328.47933	41.157	100%
	0.004	274.66184	<0.001	274.66184	41.297	100%	245.07981	<0.001	245.07981	41.312	100%
	0.007	229.84521	<0.001	229.84521	40.969	100%	195.05550	<0.001	195.05550	38.031	100%
	0.010	197.39518	<0.001	197.39518	40.969	100%	161.71828	<0.001	161.71828	39.625	100%
	0.040	79.98941	<0.001	79.98941	40.547	100%	57.77309	<0.001	57.77309	41.5	100%
	0.070	48.96107	<0.001	48.96107	41.062	100%	34.01866	<0.001	34.01866	40.625	100%
	0.100	34.73017	<0.001	34.73017	41.594	100%	23.60516	<0.001	23.60516	39.313	100%
	0.400	7.78346	<0.001	7.78346	40.968	100%	4.96690	<0.001	4.96690	41.453	100%
	0.700	4.23709	<0.001	4.23709	41.344	100%	2.74807	<0.001	2.74807	39.75	100%
1.000	2.99155	<0.001	2.99155	41.094	100%	2.01643	<0.001	2.01643	41.312	100%	

Source: created by the authors

Table 3. The ARL of extended EWMA control chart for ARMAX(2,q,r) when $\lambda_1 = 0.2$ and $\lambda_2 = 0.1, 0.15$

ARMAX	δ	$\lambda_2 = 0.1$					$\lambda_2 = 0.15$				
		$A(s)$	Time	$\hat{A}_B(s)$	Time	%Acc	$A(s)$	Time	$\hat{A}_B(s)$	Time	%Acc
		$b = 0.0746884$					$b = 0.0394201$				
(2,1,2)	0.000	370.03523	<0.001	370.03523	40.852	100%	370.17073	<0.001	370.17073	39.592	100%
	0.001	238.93704	<0.001	238.93704	41.058	100%	222.97954	<0.001	222.97954	39.674	100%
$\phi_1 = 0.1$	0.004	116.05591	<0.001	116.05591	40.150	100%	101.88408	<0.001	101.88408	38.610	100%
	0.007	76.79008	<0.001	76.79008	39.972	100%	66.15401	<0.001	66.15401	41.422	100%
$\phi_2 = -0.1$	0.010	57.46067	<0.001	57.46067	40.765	100%	49.04812	<0.001	49.04812	40.137	100%
$\theta_1 = 0.25$	0.040	16.71205	<0.001	16.71205	39.925	100%	13.98853	<0.001	13.98853	40.245	100%
$c_1 = 0.1$	0.070	10.01616	<0.001	10.01616	40.648	100%	8.35869	<0.001	8.35869	39.782	100%
	0.100	7.27204	<0.001	7.27204	40.018	100%	6.06524	<0.001	6.06524	40.625	100%
$c_2 = 0.25$	0.400	2.42667	<0.001	2.42667	40.267	100%	2.06592	<0.001	2.06592	38.753	100%
	0.700	1.76024	<0.001	1.76024	41.036	100%	1.53841	<0.001	1.53841	39.814	100%
	1.000	1.5060	<0.001	1.5060	39.428	100%	1.34451	<0.001	1.34451	39.778	100%
		$b = 0.01641722$					$b = 0.00875963$				
(2,2,2)	0.000	370.06556	<0.001	370.06556	41.266	100%	370.07311	<0.001	370.07311	40.484	100%
	0.001	199.59048	<0.001	199.59048	41.391	100%	188.10824	<0.001	188.10824	40.954	100%
$\phi_1 = -0.2$	0.004	83.99005	<0.001	83.99005	41.243	100%	76.18104	<0.001	76.18104	41.168	100%
$\phi_2 = -0.3$	0.007	53.30608	<0.001	53.30608	40.984	100%	47.87008	<0.001	47.87008	40.025	100%
$\theta_1 = 0.25$	0.010	39.10795	<0.001	39.10795	40.552	100%	34.95859	<0.001	34.95859	41.562	100%
	0.040	10.95485	<0.001	10.95485	41.412	100%	9.70608	<0.001	9.70608	40.392	100%
$\theta_2 = -0.3$	0.070	6.55394	<0.001	6.55394	40.953	100%	5.80451	<0.001	5.80451	40.531	100%
$c_1 = 0.5$	0.100	4.77387	<0.001	4.77387	41.386	100%	4.23273	<0.001	4.23273	41.185	100%
	0.400	1.71909	<0.001	1.71909	41.141	100%	1.56868	<0.001	1.56868	40.959	100%
$c_2 = 0.5$	0.700	1.33820	<0.001	1.33820	40.525	100%	1.25222	<0.001	1.25222	40.064	100%
	1.000	1.20509	<0.001	1.20509	41.238	100%	1.14632	<0.001	1.14632	41.054	100%
		$b = 0.1313359$					$b = 0.06859128$				
(2,3,1)	0.000	370.07644	<0.001	370.07644	39.956	100%	370.02798	<0.001	370.02798	40.026	100%
	0.001	256.92550	<0.001	256.92550	39.487	100%	237.83861	<0.001	237.83861	40.247	100%
$\phi_1 = 0.3$	0.004	134.24918	<0.001	134.24918	40.352	100%	115.01917	<0.001	115.01917	39.854	100%
$\phi_2 = 0.4$	0.007	91.02982	<0.001	91.02982	40.294	100%	75.99215	<0.001	75.99215	39.636	100%
$\theta_1 = 0.1$	0.010	68.95591	<0.001	68.95591	41.239	100%	56.81909	<0.001	56.81909	40.932	100%
$\theta_2 = 0.1$	0.040	20.57805	<0.001	20.57805	41.528	100%	16.48470	<0.001	16.48470	40.265	100%
	0.070	12.37332	<0.001	12.37332	40.284	100%	9.86887	<0.001	9.86887	41.014	100%
$\theta_3 = -0.25$	0.100	8.98588	<0.001	8.98588	40.656	100%	7.15921	<0.001	7.15921	40.176	100%
$c_1 = -0.5$	0.400	2.93470	<0.001	2.93470	39.745	100%	2.38319	<0.001	2.38319	39.438	100%
	0.700	2.07467	<0.001	2.07467	38.956	100%	1.73071	<0.001	1.73071	39.564	100%
	1.000	1.73711	<0.001	1.73711	39.937	100%	1.48330	<0.001	1.48330	40.257	100%

Source: created by the authors

Table 4. Performance comparison on extended EWMA, EWMA and CUSUM control charts for
ARMAX(1,3,1) process when $\lambda_1 = 0.05$

δ		$\phi_1 = 0.15, \theta_1 = 0.15, \theta_2 = 0.3, \theta_3 = 0.3, c_1 = 0.25$					$\phi_1 = -0.15, \theta_1 = -0.15, \theta_2 = 0.3, \theta_3 = 0.3, c_1 = 0.5$				
		Extended EWMA			EWMA	CUSUM	Extended EWMA			EWMA	CUSUM
		$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$			$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$		
0.000	UCL	0.03884	0.02651	0.01243	0.12682	2.6805	0.01046	0.00718	0.00338	0.03278	4.3977
	ARL_{IC}	370.0678	370.0263	370.0083	370.0099	370.0105	370.0187	370.0802	370.0125	370.0097	370.0139
	$SDRL_{IC}$	369.5674	369.5260	369.5080	369.5095	369.5101	369.5184	369.580	369.5122	369.5093	369.5135
	MRL_{IC}	256.1647	256.1360	256.1235	256.1246	256.1250	256.1307	256.1733	256.1264	256.1244	256.1274
0.001	ARL_{OOC}	337.7175	333.0239	324.7634	366.2475	367.7584	320.9765	316.9508	309.0554	334.1927	367.4082
	$SDRL_{OOC}$	337.2171	332.5236	324.2630	365.7471	367.2581	320.4761	316.4505	308.5550	333.6924	366.9079
	MRL_{OOC}	233.7412	230.4879	224.7621	253.5167	254.5640	222.1372	219.3468	213.8741	231.2980	254.3212
0.004	ARL_{OOC}	267.2535	255.8178	237.1297	355.1092	361.1102	229.1731	220.9971	206.1897	258.6679	359.7318
	$SDRL_{OOC}$	266.7530	255.3173	236.6292	354.6088	360.6099	228.6726	220.4966	205.6891	258.1674	359.2314
	MRL_{OOC}	184.8992	176.9726	164.0190	245.7962	249.9558	158.5039	152.8367	142.5730	178.9481	249.0003
0.007	ARL_{OOC}	220.8140	207.3246	186.3054	344.2110	354.6203	177.7674	169.1670	154.1622	210.6740	352.2611
	$SDRL_{OOC}$	220.3135	206.8240	185.8047	343.7107	354.120	177.2667	168.6662	153.6614	210.1734	351.7607
	MRL_{OOC}	152.7098	143.3596	128.7902	238.2422	245.4574	122.8721	116.9107	106.5101	145.6812	243.8220
0.010	ARL_{OOC}	187.9062	174.0432	153.1303	333.5709	348.2842	144.9096	136.7223	122.7614	177.4825	344.9896
	$SDRL_{OOC}$	187.4056	173.5425	152.6295	333.0705	347.7838	144.4087	136.2214	122.2604	176.9817	344.4892
	MRL_{OOC}	129.8998	120.2906	105.7949	230.8670	241.0655	100.0967	94.4217	84.7447	122.6746	238.7818
0.040	ARL_{OOC}	73.6257	64.9451	53.1206	242.9786	292.4776	49.0436	45.0015	38.5419	67.1913	281.9531
	$SDRL_{OOC}$	73.1240	64.4431	52.6182	242.4781	291.9771	48.5410	44.4987	38.0386	66.6894	281.4527
	MRL_{OOC}	50.6861	44.6690	36.4727	168.0731	202.3832	33.6467	30.8448	26.3671	46.2260	195.0883
0.070	ARL_{OOC}	44.6264	38.780	30.9870	179.4197	248.0067	28.4549	25.8747	21.8147	40.3756	233.1538
	$SDRL_{OOC}$	44.1235	38.2767	30.4829	178.9190	247.5062	27.9504	25.3698	21.3088	39.8725	232.6533
	MRL_{OOC}	30.5848	26.5322	21.1301	124.0174	171.5583	19.3748	17.5861	14.7715	27.6382	161.2631
0.100	ARL_{OOC}	31.4944	27.1374	21.3785	135.8658	212.1826	19.5886	17.7112	14.7814	28.3849	194.9076
	$SDRL_{OOC}$	30.9904	26.6327	20.8725	135.3648	211.6821	19.0821	17.2040	14.2727	27.8804	194.4071
	MRL_{OOC}	21.4818	18.4615	14.4691	93.8280	146.7270	13.2282	11.9266	9.8951	19.3262	134.7529
0.400	ARL_{OOC}	6.9776	5.8668	4.4221	24.8751	64.370	4.0720	3.6263	2.9599	6.2780	51.2309
	$SDRL_{OOC}$	6.4583	5.3435	3.8901	24.370	63.8681	3.5368	3.0860	2.4085	5.7564	50.7284
	MRL_{OOC}	4.4810	3.7092	2.7038	16.8932	44.2704	2.4596	2.1484	1.6813	3.9950	35.1628
0.700	ARL_{OOC}	3.8145	3.2245	2.4661	11.0669	29.8632	2.3038	2.0777	1.7504	3.4661	22.8889
	$SDRL_{OOC}$	3.2766	2.6782	1.9015	10.5551	29.3590	1.73309	1.4964	1.1461	2.9236	22.3833
	MRL_{OOC}	2.2799	1.8671	1.3329	7.31897	20.3511	1.2176	2.1484	1.6813	3.9950	35.1628
1.000	ARL_{OOC}	2.7155	2.3278	1.8352	6.8074	17.5396	1.7375	1.5947	1.3932	2.4952	13.5822
	$SDRL_{OOC}$	2.1584	1.7581	1.2381	6.2875	17.0323	1.1320	0.9738	0.7401	1.9315	13.0727
	MRL_{OOC}	1.5092	1.2347	0.8805	4.3628	11.8076	0.8089	0.7027	0.5479	1.3535	9.0635
AEQL		0.5806	0.4949	0.3846	1.7176	4.2123	0.3599	0.3268	0.2786	0.5291	3.3268
RMI		0.3166	0.1818	0	2.4721	5.1355	0.2113	0.1288	0	0.5959	6.3047
ESDRL		140.1262	134.2605	125.4398	214.9656	241.8824	122.0289	118.5494	112.5084	135.7798	235.1455
EMRL		97.1307	93.0655	86.9526	149.0036	167.6605	84.5888	82.1776	77.9918	94.1183	162.9909

Source: created by the authors

Table 5. Performance comparison on extended EWMA, EWMA and CUSUM control charts for
ARMAX(2,1,1) process when $\lambda_1 = 0.1$

δ		$\phi_1 = 0.2, \phi_2 = 0.2, \theta_1 = 0.3, c_1 = 0.4$					$\phi_1 = -0.2, \phi_2 = -0.2, \theta_1 = 0.3, c_1 = 0.5$				
		Extended EWMA			EWMA	CUSUM	Extended EWMA			EWMA	CUSUM
		$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$			$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$		
	UCL	0.03986	0.02723	0.01279	0.12972	3.392	0.019673	0.0134806	0.006349	0.0622417	4.5788
0.000	ARL_{IC}	370.0940	370.0145	370.0317	370.1088	370.0969	370.0081	370.0105	370.0044	370.0507	370.0040
	$SDRL_{IC}$	369.5936	369.5142	369.5314	369.6085	369.5966	369.5078	369.5101	369.5040	369.5503	369.5037
	MRL_{IC}	256.1829	256.1278	256.1397	256.1932	256.1849	256.1234	256.1250	256.1207	256.1529	256.1205
0.001	ARL_{OOC}	278.0292	269.9775	255.5266	312.1373	367.7543	261.6558	254.430	241.2030	287.2135	367.3241
	$SDRL_{OOC}$	277.5287	269.4771	255.0261	311.6370	367.2540	261.1553	253.9295	240.7025	286.7130	366.8237
	MRL_{OOC}	192.3684	186.7874	176.7708	216.0104	254.5611	181.0192	176.0106	166.8424	198.7344	254.2629
0.004	ARL_{OOC}	159.1903	149.0210	132.4128	212.3561	360.8422	139.2315	131.2649	117.8663	171.8409	359.4322
	$SDRL_{OOC}$	158.6895	148.5201	131.9119	211.8555	360.3418	138.7306	130.7639	117.3652	171.3402	358.9319
	MRL_{OOC}	109.9953	102.9465	91.43457	146.8472	249.770	96.1610	90.6389	81.3516	118.7641	248.7927
0.007	ARL_{OOC}	111.5079	102.8806	89.2951	160.920	354.0997	94.8147	88.3970	77.9125	122.6097	351.7569
	$SDRL_{OOC}$	111.0068	102.3793	88.7937	160.4192	353.5994	94.3134	87.8955	77.4109	122.1087	351.2565
	MRL_{OOC}	76.9443	70.9642	61.5475	111.1943	245.0965	65.3734	60.9249	53.6575	84.6396	243.4725
0.010	ARL_{OOC}	85.7995	78.540	67.3263	129.5462	347.5219	71.8636	66.6083	58.1487	95.3154	344.2911
	$SDRL_{OOC}$	85.2980	78.0384	66.8244	129.0452	347.0215	71.3618	66.1064	57.6465	94.8141	343.7907
	MRL_{OOC}	59.1244	54.0925	46.3196	89.4475	240.5371	49.4646	45.8219	39.9580	65.7204	238.2976
0.040	ARL_{OOC}	25.9501	23.2824	19.3397	43.9630	289.8121	20.9515	19.1372	16.3117	29.620	279.7871
	$SDRL_{OOC}$	25.4452	22.7770	18.8330	43.4602	289.3116	20.4454	18.6305	15.8037	29.1157	279.2867
	MRL_{OOC}	17.6384	15.7890	13.0556	30.1250	200.5356	14.17306	12.9152	10.9561	20.1824	193.5869
0.070	ARL_{OOC}	15.3115	13.6673	11.2585	26.5256	244.1513	12.2618	11.1574	9.4496	17.6048	230.1577
	$SDRL_{OOC}$	14.8031	13.1578	10.7468	26.0208	243.6508	11.7512	10.6456	8.9356	17.0975	229.6572
	MRL_{OOC}	10.2626	9.1225	7.4518	18.0373	168.8860	8.1478	7.3817	6.1969	11.8528	159.1864
0.100	ARL_{OOC}	10.8901	9.6935	7.9488	19.0339	207.6184	8.6878	7.8893	6.6601	12.5742	191.4874
	$SDRL_{OOC}$	10.3780	9.1799	7.4320	18.5272	207.1178	8.1725	7.3724	6.1397	12.0638	190.9867
	MRL_{OOC}	7.19628	6.3661	5.1553	12.8436	143.5633	5.6683	5.1141	4.2604	8.3644	132.3820
0.400	ARL_{OOC}	3.0795	2.7428	2.2661	5.2767	60.4681	2.4858	2.2695	1.9476	3.5776	49.0030
	$SDRL_{OOC}$	2.5306	2.1863	1.6939	4.7505	59.9661	1.9219	1.6974	1.3585	3.0367	48.5004
	MRL_{OOC}	1.7653	1.5285	1.1908	3.2988	41.5658	1.3469	1.1932	0.9621	2.1143	33.6185
0.700	ARL_{OOC}	2.0489	1.8514	1.5789	3.3227	27.620	1.7079	1.5851	1.4076	2.3497	21.8441
	$SDRL_{OOC}$	1.4660	1.2555	0.9560	2.7780	27.1154	1.0996	0.9630	0.7575	1.7809	21.3382
	MRL_{OOC}	1.0352	0.8923	0.6908	1.9359	18.7960	0.7871	0.6955	0.5593	1.2503	14.7919
1.000	ARL_{OOC}	1.6716	1.5326	1.5789	2.5685	16.1851	1.4347	1.3505	1.2319	1.8879	13.0504
	$SDRL_{OOC}$	1.0596	0.9035	0.9560	2.0071	15.6772	0.7898	0.6880	0.5345	1.2947	12.5404
	MRL_{OOC}	0.7602	0.6558	0.6908	1.4054	10.8684	0.5805	0.5138	0.4150	0.9188	8.6947
	AEQL	0.3101	0.2814	0.2418	0.4960	3.9262	0.2604	0.2425	0.2167	0.3536	3.1948
	RMI	0.2532	0.1510	0	0.9017	11.1177	0.2054	0.1250	0	0.5879	11.5038
	ESDRL	96.1636	92.4899	86.5846	116.3736	240.0593	89.0227	86.2002	81.4690	100.8105	233.8742
	EMRL	66.6612	64.1157	60.0242	80.6671	166.3968	61.7132	59.7577	56.4801	69.8813	162.1097

Source: created by the authors

Table 6. Performance comparison on extended EWMA, EWMA and CUSUM control charts for ARMAX(3,2,2) process when $\lambda_1 = 0.2$

δ	$\phi_1 = 0.3, \phi_2 = 0.4, \phi_3 = 0.5, \theta_1 = 0.1, \theta_2 = 0.2,$ $c_1 = 0.25, c_2 = 0.25$						$\phi_1 = -0.3, \phi_2 = -0.4, \phi_3 = -0.5, \theta_1 = -0.1, \theta_2 = 0.2,$ $c_1 = -0.25, c_2 = 0.5$				
	Extended EWMA			EWMA	CUSUM		Extended EWMA			EWMA	CUSUM
		$0.45\lambda_1$	$0.6\lambda_1$				$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$		
0.000	UCL	0.046162	0.03152	0.014803	0.151414	4.2405	0.065872	0.044853	0.0210136	0.222465	3.6632
	ARL_{IC}	370.0253	370.0116	370.0020	370.0706	370.0207	370.0276	370.0062	370.0066	370.0119	370.0131
	$SDRL_{IC}$	369.5249	369.5112	369.5017	369.5702	369.5204	369.5273	369.5058	369.5063	369.5116	369.5127
	MRL_{IC}	256.1352	256.1257	256.1191	256.1666	256.1321	256.1369	256.1220	256.1223	256.1260	256.1268
0.001	ARL_{OOC}	225.0334	216.1095	200.5965	261.0473	367.4713	234.990	225.2680	208.5929	278.6360	367.6193
	$SDRL_{OOC}$	224.5329	215.6089	200.0959	260.5468	366.9710	234.4895	224.7674	208.0923	278.1356	367.1190
	MRL_{OOC}	155.6345	149.4488	138.6960	180.5974	254.3650	162.5359	155.7970	144.2388	192.7890	254.4676
0.004	ARL_{OOC}	103.6555	96.3448	84.6908	138.8243	359.9580	112.4027	103.8518	90.5316	160.2832	360.5585
	$SDRL_{OOC}$	103.1543	95.8435	84.1893	138.3234	359.4576	111.9016	103.3506	90.0302	159.7824	360.0581
	MRL_{OOC}	71.5014	66.4340	58.3560	95.8787	249.1571	77.5645	71.6374	62.4045	110.7529	249.5733
0.007	ARL_{OOC}	67.4740	62.1174	53.7847	94.7274	352.6424	74.0146	67.6145	57.9237	112.6658	353.6740
	$SDRL_{OOC}$	66.9721	61.6154	53.2823	94.2261	352.1420	73.5129	67.1127	57.4215	112.1647	353.1736
	MRL_{OOC}	46.4219	42.7090	36.9330	65.3129	244.0863	50.9557	46.5194	39.8021	77.7469	244.8013
0.010	ARL_{OOC}	50.0918	45.9038	39.4637	71.9911	345.5183	55.2537	50.1984	42.6468	86.960	346.9606
	$SDRL_{OOC}$	49.5892	45.4011	38.9605	71.4893	345.0179	54.7514	49.6959	42.1439	86.4586	346.4602
	MRL_{OOC}	34.3732	31.4703	27.0061	49.5530	239.1483	37.9513	34.4471	29.2126	59.9288	240.1480
0.040	ARL_{OOC}	14.3416	13.0184	11.0310	21.6447	283.6016	15.9957	14.3641	11.9935	27.0023	288.1980
	$SDRL_{OOC}$	13.8326	12.5084	10.5191	21.1388	283.1012	15.4876	13.8551	11.4826	26.4976	287.6976
	MRL_{OOC}	9.5901	8.6725	7.2940	14.6537	196.2309	10.7371	9.6057	7.9616	18.3678	199.4169
0.070	ARL_{OOC}	8.5835	7.7820	6.5834	13.0335	235.4446	9.5833	8.5911	7.1565	16.3010	241.9002
	$SDRL_{OOC}$	8.0680	7.2647	6.0629	12.5236	234.9441	9.0696	8.0757	6.6377	15.7931	241.3997
	MRL_{OOC}	5.5959	5.0395	4.2072	8.6830	162.8509	6.2897	5.6012	4.6052	10.9488	167.3257
0.100	ARL_{OOC}	6.2353	5.6532	4.7852	9.4711	197.5346	6.9592	6.2371	5.1958	11.8326	205.0058
	$SDRL_{OOC}$	5.7135	5.1289	4.2560	8.9572	197.0340	6.4398	5.7152	4.6691	11.3216	204.5052
	MRL_{OOC}	3.9653	3.5607	2.9567	6.21187	136.5737	4.4682	3.9665	3.2425	7.8501	141.7524
0.400	ARL_{OOC}	2.1271	1.9560	1.7075	3.0876	53.0060	2.3368	2.1213	1.8178	3.7488	58.4344
	$SDRL_{OOC}$	1.5484	1.36745	1.0991	2.5389	52.5036	1.7675	1.5423	1.2192	3.2101	57.9323
	MRL_{OOC}	1.0914	0.9682	0.7867	1.7711	36.3933	1.2411	1.0872	0.8677	2.2340	40.1561
0.700	ARL_{OOC}	1.5784	1.4747	1.3283	2.1724	23.7458	1.7057	1.5733	1.3911	2.5712	26.5177
	$SDRL_{OOC}$	0.9555	0.8367	0.6603	1.5960	23.2404	1.0972	0.9497	0.7376	2.010	26.0129
	MRL_{OOC}	0.6905	0.6115	0.4959	1.1238	16.1102	0.7854	0.6866	0.5463	1.4073	18.0318
1.000	ARL_{OOC}	1.3744	1.2999	1.1971	1.8105	14.0297	1.4665	1.3701	1.2403	2.1001	15.5496
	$SDRL_{OOC}$	0.7174	0.6244	0.4857	1.2114	13.5205	0.8271	0.7121	0.5459	1.5199	15.0413
	MRL_{OOC}	0.5330	0.4726	0.3842	0.8624	9.3738	0.6052	0.5296	0.4223	1.0720	10.4277
AEQL		0.2387	0.2237	0.2025	0.3251	3.4352	0.2572	0.2380	0.2116	0.3835	3.7861
RMI		0.2144	0.1289	0	0.6730	15.8898	0.2407	0.1430	0	0.8637	15.5472
ESDRL		76.7826	74.1555	69.9193	89.2838	236.1321	79.8974	76.8439	72.0442	96.9459	238.9920
EMRL		53.2302	51.4103	48.4759	61.8922	163.6747	55.3883	53.2727	49.9478	67.2021	165.6571

Source: created by the authors

Table 7. Predictive models for real data

ARMAX (p,q,r)	Parameters	coefficient	Std.	t	Sig.	Model fit		
						Stationary R-square	RMSE	Normalized BIC
(1,1,1)	ϕ_1	0.951	0.035	27.259	0.000	0.822	0.567	-0.973
	θ_1	-0.279	0.112	-2.478	0.015			
	c_1	26.814	0.756	35.485	0.000			
(1,2,1)	ϕ_1	0.963	0.032	29.854	0.000	0.828	0.562	-0.937
	θ_1	-0.241	0.120	-2.004	0.049			
	θ_2	0.064	0.121	0.528	0.599			
	c_1	26.724	0.888	30.080	0.000			
(2,1,1)	ϕ_1	0.829	0.425	1.952	0.055	0.824	0.567	-0.919
	ϕ_2	0.123	0.416	0.285	0.768			
	θ_1	-0.388	0.396	-0.981	0.330			
	c_1	26.778	0.815	32.848	0.000			

Source: created by the authors

Table 8. Exponentially distributed testing for residuals

Exponential Parameter	Kolmogorov-Smirnox Z	Asymp. Sig.
0.3248	1.017	0.252

Source: created by the authors

Table 9. Performance comparison on extended EWMA, EWMA, and CUSUM control charts for ARMAX(1,1,1) process when $\lambda_1 = 0.2$, $\phi_1 = 0.951$, $\theta_1 = -0.279$, and $c_1 = 26.814$

δ	Control chart	Extended EWMA			EWMA	CUSUM
		$0.45\lambda_1$	$0.6\lambda_1$	$0.9\lambda_1$		
0.000	UCL	0.00072514	0.000228332	0.00002266	0.0240716	1.442
	ARL_{IC}	370.0887	370.0507	370.12726	370.1821	370.2404
	$SDRL_{IC}$	369.5883	369.5504	369.6269	369.6818	369.740
	MRL_{IC}	256.1792	256.1529	256.2059	256.2439	256.2843
0.001	ARL_{OOC}	242.9052	230.7670	212.4721	296.2874	367.6135
	$SDRL_{OOC}$	242.4047	230.2664	211.9715	295.787	367.1132
	MRL_{OOC}	168.0222	159.6086	146.9276	205.024	254.4635
0.004	ARL_{OOC}	119.2393	107.9524	92.710	185.2045	359.8834
	$SDRL_{OOC}$	118.7382	107.4512	92.20862	184.7038	359.3831
	MRL_{OOC}	82.30332	74.47978	63.91445	128.0271	249.1054
0.007	ARL_{OOC}	78.7129	70.1250	58.8970	134.5252	352.3539
	$SDRL_{OOC}$	78.2113	69.6232	58.39482	134.0243	351.8536
	MRL_{OOC}	54.2123	48.2596	40.4767	92.8988	243.8864
0.010	ARL_{OOC}	58.6142	51.7798	42.9751	104.7274	345.0338
	$SDRL_{OOC}$	58.1120	51.2774	42.4722	104.2261	344.5335
	MRL_{OOC}	40.2807	35.5433	29.4402	72.24434	238.8125
0.040	ARL_{OOC}	15.92931278	13.7221	10.9369	33.0588	281.5972
	$SDRL_{OOC}$	15.42121	13.2126	10.4249	32.5549	281.0968
	MRL_{OOC}	10.69104	9.1605	7.2288	22.5662	194.8416

δ	Control chart		Extended EWMA		EWMA	CUSUM
0.070	ARL_{00C}	8.9582	7.6429	5.9827	19.3879	232.5560
	$SDRL_{00C}$	8.4434	7.1254	5.4598	18.8813	232.0555
	MRL_{00C}	5.8559	4.9430	3.7898	13.0890	160.8487
0.100	ARL_{00C}	6.153430073	5.220991889	4.0462698	13.6435	194.1710
	$SDRL_{00C}$	5.6313	4.6944	3.5108	13.1340	193.6704
	MRL_{00C}	3.9084	3.2601	2.4417	9.1060	134.2422
0.400	ARL_{00C}	1.6474	1.4509	1.2307	3.4992	50.6150
	$SDRL_{00C}$	1.0327	0.8089	0.5328	2.9572	50.1125
	MRL_{00C}	0.7421	0.5931	0.4140	2.0595	34.7359
0.700	ARL_{00C}	1.2257	1.1390	1.0547	2.2064	22.5341
	$SDRL_{00C}$	0.5256	0.3980	0.2401	1.6315	22.0285
	MRL_{00C}	0.4097	0.3296	0.2342	1.1481	15.270
1.000	ARL_{00C}	1.1098	1.0616	1.0199	1.7487	13.3548
	$SDRL_{00C}$	0.3490	0.2557	0.1425	1.1442	12.8451
	MRL_{00C}	0.2996	0.2435	0.1761	0.8171	8.9058
AEQL		0.19243	0.179462	0.166343	0.335852	3.280421
RMI		0.29029	0.160759	0	1.308929	17.98336
ESDRL		81.67802	77.69669	72.27138	105.3387	234.9484
EMRL		56.62768	53.87035	50.11358	73.02038	162.8542

Source: created by the authors

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Totsaporn Muangngam carried out the methodology, simulation, software and writing.

Yupaporn Areepong was responsible for conceptualization, administration, review and editing, writing and resource.

Saowanit Sukparangsee was responsible for validation.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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