

Highly Effective Iterative Approach for Solving Fractional Riccati Equations via Ujlayan-Dixit Operator

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Abstract: The aim of this study is to review the concepts and applications related to fractional calculus. Fractional calculus and its applications constitute an important field in the study of modern mathematical models, and among its most prominent equations is the fractional Riccati equation. This equation represents a fundamental tool for modeling nonlinear phenomena in quantum mechanics and control systems. Despite its significance, traditional approaches based on the Riemann–Liouville or Caputo derivatives often lack closed-form analytical solutions and suffer from considerable computational complexity. In contrast, the Ujlayan–Dixit operator provides a simpler structure for the proportional fractional derivative of order α however, effective iterative methods for solving fractional Riccati equations based on this operator have not yet been sufficiently investigated. In light of this research gap, the present study aims to develop a form of the variational iteration method that is compatible with the Ujlayan–Dixit operator, with the objective of providing a simpler and more efficient numerical approach for solving fractional Riccati equations. The proposed method is applied to a set of numerical examples to demonstrate its effectiveness and its ability to yield accurate approximate solutions.

Key- Words: Fractional calculus, Fractional Riccati equation, Variational iteration method, Ujlayan–Dixit operator, Nonlinear differential equation, Approximate Solution.

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1 Introduction

Recent advances in many computational physics, artificial intelligence, and probabilistic modeling have highlighted the growing importance of efficient analytical and numerical techniques for solving complex mathematical problems. These developments have found applications in wave propagation, stochastic systems, energy management, and generalized probability distributions, demonstrating the broad applicability of modern mathematical tools in science and engineering, [1], [2], [3], [4].

Fractional calculus is a generalization of classical calculus, extending the concepts of differentiation and integration to non-integer

orders. Unlike traditional calculus, which is confined to integer-order derivatives, fractional calculus provides a more flexible mathematical framework for modeling systems with memory and hereditary characteristics. Interest in this field has grown rapidly in recent decades due to its effectiveness in accurately describing complex dynamics across various domains, particularly where conventional differential equations fail to capture the true behavior of materials or processes, [5], [6].

Fractional derivatives are inherently non-local, making them especially suitable for problems involving viscoelastic materials, fluid mechanics, bioengineering, control systems, signal processing,

anomalous diffusion, and more. With the expansion of this field, several definitions of fractional derivatives have emerged, each offering unique advantages depending on the problem at hand. These definitions differ in terms of non-locality, computational feasibility, and suitability for physical modeling, [7], [8], [9], [10].

Among the most established definitions is the Riemann–Liouville derivative, which is based on an extension of integer-order integrals. Although it has a strong theoretical foundation, its application in real-world problems can be challenging due to its unconventional initial condition requirements. To overcome this issue, the Caputo derivative was introduced as a modification of the Riemann–Liouville form, allowing the use of standard initial conditions and making it more applicable in engineering and physical contexts. More recently, the conformable derivative proposed by, [11] has been introduced. It retains key properties of classical calculus, such as the product rule and linearity, which help reduce the mathematical complexity typically associated with fractional derivatives.

An even newer addition is the (UD) derivative, developed to address some of the analytical and computational difficulties present in traditional definitions like those of Riemann–Liouville and Caputo. The UD derivative offers a more straightforward structure, enhancing both numerical implementation and theoretical analysis. One of its key strengths lies in its adaptability and efficiency, especially when applied to nonlinear differential equations. It maintains essential features such as memory-effect modeling and linearity while simplifying the formulation of initial conditions—an area where many existing models struggle. This makes it a promising tool in the expanding field of fractional modeling, [12], [13].

Recent studies have shown that the UD derivative contributes significantly to improving the convergence and stability of numerical approaches, particularly when applied alongside the VIM. The VIM is a powerful semi-analytical tool that has gained wide acceptance for solving various types of differential equations, including those involving fractional-order derivatives. Because fractional differential equations (FDEs) involve memory and nonlocal characteristics, traditional numerical techniques often struggle with issues related to convergence, stability, and the management of initial conditions.

The VIM overcomes many of these challenges by utilizing an iterative process built on variational principles. It generates correction

functionals without relying on linearization, discretization, or perturbation methods. Each iteration improves the approximation, often leading to highly accurate results. Furthermore, the method is capable of maintaining the original form of the equation, which makes it particularly effective for fractional models where conventional techniques might introduce significant numerical errors.

In this research, our main attention is directed towards the fractional Riccati equation, a nonlinear model that includes both quadratic terms and memory effects, adding to its complexity. Traditional methods often find such equations difficult to solve due to their fractional and nonlinear nature. To address this, we propose a computational framework that combines the strengths of VIM and the UD derivative. This hybrid approach simplifies the solution process, enhances numerical stability and accuracy, and allows for more efficient handling of initial conditions. The main objective is to present a practical and robust technique for solving the fractional Riccati equation, especially in contexts.

2 Preliminaries

In this section, we review some basic notions and fundamental results associated with fractional calculus.

Definition 1. [12]. For a given function $f: [0, \infty) \rightarrow \mathbb{R}$ and $\alpha \in [0, 1]$, the UD fractional derivative of order α is defined as

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{e^{\epsilon(1-\alpha)} f(xe^{\frac{\epsilon}{x}}) - f(x)}{\epsilon}, \quad (1)$$

if this limit exists, then $D^\alpha f(x)$ is called the UD derivative of f for $\alpha \in [0, 1]$, with the understanding that

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}.$$

Also, if f is UD differentiable on the interval $(0, x)$, for $x > 0$, and $\alpha \in [0, 1]$ such that $\lim_{x \rightarrow 0^+} D^\alpha f(x)$ exists, then we have

$$f^\alpha(0) = \lim_{x \rightarrow 0^+} f^\alpha(x).$$

Theorem 1. [12], [14]. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function and $\alpha \in [0, 1]$. Then, f is UD differentiable.

Proof. By Definition (1), we have

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{e^{\epsilon(1-\alpha)} f(xe^{\frac{\epsilon}{x}}) - f(x)}{\epsilon}.$$

By using Taylor's expansion, we can get

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{[1 + \epsilon(1 - \alpha) + o(\epsilon^2)] [f(x + \epsilon\alpha + o(\epsilon^2))] - f(x)}{\epsilon},$$

which implies

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{[1 + \epsilon(1 - \alpha)] [f(x) + f'(x)(\epsilon\alpha)] - f(x)}{\epsilon}.$$

Consequently, we have

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x) + \epsilon(1 - \alpha)f(x) + \epsilon\alpha f'(x) - f(x)}{\epsilon}.$$

Then, we have

$$D^\alpha f(x) = (1 - \alpha)f(x) + \alpha f'(x), \quad (2)$$

where $\alpha \in [0, 1]$. So, f is differentiable. $\square$

Remark 1. [12]. The UD derivatives of order α , $\alpha \in [0, 1]$, of some elementary real-valued differentiable functions in $[0, \infty)$, can be given as follows:

- (i) $D^\alpha(\lambda) = (1 - \alpha)\lambda$, for any constant $\lambda \in \mathbb{R}$,
- (ii) $D^\alpha((ax + b)^n) = (1 - \alpha)(ax + b)^n + \alpha n(ax + b)^{n-1}$, for all $a, b \in \mathbb{R}$,
- (iii) $D^\alpha(e^{ax+b}) = ((1 - \alpha) + \alpha a)e^{ax+b}$, for all $a, b \in \mathbb{R}$,
- (iv) $D^\alpha(\sin(ax + b)) = (1 - \alpha)\sin(ax + b) + \alpha a \cos(ax + b)$, for all $a, b \in \mathbb{R}$,
- (v) $D^\alpha(\cos(ax + b)) = (1 - \alpha)\cos(ax + b) - \alpha a \sin(ax + b)$, for all $a, b \in \mathbb{R}$,
- (vi) $D^\alpha(\log(ax + b)) = (1 - \alpha)\log(ax + b) + \alpha a(ax + b)^{-1}$, for all $a, b \in \mathbb{R}$.

Definition 2. [12]. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be an n times differentiable function. Then, the UD derivative of order α , $\alpha \in (n, n + 1]$, is defined as

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{e^{\epsilon(1-\alpha)} f^{[\alpha]-1}\left(xe^{\frac{\epsilon\alpha}{x}}\right) - f^{[\alpha]-1}(x)}{\epsilon},$$

where $[\alpha]$ represents the smallest integer greater than or equal to α .

Remark 2. [12] It is important to note that the UD derivative of order $\alpha \in [0, 1]$, as defined in Definition 1, does not satisfy certain classical properties of integer-order derivatives. In particular, it does not obey the standard Leibniz rule for the derivative of a product, i.e.,

$$D^\alpha(fg) \neq g D^\alpha f + f D^\alpha g,$$

and it also fails to satisfy the conventional law of indices for the composition of derivatives,

$$D^\alpha(D^\beta f) \neq D^{\alpha+\beta} f.$$

Remark 3. [12], [14]. The equation (2) indicates that the UD derivative of order $\alpha \in [0, 1]$ of a differentiable function can be expressed as a convex combination of the function itself and its first derivative. Specifically, for $\alpha = 0$, it reduces to the original function, i.e.,

$$D^0 f(t) = f(t),$$

while for $\alpha = 1$, it becomes the classical first derivative,

$$D^1 f(t) = f'(t).$$

Therefore, the UD derivative retains the conformable property often observed in conformable fractional derivatives.

Definition 3 (UD Integral). [12]. Let f be a continuous function on $[a, b]$. The UD integral, $I_a^\alpha f$, is defined as follows:

$$I_a^\alpha f(x) = \frac{1}{\alpha} \int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f(t) dt, \quad \text{where } \alpha \in (0, 1]. \quad (3)$$

3 Main Results

The VIM is an analytical approach designed to handle nonlinear problems in differential equations. Unlike purely numerical techniques, VIM constructs iterative corrections that converge efficiently to an accurate solution. The method is built on principles from the calculus of variations, where functionals and correction functionals play a central role. Its main advantage lies in its simplicity, as it avoids linearization and small parameter assumptions often required in other methods. The VIM has been proven effective for a variety of scientific and engineering model, demonstrating both flexibility and reliability. This makes it a valuable tool for researchers seeking approximate or exact solutions to complex nonlinear systems.

3.1 Foundations and Concepts

To understand VIM, we should first recall some key concepts from differential equations and the calculus of variations.

Definition 4. A functional $J[y]$ is a expression that associates a real number with every function $y(x)$ within a specified class. Functionals are often expressed as definite integrals involving $y(x)$ and its derivatives, [15].

To illustrate the concept of functional, we take the so-called arc length functional, which has the form

$$J[y] = \int_a^b \sqrt{1 + (y'(x))^2} dx.$$

Definition 5. (Variation of a Functional). The variation of a functional $J[y]$, symbolized by δJ , measures the changes in J due to a small change δy in the function y , while the independent variable x is fixed. In particular, for the functional

$$J[y] = \int L(x, y, y') dx,$$

the variation is given by

$$\delta J = \int \left(\frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial y'} \delta y' \right) dx.$$

Definition 6. (Restricted Variation). Suppose that $\delta g(x, y)$ is the variation of the function $g(x, y)$. If a function within a functional is considered a restricted variation, then its variation is taken to be zero, i.e., we have, [15]:

$$\delta \tilde{g} = 0.$$

Theorem 2. If we have the following equation, [15]:

$$\int_a^b M(\xi) \phi(\xi) d\xi = 0,$$

for every continuous function $\phi(\xi)$, then

$$M(\xi) = 0, \quad \text{for all } \xi \in [a, b].$$

Theorem 3. (Lagrange Multiplier Method). To find the extremum of a functional

$$\mathcal{J}[y] = \int_a^b F(x, y, y') dx,$$

under a given constraint

$$G(x, y, y') = 0,$$

one constructs a new functional

$$\mathcal{J}^*[y] = \int_a^b \left(F(x, y, y') + \lambda(x) G(x, y, y') \right) dx,$$

where $\lambda(x)$ is the Lagrange multiplier, [15]. The necessary condition for an extremum is

$$\delta \mathcal{J}^* = 0.$$

3.2 VIM Procedure

The VIM is a semi-analytical method developed to provide efficient solutions for differential equations, whether linear or nonlinear. Unlike traditional numerical methods, VIM generates correction functionals that improve the approximation in each iteration. The approach is rooted in the calculus of variations, where the concept of restricted variations is used to construct iterative formulas. Its main strength lies in its simplicity and wide applicability, as it avoids linearization or small parameter assumptions. Over the years, VIM has been successfully applied to diverse problems in science and engineering, establishing itself as a reliable tool for handling nonlinear systems. Let us examine the following nonlinear differential equation

$$L[y(x)] + \mathcal{N}[y(x)] = g(x), \quad (4)$$

where L is a linear operator, $\mathcal{N}$ is a nonlinear operator, and $g(x)$ is a given function such that $g(x) \neq 0$. It is common that $L[y] = y'$. So, Eq. (4) can be rewritten as

$$y'(x) = g(x) - \mathcal{N}[y(x)]. \quad (5)$$

Let us assume that

$$f(x, y) = g(x) - \mathcal{N}[y(x)],$$

then equation (5) becomes

$$y'(x) = f(x, y)$$

or

$$y'(x) - f(x, y) = 0. \quad (6)$$

Now, the core idea of any iterative method is to create a new approximation y_{n+1} from the old one y_n . This can be performed by adding a correction term, i.e.,

$$y_{n+1}(x) = y_n(x) + \text{Correction}_n(x). \quad (7)$$

From this perspective, the VIM suggests the best correction, which is the residual error. That has the following form

$$R_n(\xi) = y'_n(\xi) - f(\xi, y_n(\xi)), \quad (8)$$

where $\xi \in (0, x)$. Consequently, the $\text{Correction}_n(x)$ in (7) can be identified in order to obtain a more accurate result by assuming that

$$\begin{aligned} \text{Correction}_n(x) &= \int_0^x \lambda(\xi) (y'_n(\xi) - f(\xi, y_n(\xi))) d\xi, \end{aligned}$$

where $\lambda(\xi)$ is the Lagrange multiplier or weighting function or equivalently.

$$\text{Correction}_n(x) = \int_0^x \lambda(\xi) R_n(\xi) d\xi. \quad (9)$$

Consequently, setting (9) in (7) yields

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) R_n(\xi) d\xi, \quad (10)$$

or

$$\begin{aligned} y_{n+1}(x) &= y_n(x) \\ &+ \int_0^x \lambda(\xi) (y'_n(\xi) - f(\xi, y_n(\xi))) d\xi. \end{aligned} \quad (11)$$

Now, the most critical point here is to set the variation of some terms to zero, i.e.,

$$\delta \tilde{f} = 0 \quad \text{and} \quad \delta \tilde{y}_n = 0,$$

which is previously called the restricted variational step. In fact, such a step is considered the core of the VIM. Thus, in order to get a benefit from this step, we rewrite (11) as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) ((y'_n(\xi) - \tilde{f}(\xi, \tilde{y}_n(\xi))) d\xi. \quad (12)$$

Now, the next significant step is to identify the Lagrange multiplier $\lambda(\xi)$. This can be carried out by the calculus of variations. Thus, if one takes the variation to both sides of (12), we get

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) \\ &+ \delta \int_0^x \lambda(\xi) (y'_n(\xi) - \tilde{f}(\xi, \tilde{y}_n(\xi))) d\xi. \end{aligned} \quad (13)$$

But $\delta \tilde{f} = \delta \tilde{y}_n = 0$. Then, (13) becomes

$$\delta y_{n+1}(x) = \delta y_n(x) + \int_0^x \lambda(\xi) \delta y'_n(\xi) d\xi.$$

or

$$\delta y_{n+1}(x) = \delta y_n(x) + \int_0^x \lambda(\xi) (\delta y_n(\xi))' d\xi. \quad (14)$$

Consequently, by integrating by parts, we can get

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) \\ &+ [\lambda(\xi) \delta y_n(\xi)]_0^x - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi. \end{aligned}$$

Now, by assuming that the initial condition at each iteration satisfies

$$\delta y_n(0) = 0,$$

we arrive at

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) \\ &+ \lambda(x) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi \end{aligned}$$

or equivalently

$$\delta y_{n+1}(x) = (1 + \lambda(x)) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi. \quad (15)$$

Now, in order to reach the optimality, assume that

$$\delta y_{n+1}(x) = 0.$$

Consequently, this yields

$$(1 + \lambda(x)) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi = 0. \quad (16)$$

Thus, we have the following claim: We have

$$\int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi = 0.$$

To see this, suppose not, i.e., $\lambda'(\xi) \neq 0$, whereas $\delta y_n(\xi) \neq 0$ because it is a variation of the variable y , for $\xi \in (0, x)$. Now, since

$$\int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi \neq 0,$$

then equation (16) becomes

$$(1 + \lambda(x)) \delta y_n(x) = \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi. \quad (17)$$

By integrating by parts, equation (17) becomes

$$\begin{aligned} (1 + \lambda(x)) \delta y_n(x) &= [\lambda(\xi) \delta y_n(\xi)]_0^x \\ &- \int_0^x \lambda(\xi) \delta y'_n(\xi) d\xi, \end{aligned}$$

which implies

$$\begin{aligned} \delta y_n(x) + \lambda(x) \delta y_n(x) &= \lambda(x) \delta y_n(x) \\ &- \int_0^x \lambda(\xi) \delta y'_n(\xi) d\xi \end{aligned}$$

or

$$y_n(x) = - \int_0^x \lambda(\xi) \xi y'_n(\xi) d\xi. \quad (18)$$

By comparing both sides of equation (18), we find a contradiction. Thus

$$\int_0^x \lambda'(\xi) (\delta y_n(\xi)) d\xi = 0$$

and hence equation (16) becomes

$$(1 + \lambda(x)) (\delta y_n(x)) = 0,$$

which implies

$$\lambda(x) = -1.$$

Therefore, by substituting this value of $\lambda(x)$ into equation (11), we obtain

$$y_{n+1}(x) = y_n(x) - \int_0^x (y'_n(\xi) - f(\xi, y_n(\xi))) d\xi$$

or equivalently

$$y_{n+1}(x) = y_n(x) - \int_0^x y'_n(\xi) d\xi + \int_0^x f(\xi, y_n(\xi)) d\xi,$$

which implies

$$y_{n+1}(x) = y_n(0) + \int_0^x f(\xi, y_n(\xi)) d\xi. \quad (19)$$

Therefore, if the initial guess is chosen to satisfy the initial condition at each iteration ($y_n(0) = y_0$), then the final VIM iteration formula is

$$y_{n+1}(x) = y_0 + \int_0^x f(\xi, y_n(\xi)) d\xi.$$

Finally, the exact solution can be obtained by accordingly considering that

$$y(x) = \lim_{n \rightarrow \infty} y_n(x).$$

3.3 UD-Fractional Riccati Equation

It is known that the classical Riccati equation is a nonlinear differential equation of the form

$$y'(x) = P(x) + Q(x)y(x) + R(x)y^2(x), \quad (20)$$

with initial condition

$$y(0) = y_0,$$

where $P(x), Q(x), R(x)$ are given functions. From this point, we can identify the fractional Riccati equation using the UD operator by first reformulating (20) once again to be as follows:

$$D^\alpha y(x) = P(x) + Q(x)y(x) + R(x)y^2(x), \quad (21)$$

where $\alpha \in (0, 1)$. Now, by the property

$$D^\alpha y = \alpha y' + \beta y, \quad \beta = 1 - \alpha,$$

Eq. (21) becomes

$$\alpha \frac{dy}{dx} + \beta y(x) = P(x) + Q(x)y(x) + R(x)y^2(x). \quad (22)$$

From the previous result, we have

$$\alpha \frac{dy}{dx} = P(x) + (Q(x) - \beta)y(x) + R(x)y^2(x)$$

or

$$\frac{dy}{dx} = \frac{1}{\alpha} P(x) + \frac{1}{\alpha} (Q(x) - (1 - \alpha))y(x) + \frac{1}{\alpha} R(x)y^2(x), \quad (23)$$

with initial condition

$$y(0) = y_0,$$

Now, to deal with equation (24) via the VIM, we may consider the following assumptions:

$$A(x) = \frac{1}{\alpha} P(x), \quad B(x) = \frac{1}{\alpha} (Q(x) - (1 - \alpha)), \\ C(x) = \frac{1}{\alpha} R(x).$$

Consequently, Eq. (23) becomes

$$y'(x) = A(x) + B(x)y(x) + C(x)y^2(x), \quad (24)$$

with initial condition

$$y(0) = y_0.$$

As a consequence, the VIM assumes the correction functional for the equation

$$y' - f(x, y) = 0, \quad (25)$$

such that

$$f(x, y) = A(x) + B(x)y + C(x)y^2. \quad (26)$$

Such a correction functional is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) (y'_n(\xi) - \tilde{f}(\xi, \tilde{y}_n(\xi))) d\xi, \quad (27)$$

where $\lambda(\xi)$ is the Lagrange multiplier and $\tilde{f}, \tilde{y}_n$ denote the restricted variation so that

$$\delta \tilde{f} = \delta \tilde{y}_n = 0.$$

Consequently, by taking the variation of the correction functional given in (27), we determine

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\xi) (y'_n(\xi) - \tilde{f}(\xi, \tilde{y}_n(\xi))) d\xi, \quad (28)$$

or equivalently

$$\delta y_{n+1}(x) = \delta y_n(x) + \int_0^x \lambda(\xi) (\delta y_n(\xi))' d\xi.$$

As a result, by taking integration by parts in (28), we get

$$\delta y_{n+1}(x) = \delta y_n(x) + \lambda(\xi) \delta y_n(\xi) \Big|_0^x - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi.$$

By assuming that $y_n(0) = 0, \forall n$, we get

$$\delta y_{n+1}(x) = \delta y_n(x) + \lambda(x) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi$$

or

$$\delta y_{n+1}(x) = (1 + \lambda(x)) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi. \quad (29)$$

Now, to obtain the optimal correction, we assume that

$$(1 + \lambda(x)) \delta y_n(x) - \int_0^x \lambda'(\xi) \delta y_n(\xi) d\xi = 0. \quad (30)$$

By tracking the same procedure we discussed previously in the classical case, we obtain

$$\lambda(\xi) = -1,$$

and hence Eq. (27) becomes

$$y_{n+1}(x) = y_n(x) - \int_0^x (y'_n(\xi) - f(\xi, y_n(\xi))) d\xi$$

or

$$y_{n+1}(x) = y_n(x) - \int_0^x y'_n(\xi) d\xi + \int_0^x f(\xi, y_n(\xi)) d\xi$$

This implies

$$y_{n+1}(x) = y_n(0) + \int_0^x f(\xi, y_n(\xi)) d\xi$$

or

$$y_{n+1}(x) = y_0 + \int_0^x f(\xi, y_n(\xi)) d\xi,$$

or

$$y_{n+1}(x) = y_0 + \int_0^x [A(\xi) + B(\xi)y_n(\xi) + C(\xi)y_n^2(\xi)] d\xi.$$

Exact results are obtainable by taking the limit of this iterative step as $n \rightarrow \infty$, i.e.,

$$y(x) = \lim_{n \rightarrow \infty} y_n(x).$$

4 Numerical Simulation

In this section, we provide three illustrative examples to demonstrate the validity of our proposed technique, the VIM to solve the UD fractional Riccati problem.

Example 1. Consider the following fractional Riccati problems:

$$D^\alpha y = 1 - y^2, \quad (31)$$

with IC

$$y(0) = 0, \quad (32)$$

where D^α is the UD fractional operator so that

$$D^\alpha y = \alpha y' + (1 - \alpha) y. \quad (33)$$

To address this problem with the use of VIM, we have to use the linear combination (33) in (31) to obtain

$$\alpha y' + (1 - \alpha)y = 1 - y^2.$$

Then, we have

$$\alpha y' = 1 - y^2 - (1 - \alpha)y,$$

which implies

$$y' = \frac{1}{\alpha} - \frac{1 - \alpha}{\alpha} y - \frac{1}{\alpha} y^2. \quad (34)$$

Let

$$A = \frac{1}{\alpha}, \quad B = -\frac{1 - \alpha}{\alpha}, \quad \text{and} \quad C = -\frac{1}{\alpha},$$

then (34) becomes

$$y' = A + By + Cy^2,$$

i.e., we have

$$f(x, y) = A + By + Cy^2 = \frac{1}{\alpha} - \frac{(1 - \alpha)}{\alpha} y - \frac{1}{\alpha} y^2.$$

Now, by using the VIM formula, we obtain

$$y_{n+1} = y_0 + \int_0^x f(\xi, y_n(\xi)) d\xi, \quad (35)$$

with the initial condition $y(0) = 0$, which means $y_0(x) = 0$. So (35) becomes

$$y_{n+1}(x) = \int_0^x \left[\frac{1}{\alpha} - \frac{(1-\alpha)}{\alpha} y_n(\xi) - \frac{1}{\alpha} (y_n(\xi))^2 \right] d\xi,$$

or

$$y_{n+1}(x) = \frac{1}{\alpha} \int_0^x [1 - (1-\alpha)y_n(\xi) - (y_n(\xi))^2] d\xi.$$

Consequently, we can have the following approximation solution to the problem at hand:

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= \frac{x}{\alpha} \\ y_2(x) &= \frac{x}{\alpha} - \frac{(1-\alpha)}{2\alpha^2} x^2 - \frac{1}{3\alpha^2} x^3 \\ y_3(x) &= \frac{x}{\alpha} - \frac{(1-\alpha)}{2\alpha^2} x^2 + \frac{(1-\alpha)^2 - 2}{6\alpha^3} x^3 \\ &\quad + \frac{(1-\alpha)}{3\alpha^4} x^4 - \frac{3(1-\alpha)^2 - 8}{60\alpha^5} x^5 \\ &\quad - \frac{(1-\alpha)}{18\alpha^6} x^6 - \frac{1}{63\alpha^7} x^7. \end{aligned}$$

For the classical case (i.e., when $\alpha = 1$), we have

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= x \\ y_2(x) &= x - \frac{1}{3} x^3 \\ y_3(x) &= x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{1}{63} x^7. \end{aligned}$$

which coincides with the exact solution

$$y(x) = \tanh x = x - \frac{x^3}{3} + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \dots$$

This actually confirms the validity of our proposed scheme.

The graph presented in Figure 1 shows a comparison between the exact solution of the Riccati equation for $\alpha = 1$, given by $y(x) = \tanh(x)$, and the approximate solution obtained using the VIM. It is evident that the VIM approximation aligns almost perfectly with the exact solution throughout the interval $0 \leq x \leq 1$. The strong agreement shown in Figure 1 clearly demonstrates the accuracy and reliability of the

proposed iterative scheme. The close overlap of the curves confirms that VIM is highly effective in solving the nonlinear Riccati equation.

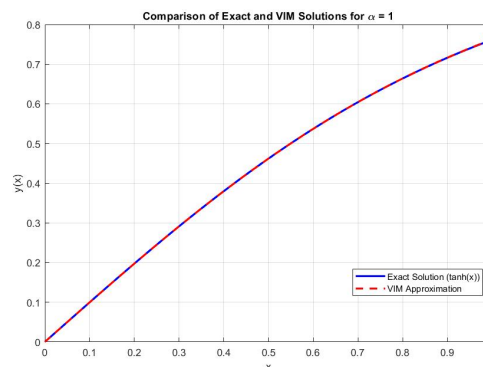


Figure 1: Comparison between the exact and VIM solutions when $\alpha = 1$.

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Figure 2 provides a comprehensive overview of the fractional Riccati equation solutions for several values of the fractional order α , specifically $\alpha = 0.7, 0.8, 0.9, 0.99$ alongside the exact solution corresponding to $\alpha = 1$. The figure shows that the fractional-order solutions gradually converge toward the exact solution as α approaches 1. Smaller values of α produce curves with slightly more pronounced curvature, reflecting the influence of the fractional derivative on the system's behavior. This observation confirms that the VIM approach appropriately captures the dynamics of both integer-order and fractional-order Riccati equations, demonstrating its robustness and effectiveness in handling fractional models.

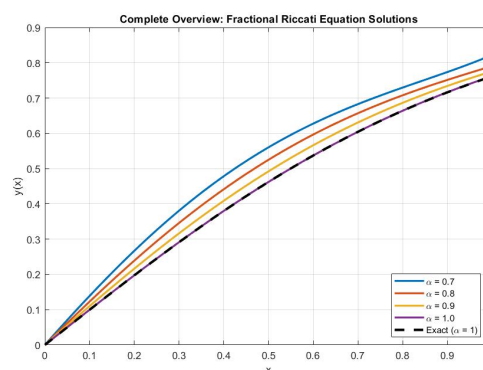


Figure 2: Fractional Riccati equation solutions for different values of α .

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Figure 3 presents the absolute error between the

exact solution and the VIM approximation over the interval $0 \leq x \leq 1$. The error remains extremely small for most of the domain and increases slightly as x approaches 1. This behavior is typical for numerical approximation methods. Overall, the graph demonstrates that the VIM provides a highly accurate approximation with minimal deviation from the exact solution.

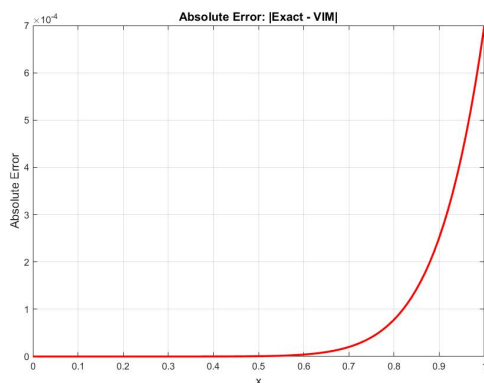


Figure 3: The absolute error between the exact and VIM solutions.

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In Appendix, Table 1 compares the exact solution with the VIM approximation at several points in the interval $0 \leq x \leq 1$. The numerical values show that the VIM results closely match the exact solution, with absolute error values remaining extremely small. The error is nearly zero for most points and increases slightly as x approaches 1, yet it still remains within a very small range. These results confirm the accuracy and reliability of the VIM in approximating the solution.

Example 2. Consider the following fractional Riccati equation:

$$D^\alpha y = y^2 - 2 \sin(x)y + \sin^2(x) + \alpha \cos(x) + (1 - \alpha) \sinh(x), \quad (36)$$

with the IC

$$y(0) = 0. \quad (37)$$

The exact solution of the above problem when $\alpha = 1$ is

$$y(x) = \sinh(x).$$

Now, we intend to solve this problem for any α via the VIM. To do so, we transform equation (36) into its classical form, i.e.,

$$\begin{aligned} D^\alpha y &= \alpha y' + (1 - \alpha)y \\ &= y^2 - 2 \sin(x)y + \sin^2(x) \\ &\quad + \alpha \cos(x) + (1 - \alpha) \sin(x). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} y' &= \frac{1}{\alpha} y^2 - \frac{2 \sin(x)}{\alpha} y + \frac{1}{\alpha} \sin^2(x) \\ &\quad + \cos(x) + \frac{(1 - \alpha)}{\alpha} \sin(x) - \frac{(1 - \alpha)}{\alpha} y \end{aligned}$$

or

$$\begin{aligned} y' &= \frac{1}{\alpha} y^2 - \left(\frac{2 \sin(x)}{\alpha} + \frac{1 - \alpha}{\alpha} \right) y \\ &\quad + \left(\frac{1}{\alpha} \sin^2(x) + \cos(x) + \frac{1 - \alpha}{\alpha} \sin(x) \right). \end{aligned} \quad (38)$$

Assume

$$\begin{aligned} A(x) &= \frac{1}{\alpha} \sin^2(x) + \cos(x) + \frac{(1 - \alpha)}{\alpha} \sin(x), \\ B(x) &= -\frac{2 \sin(x) - (1 - \alpha)}{\alpha}, \text{ and } C(x) = \frac{1}{\alpha}. \end{aligned}$$

Then (38) becomes

$$y' = A(x) + B(x)y + C(x)y^2. \quad (39)$$

Now, by using the VIM formula, we obtain

$$\begin{aligned} y_{n+1}(x) &= y_0 \\ &\quad + \int_0^x \left[A(\xi) + B(\xi)y_n(\xi) + C(\xi)(y_n(\xi))^2 \right] d\xi, \end{aligned}$$

with $y(0) = 0$, which implies $y_0(x) = 0$.

Now, as a result, we can get the approximate solution to the problem at hand. However, Figure 4 presents a comparison between the exact solution and the VIM approximation for the case $\alpha = 1$. Both curves follow nearly identical behavior throughout the interval, demonstrating that the VIM method closely matches the exact analytical solution with only minimal visible deviation.

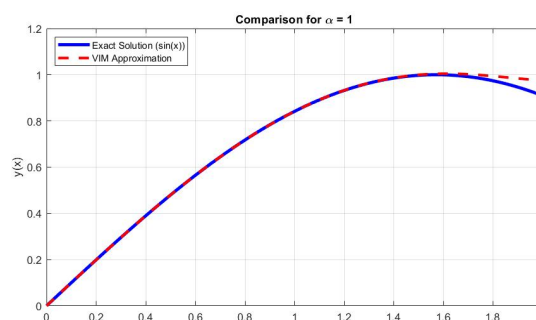


Figure 4: Comparison between the exact and VIM solutions when $\alpha = 1$.

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Figure 5 shows the VIM solutions for several different values of α . The curves illustrate how the

solution behavior changes as the parameter varies. The comparison highlights the sensitivity of the system to changes in α , as well as the ability of VIM to produce consistent approximations across different fractional order values of (α) .

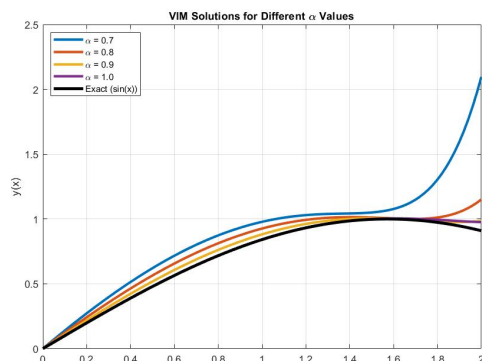


Figure 5: VIM solutions for different α values.
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Figure 6 displays the absolute error between the VIM solution and the exact solution for $\alpha = 1$. The error remains very small across most of the interval and grows noticeably only near the endpoint $x = 2$. The maximum and mean error values shown in the figure confirm that the VIM method achieves high accuracy overall.

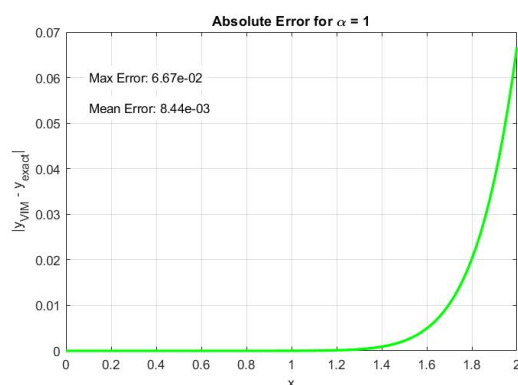


Figure 6: The absolute error between the exact and VIM solutions.
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In Appendix, Table 2 presents a comparison between the exact solution and the VIM approximation for the case $\alpha = 1$. It lists several values of x along with their corresponding exact solution and VIM approximate values. The last column shows the absolute error, which is the absolute difference between the exact and

approximate solutions. As seen from the table, the VIM approximation matches the exact solution very closely for small values of x , with the error gradually increasing as x becomes larger. Overall, the results demonstrate that the VIM provides a highly accurate approximation for the given problem.

5 Conclusion

This study aims to revisit the fundamental mathematical concepts and definitions related to fractional calculus. Building on this foundation, a fractional form of the Riccati equation was introduced by employing the Ujlayan–Dixit operator. This formulation was subsequently applied to obtain numerical solutions for the fractional initial value problems through the VIM. To demonstrate the reliability and efficiency of the proposed numerical technique, several computational examples were carried out. In the future, I will conduct studies to improve the structure of the Variational Iteration Method by refining its correction functional and Lagrange multipliers. I will also extend the application of the method to more advanced differential equations and nonlinear systems. I will perform comparative evaluations with other analytical and numerical techniques to clarify the method's accuracy and convergence behavior. Additionally, I will carry out further theoretical analysis to investigate the conditions that ensure convergence more rigorously. Moreover, I plan to develop computational algorithms that integrate VIM with modern numerical or data-driven approaches to enhance its overall performance. I will also explore new physical and engineering applications of the method and study the impact of the initial guess on convergence as part of my future research plans.

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APPENDIX

Table 1: Comparison between exact solution and VIM approximation when $\alpha = 1$

x	Exact Solution	VIM Approximation	Absolute Error
0	0	0	0
0.1	0.099668	0.099668	1.5238e-14
0.2	0.19738	0.19738	3.0403e-11
0.3	0.29131	0.29131	2.5151e-09
0.4	0.37995	0.37995	5.6014e-08
0.5	0.46212	0.46212	6.0387e-07
0.6	0.53705	0.53705	4.0961e-06
0.7	0.60437	0.60439	2.0119e-05
0.8	0.66404	0.66411	7.7871e-05
0.9	0.71630	0.71655	0.00025093
1.0	0.76159	0.76229	0.00069918

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Table 2: Comparison Table between exact solution and VIM approximation when $\alpha = 1$

x	Exact Solution	VIM Approximation	Absolute Error
0.00	0.000000	0.000000	0.00e+00
0.20	0.198669	0.198669	5.27e-16
0.40	0.389418	0.389418	1.65e-11
0.60	0.564642	0.564642	6.48e-09
0.80	0.717356	0.717357	4.17e-07
1.00	0.841471	0.841481	9.74e-06
1.20	0.932039	0.932158	1.19e-04
1.40	0.985450	0.986357	9.07e-04
1.60	0.999574	1.004479	4.91e-03
1.80	0.973848	0.994040	2.02e-02
2.00	0.909297	0.975997	6.67e-02

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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