

New traveling wave solutions for the extended (3+1)-dimensional KdV-Calogero-Bogoyavlenskii-Schiff and the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur water wave equations using efficient techniques

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Abstract: This study develops novel solitary wave solutions for the recently introduced extended (3+1)-dimensional KdV-Calogero-Bogoyavlenskii-Schiff (KdV-CBS) equation and the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur (AKNS) water wave equation with a perturbation parameter. We employ the improved generalized Riccati equation method on the KdV-CBS equation and the generalized Kudryashov method on the AKNS equation to derive fascinating and robust wave solutions, including hyperbolic, trigonometric, and rational exponential functions. Maple software is utilized to manage the complex symbolic computations involved in these methods. The results demonstrate that the derived solutions are more general and effective than those found in existing literature, offering a broader range of solutions compared to previous methodologies. Furthermore, selected solutions are visualized to elucidate various nonlinear wave phenomena, including kink, breather, dark, and bright waves. These findings provide significant insights into the dynamics of water waves, fluid dynamics, plasma physics, and oceanography.

Key-Words: - solitary wave solutions, KdV-CBS equation, AKNS water wave equation, generalized Kudryashov method, improved generalized Riccati equation method, breather wave

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1 Introduction

Differential equations of various classes and orders have performed important and efficient roles in understanding complex and ambiguous wave propagation and other nonlinear phenomena in science and technology. These subjects include physics, chemistry, chemical engineering, ecology, biological science, marine engineering, and fluid dynamics, [1], [2]. Researchers in these domains frequently model their problems using various variables to better understand the underlying processes and situations for simulation purposes. This allows us to grasp better the important processes that are associated with these mathematical models. Wave propagation is one of the most complex and difficult problems. Dynamical generation and wave propagation are nonlinear phenomena having

numerous applications in oceans, optics, plasma physics, and other fields. There has recently been a significant increase in the study of various solutions to higher-order nonlinear integrable differential equations in an attempt to elucidate and define the effect of balancing nonlinearity and dispersion in the area of scientific disciplines. Some of these models and integrable equations exhibit a distinct wave structure known as solitons. Over decades, several soliton solutions to integrable equations and their variations arose, yielding promising results capable of interpreting important results in engineering, fluid mechanics, communications, plasma physics, and other fields. These resulted in outstanding theoretical and experimental advances in integrable equations. In soliton theory, the most frequently investigated aspects of these equations

are wave structures classified as lump, bright, and dark waves, among others, [3], [4]. These wave structures can be found and derived using a variety of techniques, including the Hirota bilinear method, [5], the Bäcklund transformation method, [6], the inverse scattering method, [7], the Darboux transform, [8], the direct algebra method, [9], the Kudryashov method, [10], the sine-cosine function method, [11], the tanh function method, [12], and the improved generalized Riccati equation mapping method, [13].

Moreover, some of the most studied integrable equations are the Kortweg-de Vries equation, the Schrödinger equation, [14], Percus-Yevick integral equation, [15], the Hu-Paz-Zhang master equation, [16], the KdV-Burgers' equation, [17], the stochastic Ginzburg-Landau equation, [18], the local fractional Gardner's equation, [19], the Kuralay equation, [20], the stochastic Konno-Oono equation, [21], the coupled Drinfeld-Sokolov-Wilson equation, [22], the Radhakrishnan-Kundu-Lakshmanan equation, [23], the higher-dimensional Fokas equation, [24], the geophysical KdV equation, [25], the Schamel's equation, [26], the Mikhailov-Novikov-Wang equation, [27], the Date-Jimbo-Kashiwara-Miwa equation, [28], the Kepler's equation for elliptical orbits, [29], the sine-Gordon equation, [30], and the Benjamin-Bona-Mahony equation, [31].

The KdV equation and its modifications, such as the modified third- and fifth-order planar and non-planar KdV equations, are among the essential equations capable of explaining numerous important results in oceanography and other related areas, [18], [32]. These families of equations are used to describe wave propagation in different dimensions, especially when perturbations are considered. Furthermore, these families spanned to well-known ZK-type equations, [33], [34]. Physical events, such as the collision of forces among charged or neutral particles in plasma physics, caused the curvature effect. In that situation, the equations become completely non-integrable. KdV and KdV-like equations have proven to be resilient in numerous applications such as ocean, Bose-Einstein, and optical fibers, [35]. In this context, the curvature effect refers to spatial wavefront effects that alter nonlinear interactions during soliton collisions. The extended (3+1)-dimensional KdV-Calogero-Bogoyavlenskii-Schiff (KdV-CBS) equation is an extension of the well-known KdV equation, with curvature appearing through the transverse derivatives, resulting in phase shift after soliton collision, deformation of the soliton profiles, and energy distribution in the transverse directions. Furthermore, when pulses propagate in multimode fibers, the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur (AKNS) water wave

equation, which is closely related to the nonlinear Schrödinger equation, exhibits curvature, resulting in beam self-bending, filament formation, and transverse instabilities. In Bose-Einstein condensates (BECs), curvature arises from trap geometry, vortex rings, and curved soliton sheets, all of which dictate collision phase shifts, vortex formation, and wavefront bending (see, [35] and references therein).

In this study, we will derive and provide novel explicit wave solutions for the extended (3+1)-dimensional KdV-CBS and the fourth-order nonlinear AKNS water wave equations. Their solutions generally give various wave structures such as breathers, dark, kink, and bright wave structures that are spatially localized and change frequently in time, [36]. Kink waves, representing transitions between distinct media states, serve as effective models for internal wavefronts and density transition layers in stratified oceans, elucidating abrupt shifts in water density propagation. Breather waves are localized wave packets characterized by oscillating amplitudes within a spatially restricted envelope. These structures are fundamental to the modulation instability mechanisms that trigger rogue waves in the ocean. Breather solutions elucidate the formation and evolution of extreme wave events, which pose significant risks to maritime vessels, offshore platforms, and coastal infrastructure. Solitons, both bright and dark, are confined wave formations. These structures represent energy deficits or phase-shifted disruptions within wave trains, arising in both surface and internal wave fields under specific nonlinear conditions. Their findings provide deeper insights into wave stability and the interplay between baseline wave energy and localized disruptions in marine environments, see, [36], [37]. Experimental results on breathers and their applications in numerous scientific areas may be found in, [37] and its references. We will consider two prominent equations with applications in water wave and oceanography, namely, the extended (3+1)-dimensional KdV-Calogero-Bogoyavlenskii-Schiff (KdV-CBS) equation

$$q_{yt} + q_{xxxxy} + 3(q_x q_y)_x + 3q_{xx} + \alpha q_{xy} + \beta q_{yy} + \gamma q_{yz} = 0, \quad (1)$$

where α , β and γ depend on the physicality of the considered model, [32]. The dependent variable $q = q(x, y, z)$ in Eq. (1) represents the physical field or the state of the system being modeled. The fourth-order nonlinear Ablowitz-Kaup-Newell-Segur (AKNS) water wave equation with a perturbation parameter β , [38] is given by

$$4q_{xt} + q_{xxxt} + 8q_x q_{xy} + 4q_{xx} q_y - \beta q_{xx} = 0. \quad (2)$$

The variable $q = q(x, y, t)$ in Eq. (2) represents the

physical state of the wave at a specific point in space and time. There are not many solutions for Eq. (1) in the literature. The only one we are aware of is the application of the Hirota bilinear approach, which was used in, [32] to demonstrate integrability and derive several solitary solutions to Eq. (1). Whereas for the AKNS water wave equation, the authors in, [38] have applied the simple equation and modified simple equation methods to construct the analytical traveling wave solutions for the equation. The Lie symmetry approach was applied for different exact solutions and the dynamical behavior of the derived solutions, [39]. The fractal variational principle method is used to attain new solitary wave solutions for the AKNS water wave equation, [40]. Abundant exact wave solutions for the fourth-order AKNS water wave equation were derived using the improved $\tanh\left(\frac{\phi(\xi)}{2}\right)$ -expansion method, [41]. Some wave solutions were generated using the simple equation and Riccati equation methods for the fourth-order fractional water wave equation, [42]. A few solutions for the AKNS water wave equation were derived using the double $\left(\frac{G'}{G}, \frac{1}{G}\right)$ -expansion method, [43]. Lastly, the application of the generalized projective Riccati equation, the new improved generalized exponential rational function, and the modified F-expansion methods has been used to derive various traveling wave solutions, [44]. The primary goal in solving Eq. (1) is to provide many new and more general analytical solutions for this model via the improved generalized Riccati equation approach. Whereas for Eq. (2), we will apply the generalized Kudryashov method to derive different solutions for the fourth-order AKNS water wave equation that are more general and robust than the existing ones.

The following part will provide a brief description of the methods, while Section 3 will use the methods to derive new and more general solitary wave solutions to Eqs. (1) and (2). Graphical illustrations of some selected solutions of the problems are shown using contour, 3D, and 2D graphs. Finally, the paper ends with the concluding section.

2 The brief description of the methods

This section provides a brief overview of the methods for solving higher-order nonlinear partial differential equations (NPDEs). Given a nonlinear partial differential equation as follows:

$$A(q, q_x, q_y, q_z, q_t, q_{xx}, \dots) = 0. \quad (3)$$

Applying any appropriate wave transformation to Eq. (3), we then have an ordinary differential equation of

the following form.

$$B(U(\xi), U'(\xi), U''(\xi), \dots) = 0. \quad (4)$$

2.1 The IGREM

Here, we make an educated guess for the solution of Eq. (4), which takes the form

$$U(\xi) = \sum_{i=0}^N c_i F^i(\xi), \quad (5)$$

where c_i for $i = 0, 1, 2, \dots, N$ are constants with $c_N \neq 0$. These constants will be determined later, and N will be calculated by balancing the nonlinear term with the highest derivative term. The function $F(\xi)$ satisfies the following ordinary equation

$$F'(\xi) = \eta_2 F^2(\xi) + \eta_1 F(\xi) + \eta_0, \quad (6)$$

where η_2, η_1 , and η_0 are constants to be obtained later. The author in, [45] proposed twenty-five non-trivial solutions for the auxiliary equation (6) which are ignored here to avoid repetition. Upon substituting Eq. (5) into Eq. (4) with considering Eq. (6), we can generate a system of equations that will be used to obtain the unknown variables c_i, η_2, η_1 , and η_0 . Substituting the unknown variables into the solution form, we finally derive nontrivial solitary wave solutions for the NPDE in Eq. (3).

2.2 The generalized Kudryashov method

The generalized Kudryashov method, [46] assumes the solution of Eq. (4) to be

$$U(\xi) = \frac{\sum_{i=0}^N a_i \Theta^i(\xi)}{\sum_{j=0}^M b_j \Theta^j(\xi)}, \quad (7)$$

where the coefficients $a_i (i = 0, 1, 2, \dots, N)$ and $b_j (j = 0, 1, \dots, M)$ are constants to be determined later and $a_N \neq 0$ and $b_M \neq 0$. The integer values N and M are to be determined by balancing the highest-order derivative and its corresponding nonlinear term of Eq. (4). The function $\Theta(\xi)$ is given as

$$\Theta(\xi) = \frac{1}{1 + \exp(\xi)}. \quad (8)$$

It is obvious that Eq. (8) is the solution of the following ODE:

$$\frac{d\Theta(\xi)}{d\xi} = \Theta^2(\xi) - \Theta(\xi). \quad (9)$$

Substituting Eq. (7) into Eq. (4) and determining the unknown values, after which we get the solution of the given nonlinear PDE in Eq. (4) by replacing ξ with the used complex wave transformation.

3 The application of the methodologies to the problems

This section will apply the IGREM to solve the extended KdV-CBS equation in Eq. (1) and the generalized Kudryashov, [46] method to solve the fourth-order AKNS water wave equation.

3.1 The proposed solutions for the extended KdV-CBS equation

In this section, we will apply the IGREM to derive new and more general solitary wave solutions to the extended (3+1)-dimensional KdV-CBS equation in Eq. (1). Applying the wave variable $q(x, y, z, t) = U(\xi)$ with $\xi = kx + ry + wz - ct$, where k, r, w , and c are constants to convert Eq. (1) into an ODE and then integrating the resulting equation once, we get

$$(\alpha kr + \beta r^2 + \gamma rw - cr + 3k^2) U' + 3k^2 r (U')^2 + rk^3 U''' = 0. \quad (10)$$

Balancing the term U''' with the term $(U')^2$, we obtain the value of $N = 1$. Therefore, the solution for Eq. (10), based on the form (5), can be written as

$$U(\xi) = c_0 + c_1 F(\xi). \quad (11)$$

Substituting Eq. (11) into Eq. (10), we generate the following equations by equating the coefficient of $F^i(\xi)$ to zero:

$$\begin{aligned} F^0(\xi) : & \quad 2k^3 rc_1 \eta_0^2 \eta_2 + k^3 rc_1 \eta_0 \eta_1^2 + 3k^2 rc_1^2 \eta_0^2 \\ & \quad + \alpha kr c_1 \eta_0 + \beta r^2 c_1 \eta_0 + \gamma rw c_1 \eta_0 - cr c_1 \eta_0 \\ & \quad + 3k^2 c_1 \eta_0 = 0, \\ F^1(\xi) : & \quad 8k^3 rc_1 \eta_0 \eta_1 \eta_2 + k^3 rc_1 \eta_1^3 + 6k^2 rc_1^2 \eta_0 \eta_1 \\ & \quad + \alpha kr c_1 \eta_1 + \beta r^2 c_1 \eta_1 + \gamma rw c_1 \eta_1 - cr c_1 \eta_1 \\ & \quad + 3k^2 c_1 \eta_1 = 0, \\ F^2(\xi) : & \quad 8k^3 rc_1 \eta_0 \eta_2^2 + 7k^3 rc_1 \eta_1^2 \eta_2 + \beta r^2 c_1 \eta_2 \\ & \quad + 3k^2 rc_1^2 \eta_1^2 + \alpha kr c_1 \eta_2 + 6k^2 rc_1^2 \eta_0 \eta_2 \\ & \quad + \gamma rw c_1 \eta_2 - cr c_1 \eta_2 + 3k^2 c_1 \eta_2 = 0, \\ F^3(\xi) : & \quad 12k^3 rc_1 \eta_1 \eta_2^2 + 6k^2 rc_1^2 \eta_1 \eta_2 = 0, \\ F^4(\xi) : & \quad 6k^3 rc_1 \eta_2^3 + 3k^2 rc_1^2 \eta_2^2 = 0. \end{aligned} \quad (12)$$

Solving these equations, we have

$$\begin{aligned} c_0 &= c_0, \quad c_1 = -2k\eta_2, \quad \eta_1 = \eta_1, \\ \eta_0 &= \frac{1}{4} \frac{k^3 r \eta_1^2 + \alpha kr + \beta r^2 + \gamma rw - cr + 3k^2}{k^3 r \eta_2}, \\ \eta_2 &= \eta_2. \end{aligned} \quad (13)$$

Substituting Eq. (13) into Eq. (11) by considering the solution of Eq. (6), we obtain nontrivial solutions for

Eq. (10) as follows.

For $\frac{\Psi}{k^3 r} < 0$ where $\Psi = \alpha kr + \beta r^2 + \gamma rw - cr + 3k^2$, we get

$$U_1(\xi) = \tanh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \times \sqrt{-\frac{\Psi}{k^3 r}} k + k\eta_1 + c_0, \quad (14)$$

$$U_2(\xi) = \coth \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) k \times \sqrt{-\frac{\Psi}{k^3 r}} k\eta_1 + c_0, \quad (15)$$

$$U_3(\xi) = \sqrt{-\frac{\Psi}{k^3 r}} k \left(\operatorname{isech} \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) + \tanh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right) + k\eta_1 + c_0, \quad (16)$$

$$U_4(\xi) = \sqrt{-\frac{\Psi}{k^3 r}} k \left(\coth \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) + \operatorname{csch} \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right) + k\eta_1 + c_0, \quad (17)$$

$$U_5(\xi) = \sqrt{-\frac{\Psi}{k^3 r}} k \left(\frac{1}{2} \tanh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) + \frac{1}{2} \coth \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right) + k\eta_1 + c_0, \quad (18)$$

$$U_6(\xi) = \left(\frac{-\sqrt{-\frac{\Psi}{k^3 r}} (p^2 + q^2) + 2p \sqrt{-\frac{\Psi}{k^3 r}} \cosh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right)}{\eta_2 (p \sinh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) + q)} + \frac{\eta_1}{\eta_2} \right) k\eta_2 + c_0, \quad (19)$$

$$U_7(\xi) = c_0 - \frac{(k^3 r \eta_1^2 + \alpha kr + \beta r^2 + \gamma rw - cr + 3k^2) \cosh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right)}{k^2 r \left(\sqrt{-\frac{\Psi}{k^3 r}} \sinh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) - \eta_1 \cosh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right)}, \quad (20)$$

$$U_8(\xi) = c_0 + \frac{(k^3 r \eta_1^2 + \alpha kr + \beta r^2 + \gamma rw - cr + 3k^2) \sinh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right)}{k^2 r \left(\eta_1 \sinh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) - \sqrt{-\frac{\Psi}{k^3 r}} \cosh \left(\frac{1}{2} \sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right)}, \quad (21)$$

$$U_9(\xi) = c_0 - \frac{(k^3 r \eta_1^2 + \alpha kr + \beta r^2 + \gamma rw - cr + 3k^2) \cosh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right)}{k^2 \left(\left(\sinh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) + i \right) \sqrt{-\frac{\Psi}{k^3 r}} - \eta_1 \cosh \left(\sqrt{-\frac{\Psi}{k^3 r}} (\xi + C) \right) \right) r}, \quad (22)$$

$$U_{10}(\xi) = c_0 + \frac{(\beta r^2 - cr + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2) \sinh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right)}{\left(\eta_1 \sinh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) - (\cosh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) + 1)\sqrt{-\frac{\Psi}{k^3 r}}\right) k^2 r}, \quad (23)$$

$$U_{11}(\xi) = \frac{2(k^3 \eta_1^2 + \alpha k r + \beta r^2 + \gamma r w - cr + 3k^2) \sinh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) \cosh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right)}{k^2 r \left(\sqrt{-\frac{\Psi}{k^3 r}} \left(\cosh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right)\right)^2 - 2\eta_1 \sinh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) \cosh\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) - \sqrt{-\frac{\Psi}{k^3 r}}\right)} - c_0, \quad (24)$$

where C is an arbitrary constant.

For $\frac{\Psi}{k^3 r} > 0$ where $\Psi = \alpha k r + \beta r^2 + \gamma r w - cr + 3k^2$, we have the following solutions

$$U_{12}(\xi) = c_0 - \sqrt{\frac{\Psi}{k^3 r}} k \tan\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) + k \eta_1, \quad (25)$$

$$U_{13}(\xi) = c_0 + \sqrt{\frac{\Psi}{k^3 r}} \cot\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) k + k \eta_1, \quad (26)$$

$$U_{14}(\xi) = -\sqrt{\frac{\Psi}{k^3 r}} k \left(\sec\left(\sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) - \tan\left(\sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) \right) + k \eta_1 + c_0, \quad (27)$$

$$U_{15}(\xi) = \sqrt{\frac{\Psi}{k^3 r}} k \left(\cot\left(\sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) + \csc\left(\sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) \right) + k \eta_1 + c_0, \quad (28)$$

$$U_{16}(\xi) = -\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}} k \left(\tan\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) + \cot\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}} (\xi + C)\right) \right) + k \eta_1 + c_0, \quad (29)$$

$$U_{17}(\xi) = \left(\frac{-\sqrt{-\frac{(\Psi)(p^2+q^2)}{k^3 r}} + 2p\sqrt{-\frac{\Psi}{k^3 r}} \cos\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right)}{\eta_2 \left(p \sin\left(\sqrt{-\frac{\Psi}{k^3 r}}(\xi + C)\right) + q\right)} + \frac{\eta_1}{\eta_2} \right) k \eta_2 + c_0, \quad (30)$$

$$U_{18}(\xi) = c_0 + \frac{(-cr + \beta r^2 + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2) \cos\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)}{k^2 r \left(\sqrt{\frac{\Psi}{k^3 r}} \sin\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) + \eta_1 \cos\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)\right)}, \quad (31)$$

$$U_{19}(\xi) = c_0 + \frac{(-cr + \beta r^2 + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2) \sin\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)}{k^2 r \left(\eta_1 \sin\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) - \sqrt{\frac{\Psi}{k^3 r}} \cos\left(\frac{1}{2} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)\right)}, \quad (32)$$

$$U_{20}(\xi) = c_0 + \frac{(\beta r^2 - cr + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2) \cos\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)}{k^2 \left((\sin\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) + 1)\sqrt{\frac{\Psi}{k^3 r}} + \eta_1 \cos\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)\right) r}, \quad (33)$$

$$U_{21}(\xi) = c_0 + \frac{(\beta r^2 - cr + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2) \sin\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)}{\left(\eta_1 \sin\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) - \sqrt{\frac{\Psi}{k^3 r}} (\cos\left(\sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) - 1)\right) r k^2}, \quad (34)$$

$$U_{22}(\xi) = \frac{2 \cos\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) \sin\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) (\beta r^2 - cr + (k^3 \eta_1^2 + \alpha k + \gamma w)r + 3k^2)}{k^2 r \left(2\eta_1 \cos\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) \sin\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right) - 2\sqrt{\frac{\Psi}{k^3 r}} \left(\cos\left(\frac{1}{4} \sqrt{\frac{\Psi}{k^3 r}}(\xi + C)\right)\right)^2 + \sqrt{\frac{\Psi}{k^3 r}}\right)} - c_0, \quad (35)$$

where C is an arbitrary constant.

Lastly, for $\eta_0 = 0$, the system (12) has the following solution:

$$c_0 = c_0, \quad c_1 = -2k\eta_2, \quad \eta_2 = \eta_2, \quad \eta_1 = \frac{\sqrt{-\frac{\alpha k r + \beta r^2 + \gamma r w - cr + 3k^2}{k r}}}{k}. \quad (36)$$

Substituting Eq. (36) into Eq. (11) by considering the solution of Eq. (6), we get the following exponential solution

$$U_{23}(\xi) = \frac{2 \sqrt{-\frac{\alpha k r + \beta r^2 + \gamma r w - cr + 3k^2}{k r}} \phi}{\exp\left(-\frac{\sqrt{-\frac{\alpha k r + \beta r^2 + \gamma r w - cr + 3k^2}{k r}}(\xi + C)}{k}\right)} + \phi + c_0, \quad (37)$$

where C and ϕ are arbitrary constants.

3.2 The obtained solutions for the fourth-order AKNS water wave equation

In this section, we will apply the generalized Kudryashov method to solve the fourth-order AKNS water wave equation in Eq. (2). Now, using the transformation $q(x, y, t) = U(\xi)$, where $\xi = ax + by + ct$, we get

$$4caU' + ca^3U''' + 6a^2b(U')^2 - \beta a^2U' = 0. \quad (38)$$

Balancing U''' with $(U')^2$, the method assumes the solution of Eq. (38) to be

$$U(\xi) = \frac{a_0 + a_1 \Theta(\xi) + a_2 (\Theta(\xi))^2}{b_0 + b_1 \Theta(\xi)}, \quad (39)$$

where a_0 , a_1 , a_2 , b_0 , and b_1 are constants to be determined later. Now we substitute Eq. (39) into Eq. (38) and equate each of the powers of the coefficients Θ^i to zero and then use Eq. (8) in Section 2.2 to derive the following equations:

$$\begin{aligned}\Theta(\xi) : & -a(-a^2ca_0b_0^2b_1 + a^2ca_1b_0^3 + a\beta a_0b_0^2b_1 \\ & -a\beta a_1b_0^3 - 4ca_0b_0^2b_1 + 4ca_1b_0^3) = 0, \\ \Theta^2(\xi) : & -a(6a^2ca_0b_0^2b_1 + 4a^2ca_0b_0b_1^2 - 6a^2ca_1b_0^3 \\ & -4a^2ca_1b_0^2b_1 + 8a^2ca_2b_0^3 - 6aba_0^2b_1^2 \\ & +12aba_0a_1b_0b_1 - 6aba_1^2b_0^2 \\ & +2a\beta a_0b_0b_1^2 - 2a\beta a_1b_0^2b_1 - 2a\beta a_2b_0^3 \\ & -8ca_0b_0b_1^2 + 8ca_1b_0^2b_1 \\ & +8ca_2b_0^3) + a(-a^2ca_0b_0^2b_1 + a^2ca_1b_0^3 \\ & +a\beta a_0b_0^2b_1 - a\beta a_1b_0^3 - 4ca_0b_0^2b_1 \\ & +4ca_1b_0^3) = 0, \\ \Theta^3(\xi) : & -a(-6a^2ca_0b_0^2b_1 - 6a^2ca_0b_0b_1^2 - a^2ca_0b_1^3 \\ & +6a^2ca_1b_0^3 + 6a^2ca_1b_0^2b_1 + a^2ca_1b_0b_1^2 \\ & -30a^2ca_2b_0^3 + 5a^2ca_2b_0^2b_1 + 6aba_0^2b_1^2 \\ & -12aba_0a_1b_0b_1 + 24aba_0a_2b_0b_1 + 6aba_1^2b_0^2 \\ & -24aba_1a_2b_0^2 + a\beta a_0b_1^3 - a\beta a_1b_0b_1^2 \\ & -5a\beta a_2b_0^2b_1 - 4ca_0b_1^3 + 4ca_1b_0b_1^2 \\ & +20ca_2b_0^2b_1) + a(6a^2ca_0b_0^2b_1 \\ & +4a^2ca_0b_0b_1^2 - 6a^2ca_1b_0^3 - 4a^2ca_1b_0^2b_1 \\ & +8a^2ca_2b_0^3 - 6aba_0^2b_1^2 + 12aba_0a_1b_0b_1 \\ & -6aba_1^2b_0^2 + 2a\beta a_0b_0b_1^2 - 2a\beta a_1b_0^2b_1 \\ & -2a\beta a_2b_0^3 - 8ca_0b_0b_1^2 + 8ca_1b_0^2b_1 \\ & +8ca_2b_0^3) = 0, \\ \Theta^4(\xi) : & -a(24a^2ca_2b_0^3 - 36a^2ca_2b_0^2b_1 \\ & +4a^2ca_2b_0b_1^2 - 24aba_0a_2b_0b_1 + 12aba_0a_2b_1^2 \\ & +24aba_1a_2b_0^2 - 12aba_1a_2b_0b_1 - 24aba_2^2b_0^2 \\ & -4a\beta a_2b_0b_1^2 + 16ca_2b_0b_1^2) + a(-6a^2ca_0b_0^2b_1 \\ & -6a^2ca_0b_0b_1^2 - a^2ca_0b_1^3 + 6a^2ca_1b_0^3 \\ & +6a^2ca_1b_0^2b_1 + a^2ca_1b_0b_1^2 - 30a^2ca_2b_0^3 \\ & +5a^2ca_2b_0^2b_1 + 6aba_0^2b_1^2 - 12aba_0a_1b_0b_1 \\ & +24aba_0a_2b_0b_1 + 6aba_1^2b_0^2 - 24aba_1a_2b_0^2 \\ & +a\beta a_0b_1^3 - a\beta a_1b_0b_1^2 - 5a\beta a_2b_0^2b_1 \\ & -4ca_0b_1^3 + 4ca_1b_0b_1^2 + 20ca_2b_0^2b_1) = 0, \\ \Theta^5(\xi) : & a(24a^2ca_2b_0^3 - 36a^2ca_2b_0^2b_1 + 4a^2ca_2b_0b_1^2 \\ & -24aba_0a_2b_0b_1 + 12aba_0a_2b_1^2 + 24aba_1a_2b_0^2 \\ & -12aba_1a_2b_0b_1 - 24aba_2^2b_0^2 - 4a\beta a_2b_0b_1^2 \\ & +16ca_2b_0b_1^2) - a(36a^2ca_2b_0^2b_1 \\ & -24a^2ca_2b_0b_1^2 + a^2ca_2b_1^3 - 12aba_0a_2b_1^2 \\ & +12aba_1a_2b_0b_1 + 24aba_2^2b_0^2 - 24aba_2^2b_0b_1 \\ & -a\beta a_2b_1^3 + 4ca_2b_1^3) = 0,\end{aligned}$$

$$\begin{aligned}\Theta^6(\xi) : & -a(24a^2ca_2b_0b_1^2 - 6a^2ca_2b_1^3 \\ & +24aba_2^2b_0b_1 - 6aba_2^2b_1^2) + a(36a^2ca_2b_0^2b_1 \\ & -24a^2ca_2b_0b_1^2 + a^2ca_2b_1^3 - 12aba_0a_2b_1^2 \\ & +12aba_1a_2b_0b_1 + 24aba_2^2b_0^2 - 24aba_2^2b_0b_1 \\ & -a\beta a_2b_1^3 + 4ca_2b_1^3) = 0, \\ \Theta^7(\xi) : & -a(6a^2ca_2b_1^3 + 6aba_2^2b_1^2) \\ & + a(24a^2ca_2b_0b_1^2 - 6a^2ca_2b_1^3 \\ & +24aba_2^2b_0b_1 - 6aba_2^2b_1^2) = 0, \\ \Theta^8(\xi) : & a(6a^2ca_2b_1^3 + 6aba_2^2b_1^2) = 0.\end{aligned}$$

Solving the above equations, we have:

Case 1

$$\begin{aligned}a_0 = a_0, \quad a_1 = a_1, \quad a_2 = -\frac{b_0(acb_0 + ba_1)ac}{b^2a_0}, \\ b_0 = b_0, \quad b_1 = \frac{b_0(acb_0 + ba_1)ac}{ba_0}, \quad \beta = \frac{c(a^2 + 4)}{a}.\end{aligned}\quad (40)$$

Case 2

$$\begin{aligned}a_0 = a_0, \quad a_1 = a_1, \quad a_2 = 0, \quad \beta = \frac{c(a^2 + 4)}{a} \\ b_0 = b_0, \quad b_1 = -\frac{b_0(acb_0 + ba_1)}{acb_0 - ba_0}.\end{aligned}\quad (41)$$

Case 3

$$\begin{aligned}a_0 = -\frac{1}{2}a_1, \quad a_1 = a_1, \quad a_2 = \frac{2acb_0}{b}, \\ b_1 = -2b_0, \quad b_0 = b_0, \quad \beta = \frac{4c(a^2 + 1)}{a}.\end{aligned}\quad (42)$$

Substituting Eq. (40) into Eq. (39), we get the following solution:

$$U_1(\xi) = \frac{a_0 - \frac{b_0(acb_0 + ba_1)ac}{(1+e^\xi)^2 b^2 a_0} + \frac{a_1}{1+e^\xi}}{\frac{b_0(acb_0 + ba_1)}{(1+e^\xi)ba_0} + b_0}.\quad (43)$$

Substituting Eq. (41) into Eq. (39), we obtain the following solution:

$$U_2(\xi) = \frac{\frac{a_1}{1+e^\xi} + a_0}{b_0 - \frac{b_0(acb_0 + ba_1)}{(1+e^\xi)(acb_0 - ba_0)}}.\quad (44)$$

Substituting Eq. (42) into Eq. (39), we obtain the following solution:

$$U_3(\xi) = \frac{\frac{2acb_0}{(1+e^\xi)^2 b} + \frac{a_1}{1+e^\xi} - \frac{1}{2}a_1}{b_0 - \frac{2b_0}{1+e^\xi}}.\quad (45)$$

4 Results and discussion

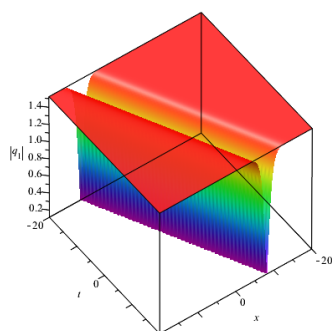
This paper proposes new solitary wave solutions for the extended (3+1)-dimensional KdV-CBS equation in Eq. (1) using the IGREM and for the fourth-order AKNS water wave equation in Eq. (2) utilizing the generalized Kudryashov method. The used methods yield new, concise, and more general solutions for the considered equations. For Eq. (1), the IGREM provides twenty-three solutions, whereas the only available related study on the the extended (3+1)-dimensional KdV-CBS equation, [32] proposed only ten solutions. The resulting solutions obtained using the IGREM method are in the form of different periodic and hyperbolic functions that are more general and efficient than those shown in, [32]. Figure 1 shows graphs of the solution $q_1(x, y, z, t) = U_1(\xi)$ in Eq. (14) using the parameters $k = -1$, $r = 1.2$, $w = 0.8$, $c = 0.5$, $y = 0.1$, $z = 0.5$, $c_0 = 0.5$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 1$, $C = 1.3$, $\eta_1 = 0.5$, $\eta_2 = 1$. Figure 1(a) represents a dark soliton obtained by plotting $|q_1(x, y, z, t)|$ while Figure 1(b) shows the kink soliton obtained by graphing $q_1(x, y, z, t)$ when $y = 0.1$, $z = 0.5$. Figure 1(c) and Figure 1(d) represent the contour and 2D plots of the kink soliton for $q_1(x, y, z, t)$, respectively. Figure 2 shows behaviors of the solution $q_2(x, y, z, t) = U_2(\xi)$ in Eq. (15) using the parameters $k = -1$, $r = 1.2$, $w = 0.8$, $c = 0.5$, $y = 0.1$, $z = 0.5$, $c_0 = 0.5$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 1$, $C = 1.3$, $\eta_1 = 0.5$, $\eta_2 = 1$. Figure 2(a) represents a bright soliton plotted using $|q_2(x, y, z, t)|$ while Figure 2(b) is the breather soliton graphed using $q_2(x, y, z, t)$. Figure 2(c) and Figure 2(d) represent the contour and 2D plots of the breather soliton for $q_2(x, y, z, t)$, respectively. In Figure 3, we use the parameters $k = -1.5$, $r = 0.2$, $w = 0.8$, $c = 0.5$, $y = 0.1$, $z = 0.5$, $c_0 = 0.5$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 1$, $C = 1$, $\eta_1 = 0.5$, $\eta_2 = 1$ for plotting physical behaviors of the solution $q_3(x, y, z, t) = U_3(\xi)$ in Eq. (16). Figure 3(a) represents the kink soliton solution for $|q_3(x, y, z, t)|$ and Figure 3(b) shows the antikink soliton solution for the real part of $q_3(x, y, z, t)$. In addition, Figure 3(c) and Figure 3(d) represent the contour and 2D plots of the kink wave for the real part of $q_3(x, y, z, t)$, respectively. Figure 4 is plotted using the parameters $k = -1$, $r = 0.2$, $w = 0.8$, $c = 0.9$, $y = 0.1$, $z = 0.5$, $c_0 = 0.5$, $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 0.6$, $C = 1$, $\eta_1 = 0.5$, $\eta_2 = 1$ for the solution $q_4(x, y, z, t) = U_4(\xi)$ in Eq. (17). Figure 4(a) and (b) represent the breather solitons plotted using $|q_4(x, y, z, t)|$ and $q_4(x, y, z, t)$, respectively when $y = 0.1$, $z = 0.5$. Figure 4(c) and Figure 4(d) show the contour and 2D plots for the breather wave of $q_4(x, y, z, t)$, respectively. In, [32], the authors were able to plot the structure of the kink soliton. However, in this work, we have demonstrated different wave

profiles including bright, dark, kink, antikink, and breather wave profiles using the IGREM.

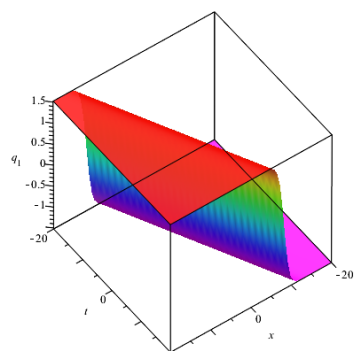
Figure 5(a) shows the evolution of the kink wave profile of the solution $q_1(x, y, t) = U_1(\xi)$ in Eq. (43) generated using $c = 0.3$, $b = 2.4$, $a = 2$, $b_0 = 2$, $a_0 = 1$, $a_1 = 1$, and $y = 2$. Figure 5(b) and Figure 5(c) show the contour and 2D graphs for the kink wave of $q_1(x, y, t)$, respectively. Figure 6(a) is the evolution of the anti-kink wave profile of the solution $q_2(x, y, t) = U_2(\xi)$ in Eq. (44) using $c = 0.2$, $b = -2.4$, $a = 2$, $b_0 = 2$, $a_0 = 1$, $a_1 = 1$, and $y = 2$. While Figure 6(b) and Figure 6(c) illustrate the contour and 2D plots for the rogue wave of $q_2(x, y, t)$, respectively. Figure 7(a) is the rogue wave solution derived using the solution $q_3(x, y, t) = U_3(\xi)$ in Eq. (45) with $c = 1.2$, $b = 2$, $a = 2.7$, $b_0 = 2.8$, $a_1 = 1.7$, and $y = 1.5$. Figure 7(b) and Figure 7(c) present the contour and 2D graphs for the solution $q_3(x, y, t)$, respectively when $y = 1.5$ is fixed. Most importantly, it is paramount to clarify that the derived solutions in this work are different from those proposed using the double $(\frac{G'}{G}, \frac{1}{G})$ -expansion method, [43] in terms of using distinct function, i.e., the fraction of exponential functions. This work varies from the existing research because the solutions described in this work may be utilized to access and investigate the dynamical influence of each parameter in the AKNS water wave equation, unlike the earlier studies that offered condition solutions.

5 Conclusion

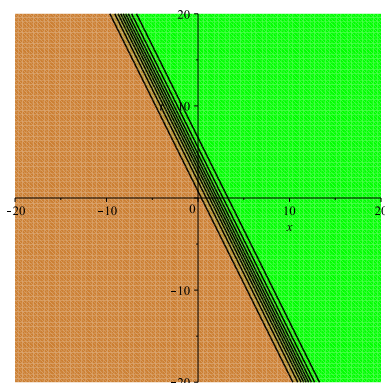
This study developed novel solitary wave solutions for the recently introduced extended (3+1)-dimensional KdV-CBS equation as expressed in Eq. (1) and the fourth-order nonlinear AKNS water wave equation with a perturbation parameter as shown in Eq. (2). The study of these equations revealed various hyperbolic, trigonometric, and exponential wave solutions utilizing the IGREM and the generalized Kudryashov method with the help of Maple. Based on the established relationships, the derived solutions are more general and effective than the existing ones in terms of different trigonometric, exponential, and hyperbolic functions. Furthermore, we graphed some of the recovered solutions using appropriate parameter values to elucidate various nonlinear wave phenomena predominantly found in water waves such as kink, breather, dark, and bright waves. The governing equations can be further studied using other efficient techniques, especially by incorporating the concept of fractional derivatives such as the truncated M-fractional derivative. The two methods utilized in this article, assisted by Maple, are efficient and reliable mathematical tools for extracting exact traveling wave solutions of emerging NPDEs.



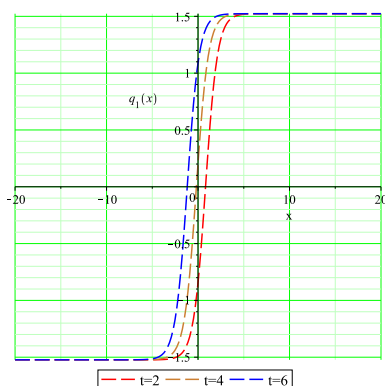
(a) The 3D absolute graph



(b) The 3D graph



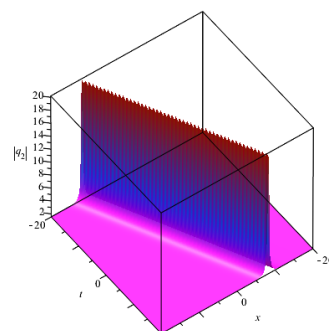
(c) The contour



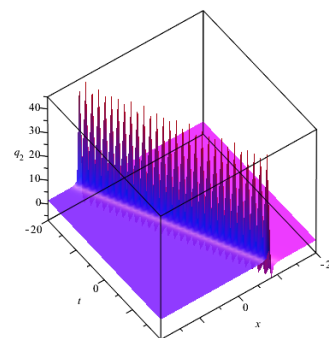
(d) The 2D graph

Fig. 1: The graphs of $|q_1(x, y, z, t)|$ and $q_1(x, y, z, t)$ where $q_1(x, y, z, t) = U_1(\xi)$ in Eq. (14).

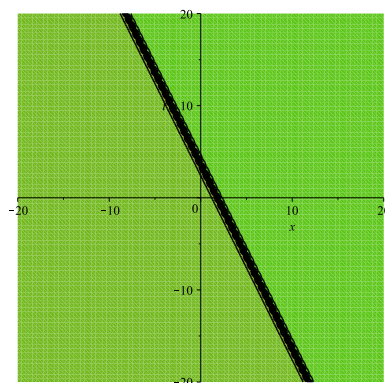
Source: created by the authors.



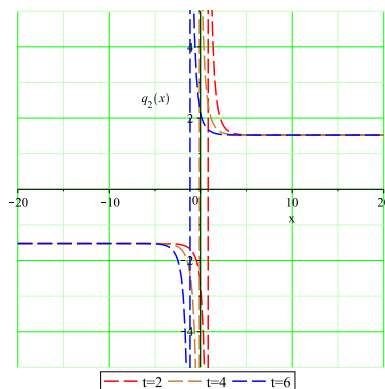
(a) The 3D absolute graph



(b) The 3D graph



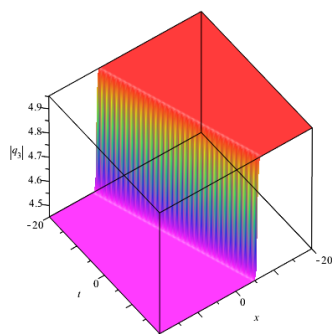
(c) The contour



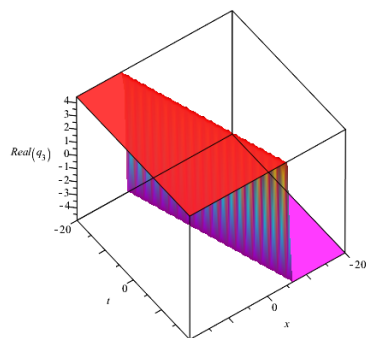
(d) The 2D graph

Fig. 2: The graphs of $|q_2(x, y, z, t)|$ and $q_2(x, y, z, t)$ where $q_2(x, y, z, t) = U_2(\xi)$ in Eq. (15).

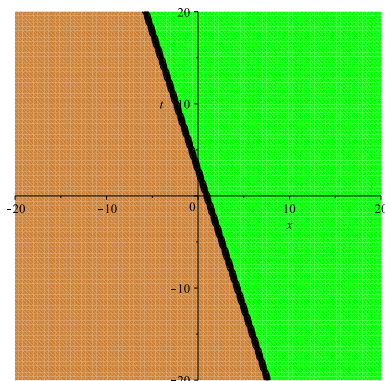
Source: created by the authors.



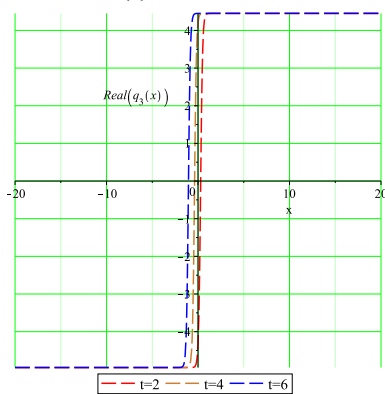
(a) The 3D absolute graph



(b) The 3D Real graph



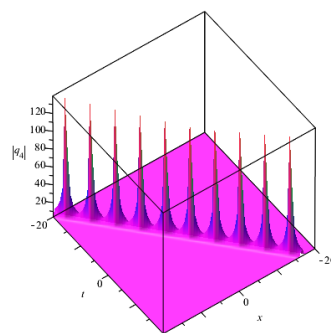
(c) The contour



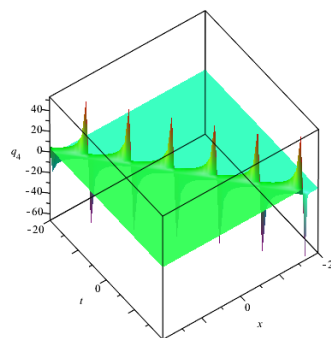
(d) The 2D graph

Fig. 3: The graphs of $|q_3(x, y, z, t)|$ and the real part of $q_3(x, y, z, t)$ where $q_3(x, y, z, t) = U_3(\xi)$ in Eq. (16).

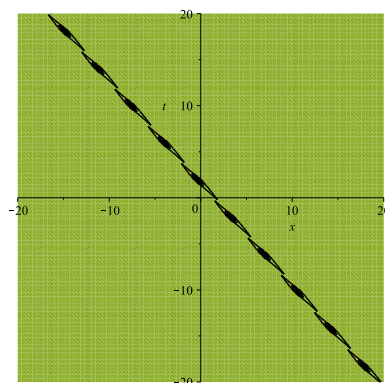
Source: created by the authors.



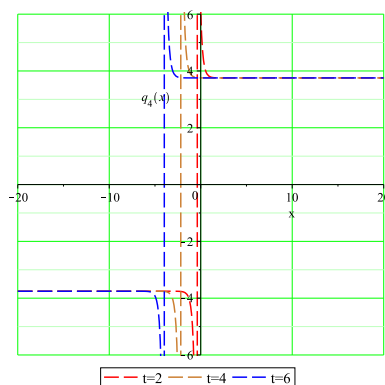
(a) The 3D absolute graph



(b) The 3D Real graph



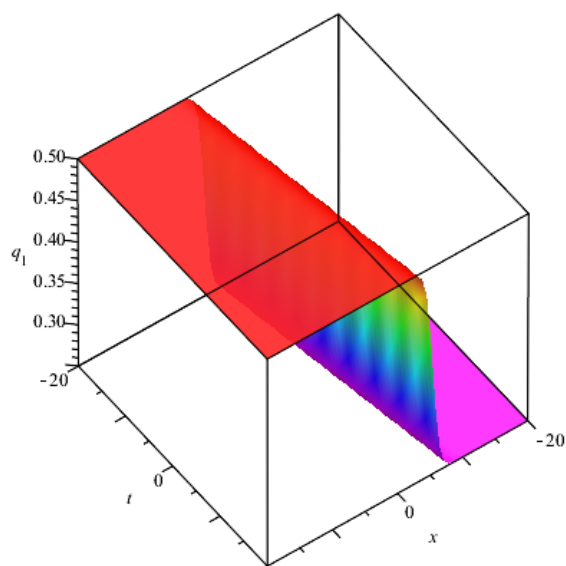
(c) The contour



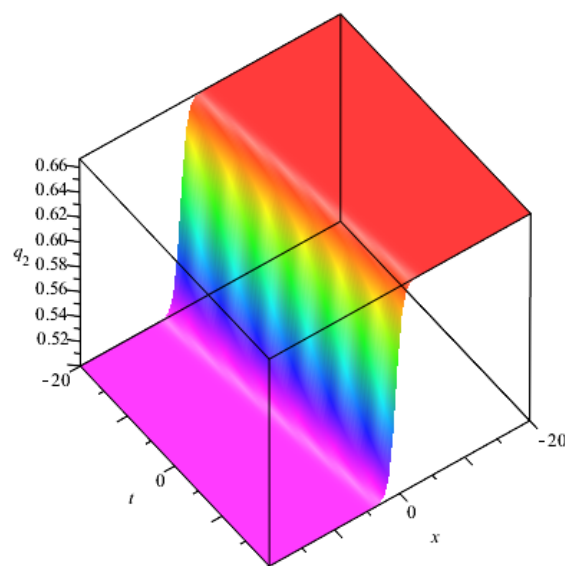
(d) The 2D graph

Fig. 4: The graphs of $|q_4(x, y, z, t)|$ and $q_4(x, y, z, t)$ where $q_4(x, y, z, t) = U_4(\xi)$ in Eq. (17).

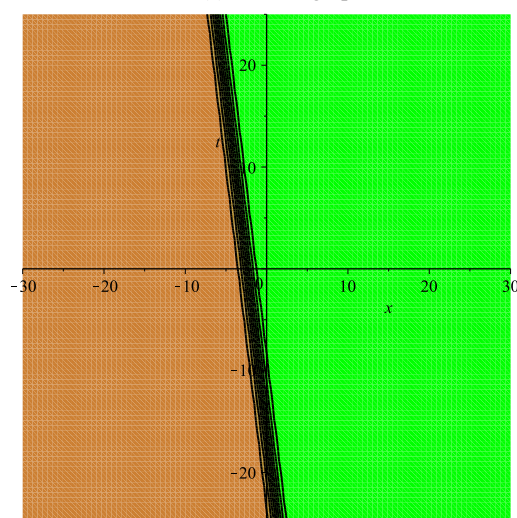
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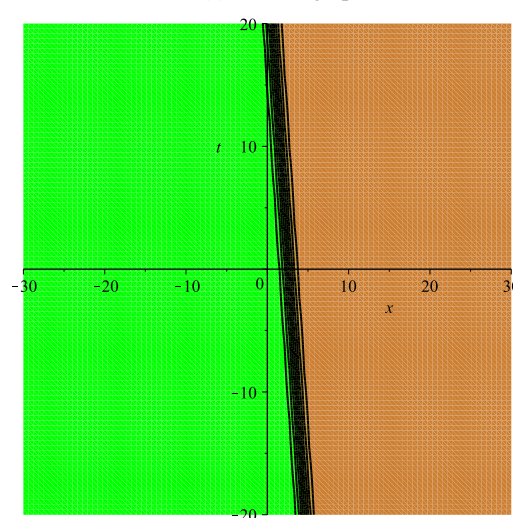
(a) The 3D graph



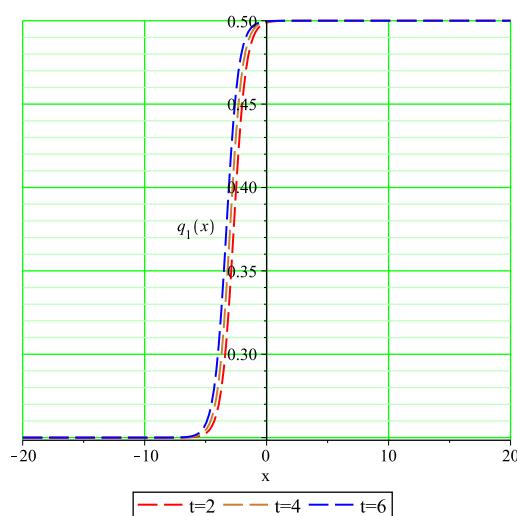
(a) The 3D graph



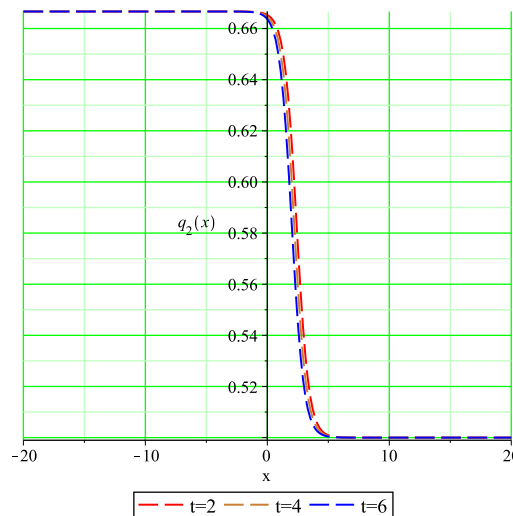
(b) The contour



(b) The contour



(c) The 2D graph



(c) The 2D graph

Fig. 5: The kink wave profile of $q_1(x, y, t)$ where $q_1(x, y, z, t) = U_1(\xi)$ in Eq. (43).
Source: created by the authors.

Fig. 6: The anti-kink wave profile for $q_2(x, y, t)$ where $q_2(x, y, z, t) = U_2(\xi)$ in Eq. (44).
Source: created by the authors.

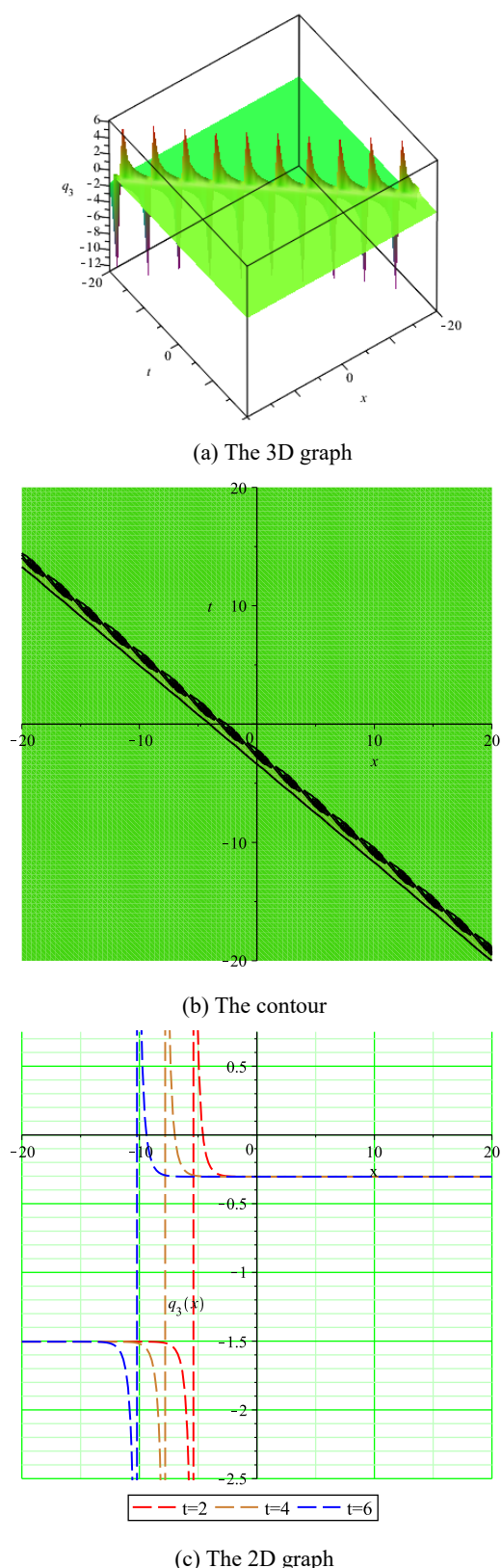


Fig. 7: The rogue wave profile for $q_3(x, y, t)$ where $q_3(x, y, z, t) = U_3(\xi)$ in Eq. (45).

Source: created by the authors.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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