

Bicomplex Extensions of Slant Hankel Operators and Their Theoretic Properties

İLKER ERYILMAZ

Department of Mathematics,
Ondokuz Mayıs University
Kurupelit, 55200, Samsun
TURKEY

Abstract: - In this paper, we introduce and study a bicomplex analog of slant Hankel operators acting on the bicomplex Hilbert space $L^2_{\mathbb{BC}}(U_{\mathbb{BC}})$. Motivated by the classical theory of slant Hankel operators defined on $L^2(\mathbb{T})$, we extend the underlying framework to the setting of bicomplex-valued functions. For a symbol $\varphi = \varphi_1 e_1 + \varphi_2 e_2$ in $L^\infty_{\mathbb{BC}}$, we define the associated slant bicomplex Hankel operator via its matrix representation with respect to the standard orthonormal basis, preserving the characteristic slant structure. We investigate fundamental operator-theoretic properties of these operators, including boundedness and norm estimates, and establish conditions for an operator to be a slant bicomplex Hankel operator. Utilizing the idempotent decomposition of bicomplex numbers, we derive representations that allow the reduction of certain problems to classical complex components.

Key-Words: - Bicomplex numbers, Bicomplex Hilbert space, Idempotent decomposition, Hyperbolic norm, Slant Hankel operator, Toeplitz operator

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1 Introduction

Vector spaces over a field are the traditional focus of functional analysis. However, a more comprehensive framework is presented by taking into account modules with bicomplex ($\mathbb{BC}$) scalars. This addition makes it possible to investigate features outside of the classical and to investigate new mathematical structures. The book [1] is a great resource for researchers interested in this field. It offers findings, methods, and applications related to functional analysis and the study of modules with bicomplex scalars. These findings include a number of functional analysis topics, including spectral theory, operator theory, and function spaces. They provide insight into how modules with bicomplex scalars behave, show links to other branches of mathematics, and may have uses in engineering, physics, or other fields.

A number of fundamental theorems pertaining to topological bicomplex modules and their systematic research are highlighted in the series of works cited in the references. The articles and their contributions are broken out as follows:

In [2], four pillars of functional analysis for bicomplex modules are introduced. In [3], a new

version of the Hahn-Banach theorem for hyperbolic and bicomplex modules is discussed and written.

The authors of [4] concentrated on bicomplex functional calculus and $\mathbb{BC}$ -bounded linear operators. Holomorphic and continuous functions, sequences and series, integration, differential equations, and geometric features unique to the bicomplex domain are just a few of the many topics it covers [5].

The study of basic theorems is extended to the context of topological bicomplex modules in [6]. The goal is to provide a greater understanding of the features of classical conclusions from functional analysis by generalizing them to the bicomplex module framework.

Together, these sources make substantial improvements to the study of functional analysis, $\mathbb{BC}$ -modules, and connected fields. For in-depth information on the corresponding fields of study, readers with an interest in these subjects might consult these references: [7], [8], and [9]. We will now give a brief summary of bicomplex numbers and some of their fundamental characteristics.

2 $\mathbb{BC}$ and $\mathbb{BC}$ -Lebesgue spaces: preliminary findings

Let i, j be imaginary units which satisfy $ij = ji, i^2 = -1 = j^2$ and $\mathbb{C}(i)$ stand for the field of complex numbers with the imaginary unit i . The set of bicomplex numbers $\mathbb{BC}$ is then defined as follows:

$$\mathbb{BC} := \{V = \vartheta_1 + j\vartheta_2 \mid \vartheta_1, \vartheta_2 \in \mathbb{C}(i)\}.$$

The standard definitions of addition and multiplication for any $Z = z_1 + jz_2 \in \mathbb{BC}, V = \vartheta_1 + j\vartheta_2 \in \mathbb{BC}$ are

$$\begin{aligned} Z + V &= (z_1 + \vartheta_1) + j(z_2 + \vartheta_2) \\ ZV &= (z_1\vartheta_1 - z_2\vartheta_2) + j(z_2\vartheta_1 + z_1\vartheta_2). \end{aligned}$$

Hence, $\mathbb{BC}$ forms a commutative ring. If $k := ij$, then a new hyperbolic unit k can be obtained such that $k^2 = 1$. All of the units in $\mathbb{BC}$ have commutative product operations. In particular,

$$ij = k, \quad jk = -i \quad \text{and} \quad ik = -j.$$

Hyperbolic numbers ($\mathbb{D}$) that is a two-dimensional extension of the $\mathbb{R}$, constitutes a number system called the hyperbolic plane (or hyperbolic plane algebra). Any element of $\mathbb{D}$ can be expressed as $\alpha = x_1 + kx_2$, where k is the hyperbolic unit and x_1 and x_2 are real numbers.

Let $V = \vartheta_1 + j\vartheta_2 \in \mathbb{BC}$ with $\vartheta_1, \vartheta_2 \in \mathbb{C}(i)$. The following conjugations are created for bicomplex numbers in [1], [5], and [10] using the notation of V with i and j :

$$\begin{aligned} V^{\dagger_1} &= \overline{\vartheta_1} + j\overline{\vartheta_2}, \\ V^{\dagger_2} &= \vartheta_1 - j\overline{\vartheta_2} \quad \text{and} \\ V^{\dagger_3} &= \overline{\vartheta_1} - j\overline{\vartheta_2} \end{aligned}$$

where the typical complex conjugates of $\vartheta_1, \vartheta_2 \in \mathbb{C}(i)$ are $\overline{\vartheta_1}$ and $\overline{\vartheta_2}$, respectively. In the same references, for any $V \in \mathbb{BC}$, specific moduli are defined as

- $|V|_i^2 = V \cdot V^{\dagger_1} = \vartheta_1^2 + \vartheta_2^2 \in \mathbb{C}(i),$
- $|V|_j^2 = V \cdot V^{\dagger_2} = (|\vartheta_1|^2 - |\vartheta_2|^2) + j(2\text{Re}(\vartheta_1\overline{\vartheta_2})) \in \mathbb{C}(j)$ and
- $|V|_k^2 = V \cdot V^{\dagger_3} = (|\vartheta_1|^2 + |\vartheta_2|^2) + k(-2\text{Im}(\vartheta_1\overline{\vartheta_2})) \in \mathbb{D}.$

Also in [1], it is shown that, $\mathbb{BC}$ is a normed space with the norm

$$\|V\|_{\mathbb{BC}} = \sqrt{|\vartheta_1|^2 + |\vartheta_2|^2}$$

for any $V = \vartheta_1 + j\vartheta_2$ in $\mathbb{BC}$. According to this,

$$\|V_1 V_2\|_{\mathbb{BC}} \leq \sqrt{2} \|V_1\|_{\mathbb{BC}} \|V_2\|_{\mathbb{BC}}$$

for any $V_1, V_2 \in \mathbb{BC}$ and it is stated in [10] that $\mathbb{BC}$ is a modified Banach algebra. If e_1 and e_2 are defined as

$$e_1 = \frac{1+k}{2} \quad \text{and} \quad e_2 = \frac{1-k}{2},$$

then it is easy to see that

$$\begin{aligned} e_1^2 &= e_1, \quad e_2^2 = e_2, \quad e_1^{\dagger_3} = e_1, \quad e_2^{\dagger_3} = e_2 \\ e_1 + e_2 &= 1, \quad e_1 \cdot e_2 = 0 \end{aligned}$$

and $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$. According to these, any $V = \vartheta_1 + j\vartheta_2 \in \mathbb{BC}$ may be uniquely expressed as a linear combination of e_1 and e_2 by utilizing the set $\{e_1, e_2\}$. In other words, $V = \vartheta_1 + j\vartheta_2$ can be expressed as follows:

$$V = \vartheta_1 + j\vartheta_2 = e_1 z_1 + e_2 z_2 \quad (1)$$

where $z_1 = \vartheta_1 - i\vartheta_2$ and $z_2 = \vartheta_1 + i\vartheta_2$ [1]. Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and (1), the preceding formula, is named with the *idempotent representation* of V .

A hyperbolic-valued norm $|V|_k$ of any bicomplex number $V = e_1 z_1 + e_2 z_2$ is defined as follows in addition to the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$:

$$|V|_k = e_1 |z_1| + e_2 |z_2|.$$

For any hyperbolic number $\alpha = x_1 + kx_2 \in \mathbb{D}$, another way to express an idempotent representation is as

$$\alpha = e_1 \alpha_1 + e_2 \alpha_2$$

where $\alpha_1 = x_1 + x_2$ and $\alpha_2 = x_1 - x_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = x_1 + kx_2 = e_1 \alpha_1 + e_2 \alpha_2 \in \mathbb{D}$, then α is called a positive hyperbolic number. Thus, the set of non-negative hyperbolic numbers

$$\begin{aligned} \mathbb{D}^+ \cup \{0\} &= \{\alpha = x_1 + kx_2: x_1^2 - x_2^2 \geq 0, x_1 \geq 0\} \\ &= \{\alpha = e_1 \alpha_1 + e_2 \alpha_2: \alpha_1, \alpha_2 \geq 0\}. \end{aligned}$$

can be defined. Suppose that α and β are any two elements of $\mathbb{D}$. A relation " $\leq$ " is defined on $\mathbb{D}$ in [1] and [5] by

$$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

This relation " $\leq$ " can construct a partial order relation on $\mathbb{D}$, as demonstrated in [1]. If the hyperbolic numbers α, β are expressed as idempotent representations, $\alpha = e_1 \alpha_1 + e_2 \alpha_2$ and $\beta = e_1 \beta_1 + e_2 \beta_2$, then $\alpha \leq \beta$ implies that $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. By $\alpha < \beta$, we mean $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. To learn more about partial order " $\leq$ " and hyperbolic numbers $\mathbb{D}$, see Section 1.5 of [1] and [5], [10].

Let U be a subset of $\mathbb{D}$. If there is a hyperbolic number δ such that $\delta \geq \alpha$ for all $\alpha \in U$, then U is called a $\mathbb{D}$ -bounded set from above. If $U \subset \mathbb{D}$ is a $\mathbb{D}$ -

bounded set from above, then $\mathbb{D}$ -supremum of U is defined as the smallest member of the set of all $\mathbb{D}$ -upper bounds of U [5].

Let $\mathbb{D} \supset U$ be $\mathbb{D}$ -bounded from above and $U_1 := \{\gamma_1: e_1\gamma_1 + e_2\gamma_2 \in U\}$, $U_2 := \{\gamma_2: e_1\gamma_1 + e_2\gamma_2 \in U\}$. Then the $\sup_{\mathbb{D}} U$ is given by

$$\sup_{\mathbb{D}} U := e_1 \sup U_1 + e_2 \sup U_2.$$

Similarly, for any set U which is $\mathbb{D}$ -bounded from below, $\mathbb{D}$ -infimum of U can be defined.

Suppose that $(Y, +)$ is an abelian group and $(Y, +, \cdot)$ is a $\mathbb{BC}$ -module. $(Y, +, \cdot)$ is called a topological $\mathbb{BC}$ -module if there exists a topology τ_Y , for which the operations $+: Y \times Y \rightarrow Y$ and $\cdot: \mathbb{BC} \times Y \rightarrow Y$ are continuous [6].

Let Y be a $\mathbb{BC}$ -module space (or $\mathbb{D}$ -module space). Then it can be decomposed as

$$Y = e_1 Y_1 + e_2 Y_2 \quad (2)$$

where $Y_1 = e_1 Y$ and $Y_2 = e_2 Y$ are $\mathbb{R}$ -vector or $(\mathbb{C}(i)$ -vector) spaces. The idempotent decomposition of the space Y is the spelling found in (2) [1].

The following is known from [6].

A $\mathbb{BC}$ -module $(Y, \|\cdot\|_Y)$ is called a bicomplex Banach module if every Cauchy sequence in Y converges to any element of it. $(Y, \|\cdot\|_Y)$ is a bicomplex Banach module, if and only if, the decomposition pairs of the space, $(Y_1, \|\cdot\|_{Y_1})$ and $(Y_2, \|\cdot\|_{Y_2})$ are complex Banach spaces [2].

Let Y be a $\mathbb{BC}$ -module with $\mathbb{D}$ -valued norm and $T: Y \rightarrow Y$. If

$$(i) T(x + y) = T(x) + T(y),$$

$$(ii) T(\alpha x) = \alpha T(x),$$

for every $x, y \in Y$ and $\alpha \in \mathbb{BC}$, then T is called a $\mathbb{BC}$ -linear operator on Y [11].

This is a well-known outcome from [4].

Let Y be a bicomplex Banach module and $T: Y \rightarrow Y$ be a linear operator. Suppose that Y has an idempotent decomposition as $Y = e_1 Y_1 + e_2 Y_2$. Then the operator T admits the idempotent representation $T = e_1 T_1 + e_2 T_2$, where

$$T_j: Y \rightarrow e_j Y \quad y \rightarrow T_j(y) := e_j T(y)$$

are linear operators for $j = 1, 2$ respectively.

Let $\mathfrak{M}$ be a σ -algebra on a non-empty set Ω and $\mu = \mu_1 e_1 + \mu_2 e_2$ be a $\mathbb{BC}$ -valued function defined on Ω . If μ_1 and μ_2 are both complex measures on $\mathfrak{M}$, then μ is called a $\mathbb{BC}$ -measure on $\mathfrak{M}$. Nevertheless, if the range of both μ_1, μ_2 are $[0, \infty]$, then μ is called

a $\mathbb{D}$ -measure on $\mathfrak{M}$. Also, if $\mu_1(\cdot), \mu_2(\cdot) \in (0, \infty)$, then μ is called a $\mathbb{D}^+$ -measure on $\mathfrak{M}$ [12].

Let $f = f_1 e_1 + f_2 e_2$ be a $\mathbb{BC}$ -valued measurable function. Then, $|f|_k = |f_1| e_1 + |f_2| e_2$ became a $\mathbb{D}$ -valued measurable. Also, the sum and multiplication of $\mathbb{BC}$ -valued measurable functions are also $\mathbb{BC}$ -measurable functions [12], [13].

Additional findings regarding $\mathbb{BC}$ -Lebesgue space, $\mathbb{D}$ -topology, include $\mathbb{D}$ -open, $\mathbb{D}$ -closed, $\mathbb{D}$ -closure, $\mathbb{D}$ -limit, $\mathbb{D}$ -continuity, $\mathbb{D}$ -Cauchy, and $\mathbb{D}$ -convergence, etc., can be found in the references in [3], [7], [8], [9], [12], [14] and [15].

Suppose that $\mathfrak{M}$ is a σ -algebra and $\mu = e_1 \mu_1 + e_2 \mu_2$ is a $\mathbb{BC}$ -measure on Ω . Any two bicomplex-valued $\mathbb{BC}$ -measurable functions $f = e_1 f_1 + e_2 f_2$ and $g = e_1 g_1 + e_2 g_2$ on Ω are called to be equal (μ -a.e.) if $f_1 = g_1$ (μ_1 -a.e.) and $f_2 = g_2$ (μ_2 -a.e.).

Let $(\Omega, \mathfrak{M}, \mu)$ be a σ -finite complete measure space with $\mu = \mu_1 e_1 + \mu_2 e_2$, $\mathfrak{F}(\Omega, \mathfrak{M})$ indicate the set of all $\mathfrak{M}$ -measurable functions on Ω and $u \in \mathfrak{F}(\Omega, \mathfrak{M})$. Let $E_M = \{x \in \Omega: |u(x)|_k > M\}$ for any $M \geq 0$. If the set A is defined as

$$\begin{aligned} A &= \{M > 0: \mu(E_M) = 0\} \\ &= \{M \in \mathbb{D}^+: |u(x)|_k \leq M \text{ } \mu - a. e. \} \end{aligned}$$

then essential $\mathbb{D}$ -supremum of u , denoted by $\text{essup}_{\mathbb{D}} u$ or $\|u\|_{\infty, \mathbb{D}}$ is defined by

$$\begin{aligned} \|u\|_{\infty, \mathbb{D}} &= \text{essup}_{\mathbb{D}} u = \inf_{\mathbb{D}}(A) \\ &= e_1 \|u_1\|_{\infty} + e_2 \|u_2\|_{\infty}. \end{aligned}$$

In recent years, wavelet transforms have been extensively studied as a powerful alternative to the Fourier transform, with applications ranging from data compression to the solution of various classes of differential equations. In this context, several researchers (see, for example [16], [17], [18] and [19]) have established connections between the smoothness of wavelets and the spectral properties of slant Toeplitz and Hankel operators.

Slant Toeplitz and slant Hankel operators are important generalizations of classical Toeplitz theory. These operators extend the study of operator behavior on function spaces. They also provide a richer framework for algebraic and functional analysis.

The algebraic structure of Toeplitz and Hankel operators was systematically studied in [20]. In their work, fundamental properties such as algebraic closure and multiplication behavior were established. These results form the basis of modern Toeplitz operator theory.

Slant Hankel operators are introduced in [21]. They investigated basic structural and boundedness properties of these operators. Their work extended classical Hankel operator theory into a more general setting. These operators extend the classical Hankel structure by incorporating a “slanted” index dependence in their matrix form. They investigated fundamental properties such as boundedness and compactness, and provided insights into the spectral behavior of these operators in [21].

These classical results describe Toeplitz, Hankel, and composition operators in complex Hilbert spaces. However, they do not cover bicomplex function spaces, which have in recent years emerged as a genuine arena for both *field theory* and *foundations of quantum mechanics*. On the foundations side, [22] and [23] formulated a generalized Schrödinger equation on a bicomplex Hilbert module; subsequent work addressed the bicomplex harmonic oscillator and finite-dimensional bicomplex Hilbert spaces [24], [25].

The theory of Hankel and Toeplitz operators has long been a central topic in functional analysis and operator theory, with deep connections to harmonic analysis, complex function theory, and applications in engineering and mathematical physics. Classical Hankel operators on the Hardy space $H^2(\mathbb{T})$ and on $L^2(\mathbb{T})$ have been extensively studied, particularly with respect to their boundedness, compactness, and spectral properties.

Recent studies have focused on extending classical operator theory to bicomplex Hilbert spaces, particularly spaces such as $L^2_{\mathbb{BC}}$ [24], [25]. In these settings, operators can often be analyzed via their idempotent decomposition, reducing complex problems to well-understood classical cases. This approach has proven effective in the study of Toeplitz-type operators and their generalizations.

However, despite the growing body of work on bicomplex operator theory, **slant Hankel operators have not yet been investigated in the bicomplex setting**. This gap motivates the present study, in which we introduce and analyze bicomplex slant Hankel operators after [26].

3 Slant $\mathbb{BC}$ -Hankel operators

Let Ω be a domain in $\mathbb{BC}$. Define $\Omega_1 = \{z_1 \mid e_1 z_1 + e_2 z_2 \in \Omega\}$ and $\Omega_2 = \{z_2 \mid e_1 z_1 + e_2 z_2 \in \Omega\}$. Then Ω_1 and Ω_2 are domains in $\mathbb{C}(i)$.

Definition 1. A function f is said to be a bicomplex holomorphic in a domain Ω of $\mathbb{BC}$ if it admits a bicomplex derivative at each point of the domain Ω [10].

Theorem 1. A bicomplex-valued function $f: \Omega \subseteq \mathbb{BC} \rightarrow \mathbb{BC}$ defined as

$$f(Z) = f(z + jw) = f(e_1 z_1 + e_2 z_2) = e_1 f_1(z_1) + e_2 f_2(z_2)$$

is a bicomplex holomorphic function on Ω if and only if f_1 and f_2 are complex holomorphic functions on Ω_1 and Ω_2 respectively [5].

Let

$$U_{\mathbb{BC}} = \{z + jw = e_1 z_1 + e_2 z_2 \mid (z_1, z_2) \in \mathbb{T}^2\}$$

be the unit disc in $\mathbb{BC}$, where $\mathbb{T}$ is the unit disk in the complex plane $\mathbb{C}(i)$ and $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. The bicomplex Hardy space $H^2_{\mathbb{BC}}(U_{\mathbb{BC}})$ is defined to be the space of all bicomplex holomorphic functions on the unit disc $U_{\mathbb{BC}}$ whose bicomplex sequence of power series coefficients is square-summable [1]. Let

$$\begin{aligned} \varphi &= \varphi_1 e_1 + \varphi_2 e_2 = \sum_{i=-\infty}^{\infty} a_i z_1^i e_1 + b_i z_2^i e_2 \\ &= e_1 \sum_{i=-\infty}^{\infty} a_i z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_i z_2^i \end{aligned}$$

be a $\mathbb{D}$ -bounded $\mathbb{BC}$ -measurable function on the unit circle $\mathbb{T}_{\mathbb{BC}}$, boundary of $U_{\mathbb{BC}}$. The notion of slant $\mathbb{BC}$ -Toeplitz operator $A_{\varphi}^{\mathbb{BC}}$ with symbol φ on the space $L^2_{\mathbb{BC}}$ can be defined as follows:

$$\begin{aligned} A_{\varphi}^{\mathbb{BC}}(Z^i) &= e_1 A_{\varphi_1}(z_1^i) + e_2 A_{\varphi_2}(z_2^i) \\ &= e_1 \sum_{j=-\infty}^{\infty} a_{2i-j} z_1^j + e_2 \sum_{j=-\infty}^{\infty} b_{2i-j} z_2^j \quad (3) \end{aligned}$$

for all $Z = e_1 z_1 + e_2 z_2 \in \mathbb{T}_{\mathbb{BC}}$ and for all j in the set of integers. Also, by using the techniques in [11], it is easy to see that if we write the $\mathbb{BC}$ -matrix of $A_{\varphi}^{\mathbb{BC}}$ with respect to the usual basis

$$\mathfrak{B}_{\mathbb{BC}} = \{Z^i = z_1^i e_1 + z_2^i e_2 : i \in \mathbb{Z}\}$$

of $L^2_{\mathbb{BC}}$, then its entries will be

$$\alpha_{ij} = (a_{2i-j})e_1 + (b_{2i-j})e_2.$$

Moreover if we define $W_{\mathbb{BC}}: L^2_{\mathbb{BC}} \rightarrow L^2_{\mathbb{BC}}$ with

$$W_{\mathbb{BC}} = e_1 W_1 + e_2 W_2$$

where W_1, W_2 act on usual L^2 as

$$W_l(z_l^{2n}) = z_l^n \quad \text{and} \quad W_l(z_l^{2n-1}) = 0$$

for $l = 1, 2$, then we get

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}}(Z^{2n}) &= W_{\mathbb{B}\mathbb{C}}((e_1 z_1 + e_2 z_2)^{2n}) \\ &= W_{\mathbb{B}\mathbb{C}}(e_1 z_1^{2n} + e_2 z_2^{2n}) \\ &= e_1 W_1(z_1^{2n}) + e_2 W_2(z_2^{2n}) \\ &= e_1 z_1^n + e_2 z_2^n = Z^n \end{aligned} \quad (4)$$

and

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}}(Z^{2n-1}) &= W_{\mathbb{B}\mathbb{C}}((e_1 z_1 + e_2 z_2)^{2n-1}) \\ &= W_{\mathbb{B}\mathbb{C}}(e_1 z_1^{2n-1} + e_2 z_2^{2n-1}) \\ &= e_1 W_1(z_1^{2n-1}) + e_2 W_2(z_2^{2n-1}) \\ &= e_1 0 + e_2 0 = 0 \end{aligned} \quad (5)$$

for each $n \in \mathbb{Z}$ by [27], Proposition 1].

Proposition 1. Let $\varphi = e_1 \varphi_1 + e_2 \varphi_2$ be in $L_{\mathbb{B}\mathbb{C}}^\infty(\mathbb{T}_{\mathbb{B}\mathbb{C}})$ and $A_\varphi^{\mathbb{B}\mathbb{C}}$ be a slant $\mathbb{B}\mathbb{C}$ -Toeplitz operator on $L_{\mathbb{B}\mathbb{C}}^2(\mathbb{T}_{\mathbb{B}\mathbb{C}})$ as in (3). Then $A_\varphi^{\mathbb{B}\mathbb{C}} = W_{\mathbb{B}\mathbb{C}} M_\varphi$ where $M_\varphi = e_1 M_{\varphi_1} + e_2 M_{\varphi_2}$ is the multiplication operator on $L_{\mathbb{B}\mathbb{C}}^2$ induced by $\varphi = \varphi_1 e_1 + \varphi_2 e_2$.

Proof. By using the definition of $A_\varphi^{\mathbb{B}\mathbb{C}}$, we can write

$$\begin{aligned} A_\varphi^{\mathbb{B}\mathbb{C}}(Z^k) &= A_\varphi^{\mathbb{B}\mathbb{C}}(e_1 z_1^k + e_2 z_2^k) \\ &= e_1 \sum_{i=-\infty}^{\infty} a_{2i-k} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{2i-k} z_2^i \end{aligned}$$

for all $e_1 z_1 + e_2 z_2 = Z \in \mathbb{T}_{\mathbb{B}\mathbb{C}}$. By using the following equality

$$\begin{aligned} M_\varphi(Z^k) &= M_\varphi(e_1 z_1^k + e_2 z_2^k) \\ &= e_1 M_{\varphi_1}(z_1^k) + e_2 M_{\varphi_2}(z_2^k) \\ &= e_1 \sum_{i=-\infty}^{\infty} a_{i-k} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{i-k} z_2^i, \end{aligned}$$

we get

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}} M_\varphi(Z^k) &= W_{\mathbb{B}\mathbb{C}} M_\varphi(e_1 z_1^k + e_2 z_2^k) = \\ &= W_{\mathbb{B}\mathbb{C}} \left(e_1 \sum_{i=-\infty}^{\infty} a_{i-k} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{i-k} z_2^i \right) \\ &= (e_1 W_1 + e_2 W_2) \left(e_1 \sum_{i=-\infty}^{\infty} a_{i-k} z_1^i \right. \\ &\quad \left. + e_2 \sum_{i=-\infty}^{\infty} b_{i-k} z_2^i \right) \\ &= e_1 W_1 \left(\sum_{i=-\infty}^{\infty} a_{i-k} z_1^i \right) + e_2 W_2 \left(\sum_{i=-\infty}^{\infty} b_{i-k} z_2^i \right) \\ &= e_1 \sum_{i=-\infty}^{\infty} a_{2i-k} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{2i-k} z_2^i \\ &= A_\varphi^{\mathbb{B}\mathbb{C}}(Z^k) \end{aligned}$$

for each $k \in \mathbb{Z}$.

The Hankel operators H_φ are usually defined on the space H^2 but they can be extended to the space $L_{\mathbb{B}\mathbb{C}}^2$ as follows.

Definition 2. The $\mathbb{B}\mathbb{C}$ -Hankel operator $S_{\mathbb{B}\mathbb{C}}^\varphi$ on $L_{\mathbb{B}\mathbb{C}}^2$ is defined as

$$S_{\mathbb{B}\mathbb{C}}^\varphi(Z^j) = e_1 \sum_{i=-\infty}^{\infty} a_{-i-j} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{-i-j} z_2^i$$

for all j in $\mathbb{Z}$.

Moreover, let $f = e_1 f_1 + e_2 f_2$ and J_1, J_2 be reflection operators on usual L^2 space. If $J_{\mathbb{B}\mathbb{C}}: L_{\mathbb{B}\mathbb{C}}^2 \rightarrow L_{\mathbb{B}\mathbb{C}}^2$ is the reflection operator defined by

$$\begin{aligned} J_{\mathbb{B}\mathbb{C}}(f(Z)) &= f(Z^{\dagger_3}) \\ &= e_1 J_1(f_1(\bar{z}_1)) + e_2 J_2(f_2(\bar{z}_2)) \end{aligned}$$

for all $Z = z_1 e_1 + z_2 e_2$, then it is easy to see that

$$S_{\mathbb{B}\mathbb{C}}^\varphi = J_{\mathbb{B}\mathbb{C}} M_\varphi \quad \text{and} \quad M_\varphi = J_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi. \quad (6)$$

Now, we introduce the concept of a slant $\mathbb{B}\mathbb{C}$ -Hankel operator on the space $L_{\mathbb{B}\mathbb{C}}^2$ as follows.

Definition 3. The slant Hankel operator $K_\varphi^{\mathbb{B}\mathbb{C}}$ on $L_{\mathbb{B}\mathbb{C}}^2$ is defined as

$$K_\varphi^{\mathbb{B}\mathbb{C}}(Z^j) = e_1 \sum_{i=-\infty}^{\infty} a_{-2i-j} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{-2i-j} z_2^i$$

for all $Z = e_1 z_1 + e_2 z_2$ and for all j in $\mathbb{Z}$.

In other words, if

$$e_1(\alpha_{ij}^1) + e_2(\alpha_{ij}^2)$$

is the matrix of $K_\varphi^{\mathbb{B}\mathbb{C}}$ with respect to the usual basis $\mathfrak{B}_{\mathbb{B}\mathbb{C}} = \{Z^i = z_1^i e_1 + z_2^i e_2 : i \in \mathbb{Z}\}$ of $L_{\mathbb{B}\mathbb{C}}^2$ then $\alpha_{ij}^1 = a_{-2i-j}$ and $\alpha_{ij}^2 = b_{-2i-j}$. Therefore if $A_\varphi^{\mathbb{B}\mathbb{C}}$ is the slant $\mathbb{B}\mathbb{C}$ -Toeplitz operator then we can easily see that $A_\varphi^{\mathbb{B}\mathbb{C}} = J_{\mathbb{B}\mathbb{C}} K_\varphi^{\mathbb{B}\mathbb{C}}$ and $K_\varphi^{\mathbb{B}\mathbb{C}} = J_{\mathbb{B}\mathbb{C}} A_\varphi^{\mathbb{B}\mathbb{C}}$. Moreover, we also observe that $J_{\mathbb{B}\mathbb{C}}$ reduces $W_{\mathbb{B}\mathbb{C}}$ as

$$\begin{aligned} J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}(Z^{2n}) &= J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}((z_1 e_1 + z_2 e_2)^{2n}) \\ &= J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}(z_1^{2n} e_1 + z_2^{2n} e_2) \\ &= J_{\mathbb{B}\mathbb{C}} (e_1 W_1(z_1^{2n}) + e_2 W_2(z_2^{2n})) \\ &= J_{\mathbb{B}\mathbb{C}} (e_1 z_1^n e_1 + e_2 z_2^n e_2) \\ &= (J_1 e_1 + J_2 e_2)(z_1^n e_1 + z_2^n e_2) \\ &= e_1(\bar{z}_1^n) + e_2(\bar{z}_2^n) = (Z^{\dagger_3})^n, \end{aligned}$$

$$\begin{aligned} J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}(Z^{2n-1}) &= J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}((z_1 e_1 + z_2 e_2)^{2n-1}) \\ &= J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}(z_1^{2n-1} e_1 + z_2^{2n-1} e_2) \\ &= J_{\mathbb{B}\mathbb{C}} (e_1 W_1(z_1^{2n-1}) \\ &\quad + e_2 W_2(z_2^{2n-1})) \\ &= J_{\mathbb{B}\mathbb{C}} (0e_1 + 0e_2) \\ &= (J_1 e_1 + J_2 e_2)(0e_1 + 0e_2) \end{aligned}$$

$$= e_1 0 + e_2 0 = 0$$

and

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}(Z^{2n}) &= W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}((z_1 e_1 + z_2 e_2)^{2n}) \\ &= W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}(z_1^{2n} e_1 + z_2^{2n} e_2) \\ &= W_{\mathbb{B}\mathbb{C}}(J_1 e_1 + J_2 e_2)(z_1^{2n} e_1 + z_2^{2n} e_2) \\ &= W_{\mathbb{B}\mathbb{C}}(J_1(z_1^{2n} e_1) + J_2(z_2^{2n} e_2)) \\ &= W_{\mathbb{B}\mathbb{C}}(\bar{z}_1^{2n} e_1 + \bar{z}_2^{2n} e_2) \\ &= e_1 W_1(\bar{z}_1^{2n}) + e_2 W_2(\bar{z}_2^{2n}) \\ &= e_1 \bar{z}_1^n + e_2 \bar{z}_2^n = (Z^{\dagger 3})^n, \end{aligned}$$

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}(Z^{2n-1}) &= W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}((z_1 e_1 + z_2 e_2)^{2n-1}) \\ &= W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}}(z_1^{2n-1} e_1 + z_2^{2n-1} e_2) \\ &= W_{\mathbb{B}\mathbb{C}}(J_1 e_1 + J_2 e_2)(z_1^{2n-1} e_1 \\ &\quad + z_2^{2n-1} e_2) \\ &= W_{\mathbb{B}\mathbb{C}}(J_1(z_1^{2n-1} e_1) \\ &\quad + J_2(z_2^{2n-1} e_2)) \\ &= W_{\mathbb{B}\mathbb{C}}(\bar{z}_1^{2n-1} e_1 + \bar{z}_2^{2n-1} e_2) \\ &= e_1 W_1(\bar{z}_1^{2n-1}) + e_2 W_2(\bar{z}_2^{2n-1}) \\ &= e_1 0 + e_2 0 = 0. \end{aligned}$$

In addition to these

$$\begin{aligned} J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}^*(Z^n) &= J_{\mathbb{B}\mathbb{C}}(Z^{2n}) = (Z^{\dagger 3})^{2n} \\ &= J_{\mathbb{B}\mathbb{C}}(Z^{2n}) = W_{\mathbb{B}\mathbb{C}}^* J_{\mathbb{B}\mathbb{C}}(Z^n). \end{aligned}$$

where $W_{\mathbb{B}\mathbb{C}}^*$ is the adjoint operator of $W_{\mathbb{B}\mathbb{C}}$. Hence

$$J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}} = W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}} \text{ and } J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}}^* = W_{\mathbb{B}\mathbb{C}}^* J_{\mathbb{B}\mathbb{C}}.$$

Theorem 2. Let $\varphi = \varphi_1 e_1 + \varphi_2 e_2$ be in $L_{\mathbb{B}\mathbb{C}}^\infty(\mathbb{T}_{\mathbb{B}\mathbb{C}})$. Then $K_\varphi^{\mathbb{B}\mathbb{C}} = W_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi$.

Proof. If $S_{\mathbb{B}\mathbb{C}}^\varphi$ is the $\mathbb{B}\mathbb{C}$ -Hankel operator on $L_{\mathbb{B}\mathbb{C}}^2$ then

$$\begin{aligned} S_{\mathbb{B}\mathbb{C}}^\varphi(Z^j) &= \sum_{i=-\infty}^{\infty} (a_{-i-j} z_1^i e_1 + b_{-i-j} z_2^i e_2) \\ &= e_1 \sum_{i=-\infty}^{\infty} a_{-i-j} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{-i-j} z_2^i. \end{aligned}$$

Therefore,

$$\begin{aligned} W_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi(Z^j) &= e_1 W_1 \left(\sum_{i=-\infty}^{\infty} a_{-i-j} z_1^i \right) \\ &\quad + e_2 W_2 \left(\sum_{i=-\infty}^{\infty} b_{-i-j} z_2^i \right) \\ &= e_1 \sum_{i=-\infty}^{\infty} a_{-2i-j} z_1^i + e_2 \sum_{i=-\infty}^{\infty} b_{-2i-j} z_2^i \\ &= \sum_{i=-\infty}^{\infty} a_{-2i-j} z_1^i e_1 + b_{-2i-j} z_2^i e_2 \\ &= K_\varphi^{\mathbb{B}\mathbb{C}}(Z^j) \end{aligned}$$

can be written for all $j \in \mathbb{Z}$. As a result we can say

$$K_\varphi^{\mathbb{B}\mathbb{C}} = W_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi = W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}} M_\varphi$$

$$= J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}} M_\varphi = J_{\mathbb{B}\mathbb{C}} A_\varphi^{\mathbb{B}\mathbb{C}}$$

by (6) and Proposition 1.

As a consequence of the above theorem, we obtain the following result.

Corollary 1. Let φ be an element of $L_{\mathbb{B}\mathbb{C}}^\infty$ with $\varphi = \varphi_1 e_1 + \varphi_2 e_2$. Then the slant $\mathbb{B}\mathbb{C}$ -Hankel operator $K_\varphi^{\mathbb{B}\mathbb{C}}$ is a $\mathbb{D}$ -bounded linear operator on $L_{\mathbb{B}\mathbb{C}}^2$ and it satisfies

$$\|K_\varphi^{\mathbb{B}\mathbb{C}}\| \leq \|\varphi\|_{\infty, \mathbb{D}}.$$

Proof. The following completes the proof:

$$\begin{aligned} \|K_\varphi^{\mathbb{B}\mathbb{C}}\| &= \|W_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi\| = \|W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}} M_\varphi\| \\ &\leq \|W_{\mathbb{B}\mathbb{C}}\| \|J_{\mathbb{B}\mathbb{C}}\| \|M_\varphi\| \\ &\leq \|M_\varphi\| = \|\varphi\|_{\infty, \mathbb{D}}. \end{aligned}$$

If we denote $L_\varphi^{\mathbb{B}\mathbb{C}}$, the compression of $K_\varphi^{\mathbb{B}\mathbb{C}}$ on the space $H_{\mathbb{B}\mathbb{C}}^2$, then $L_\varphi^{\mathbb{B}\mathbb{C}}$ can be defined as

$$\begin{aligned} L_\varphi^{\mathbb{B}\mathbb{C}}(f) &= L_\varphi^{\mathbb{B}\mathbb{C}}(e_1 f_1 + e_2 f_2) \\ &= e_1 P_1 K_{\varphi_1}(f_1) + e_2 P_2 K_{\varphi_2}(f_2) \\ &= P K_\varphi^{\mathbb{B}\mathbb{C}}(f) \end{aligned}$$

for all f in $H_{\mathbb{B}\mathbb{C}}^2$, where $P = e_1 P_1 + e_2 P_2$ is the orthogonal projection of $L_{\mathbb{B}\mathbb{C}}^2$ onto $H_{\mathbb{B}\mathbb{C}}^2$ and $K_\varphi^{\mathbb{B}\mathbb{C}} = e_1 K_{\varphi_1} + e_2 K_{\varphi_2}$. In other words, we can write the following:

$$\begin{aligned} L_\varphi^{\mathbb{B}\mathbb{C}} &= P K_\varphi^{\mathbb{B}\mathbb{C}}|_{H_{\mathbb{B}\mathbb{C}}^2} = P J_{\mathbb{B}\mathbb{C}} A_\varphi^{\mathbb{B}\mathbb{C}}|_{H_{\mathbb{B}\mathbb{C}}^2} \\ &= P J_{\mathbb{B}\mathbb{C}} W_{\mathbb{B}\mathbb{C}} M_\varphi|_{H_{\mathbb{B}\mathbb{C}}^2} \\ &= P W_{\mathbb{B}\mathbb{C}} J_{\mathbb{B}\mathbb{C}} M_\varphi|_{H_{\mathbb{B}\mathbb{C}}^2} \\ &= P W_{\mathbb{B}\mathbb{C}} S_{\mathbb{B}\mathbb{C}}^\varphi|_{H_{\mathbb{B}\mathbb{C}}^2} = W_{\mathbb{B}\mathbb{C}} P S_{\mathbb{B}\mathbb{C}}^\varphi|_{H_{\mathbb{B}\mathbb{C}}^2} \\ &= W_{\mathbb{B}\mathbb{C}} P S_{\mathbb{B}\mathbb{C}}^\varphi|_{H_{\mathbb{B}\mathbb{C}}^2} = W_{\mathbb{B}\mathbb{C}} H_{\mathbb{B}\mathbb{C}}^\varphi \end{aligned}$$

where $H_{\mathbb{B}\mathbb{C}}^\varphi$ is the $\mathbb{B}\mathbb{C}$ -Hankel operator on $H_{\mathbb{B}\mathbb{C}}^2$. Thus $L_\varphi^{\mathbb{B}\mathbb{C}} = W_{\mathbb{B}\mathbb{C}} H_{\mathbb{B}\mathbb{C}}^\varphi$. If $e_1(\alpha_{ij}^1) + e_2(\alpha_{ij}^2)$ is the matrix of $K_\varphi^{\mathbb{B}\mathbb{C}}$ with respect to the usual basis $\mathfrak{B}_{\mathbb{B}\mathbb{C}} = \{Z^i = z_1^i e_1 + z_2^i e_2 : i \in \mathbb{Z}\}$ of $L_{\mathbb{B}\mathbb{C}}^2$, then this matrix is given by

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & c_7 & c_6 & c_5 & c_4 & \cdots \\ \cdots & c_5 & c_4 & c_3 & c_2 & \cdots \\ \cdots & c_3 & c_2 & c_1 & c_0 & \cdots \\ \cdots & c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $c_k = e_1 a_k + e_2 b_k$. The lower right quarter of the matrix is the matrix of $L_\varphi^{\mathbb{B}\mathbb{C}}$. That is

$$e_1 \begin{pmatrix} a_0 & a_{-1} & \cdots \\ a_{-2} & a_{-3} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} + e_2 \begin{pmatrix} b_0 & b_{-1} & \cdots \\ b_{-2} & b_{-3} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Now we will give the necessary and sufficient conditions for an operator on $L^2_{\mathbb{BC}}$ to become a slant $\mathbb{BC}$ -Hankel operator.

Theorem 3. A bounded linear operator K on $L^2_{\mathbb{BC}}$ is a slant $\mathbb{BC}$ -Hankel operator if and only if

$$M_{Z^{\dagger 3}} K = K M_{Z^2}$$

for any $Z = e_1 z_1 + e_2 z_2$.

Proof. Let K be a slant $\mathbb{BC}$ -Hankel operator on $L^2_{\mathbb{BC}}$. Then by definition of $\mathbb{BC}$ -Hankel operator, the operator $K = W_{\mathbb{BC}} S_{\mathbb{BC}}^{\varphi}$ can be written for some φ in $L^{\infty}_{\mathbb{BC}}$. Then, for any $Z = e_1 z_1 + e_2 z_2$, since $M_{Z^{\dagger 3}} W_{\mathbb{BC}} = W_{\mathbb{BC}} M_{(Z^{\dagger 3})^2}$ by (4) and (5), we get

$$\begin{aligned} M_{Z^{\dagger 3}} K &= M_{Z^{\dagger 3}} W_{\mathbb{BC}} S_{\mathbb{BC}}^{\varphi} = W_{\mathbb{BC}} M_{(Z^{\dagger 3})^2} S_{\mathbb{BC}}^{\varphi} \\ &= W_{\mathbb{BC}} M_{(Z^{\dagger 3})^2} J_{\mathbb{BC}} M_{\varphi} \\ &= W_{\mathbb{BC}} J_{\mathbb{BC}} M_{Z^2} M_{\varphi} \\ &= W_{\mathbb{BC}} J_{\mathbb{BC}} M_{\varphi} M_{Z^2} \\ &= W_{\mathbb{BC}} S_{\mathbb{BC}}^{\varphi} M_{Z^2} = K M_{Z^2}. \end{aligned}$$

Conversely, assume that $K = e_1 K_1 + e_2 K_2$ satisfies

$$M_{Z^{\dagger 3}} K = K M_{Z^2}. \quad (7)$$

Then $K_1 M'_{z_1^2} = M'_{\bar{z}_1} K_1$ and $K_2 M''_{z_2^2} = M''_{\bar{z}_2} K_2$ can be written where $M = e_1 M_1 + e_2 M_2$. Now let f be in $L^2_{\mathbb{BC}}$ with its Fourier expansion $e_1 \sum_{i=-\infty}^{\infty} b_i z_1^i + e_2 \sum_{i=-\infty}^{\infty} d_i z_2^i$. By (7) we get

$$\begin{aligned} K(f((Z^{\dagger 3})^2)) &= K\left(e_1 \sum_{i=-\infty}^{\infty} b_i \bar{z}_1^{2i} + e_2 \sum_{i=-\infty}^{\infty} d_i \bar{z}_2^{2i}\right) \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i K_1 M'_{\bar{z}_1^{2i}}(1) + e_2 \sum_{i=-\infty}^{\infty} d_i K_2 M''_{\bar{z}_2^{2i}}(1) \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i M'_{z_1^i} K_1(1) + e_2 \sum_{i=-\infty}^{\infty} d_i M''_{z_2^i} K_2(1) \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i z_1^i K_1(1) + e_2 \sum_{i=-\infty}^{\infty} d_i z_2^i K_2(1) \\ &= (e_1 K_1(1) + e_2 K_2(1)) \left(e_1 \sum_{i=-\infty}^{\infty} b_i z_1^i + e_2 \sum_{i=-\infty}^{\infty} d_i z_2^i \right) \\ &= K(1)f(Z). \end{aligned}$$

This implies that

$$\begin{aligned} \|f(Z)K(1)\|_{\mathbb{BC}} &= \|K(f(\bar{Z}^2))\|_{\mathbb{BC}} \\ &\leq \|K\| \|f(\bar{Z}^2)\|_{\mathbb{BC}} \\ &= \|K\| \|f(Z)\|_{\mathbb{BC}} \end{aligned}$$

where

$$\|f(Z)\|_{\mathbb{BC}} = e_1 \|f_1(z_1)\|_2 + e_2 \|f_2(z_2)\|_2.$$

Now let $\varphi_0 = K(1)$ and $e_1 \varepsilon_1 + e_2 \varepsilon_2 = \varepsilon > 0$ be any positive hyperbolic number. Let $A_{\varepsilon} = \{Z: |\varphi_0(Z)|_k > \|K\| + \varepsilon\}$ and $\chi_{A_{\varepsilon}}(Z)$ denote the characteristic function of A_{ε} . Then

$$\begin{aligned} \|K(\chi_{A_{\varepsilon}})\|_{\mathbb{BC}}^2 &= \int_{\mathbb{T}_{\mathbb{BC}}} |K(\chi_{A_{\varepsilon}}(Z))|_k^2 d\mu \\ &= \int_{A_{\varepsilon}} |K(1)|_k^2 d\mu = \int_{A_{\varepsilon}} |\varphi_0|_k^2 d\mu \\ &\geq (\|K\| + \varepsilon)^2 \mu(A_{\varepsilon}) \\ &= (\|K\| + \varepsilon)^2 \|\chi_{A_{\varepsilon}}\|_{\mathbb{BC}}^2 \end{aligned} \quad (8)$$

According to (8), if $\|\chi_{A_{\varepsilon}}\|_{\mathbb{BC}} \neq 0$, then we obtain $\|K\| + \varepsilon \leq \|K\|$, which is a contradiction. Thus $\|\chi_{A_{\varepsilon}}\|_{\mathbb{BC}} = 0$ and $\mu(A_{\varepsilon}) = 0$, where $\mu = e_1 \mu_1 + e_2 \mu_2$ is the $\mathbb{D}$ -valued $\mathbb{BC}$ -Lebesgue measure on $\mathbb{T}_{\mathbb{BC}}$. Since we get this for any $\varepsilon > 0$, $\mu(A) = 0$ if $A = \{Z: |\varphi_0|_k \geq \|K\|\}$. Therefore $|\varphi_0(Z)|_k \leq \|K\|$ (μ -a.e.). This implies that φ_0 is in $L^{\infty}_{\mathbb{BC}}$.

Since

$$\begin{aligned} K(Z^{\dagger 3} f((Z^{\dagger 3})^2)) &= \\ &= K\left(e_1 \sum_{i=-\infty}^{\infty} b_i \bar{z}_1^{2i+1} + e_2 \sum_{i=-\infty}^{\infty} d_i \bar{z}_2^{2i+1}\right) \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i K_1 M'_{\bar{z}_1^{2i}} M'_{\bar{z}_1} \\ &\quad + e_2 \sum_{i=-\infty}^{\infty} d_i K_2 M''_{\bar{z}_2^{2i}} M''_{\bar{z}_2} \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i M'_{z_1^i} K_1 M'_{\bar{z}_1} \\ &\quad + e_2 \sum_{i=-\infty}^{\infty} d_i M''_{z_2^i} K_2 M''_{\bar{z}_2} \\ &= e_1 \sum_{i=-\infty}^{\infty} b_i z_1^i K_1 \bar{z}_1 + e_2 \sum_{i=-\infty}^{\infty} d_i z_2^i K_2 \bar{z}_2 \\ &= f(Z)K(Z^{\dagger 3}), \end{aligned}$$

we can see that $K(Z^{\dagger 3})$ is also $\mathbb{D}$ -bounded. Let $\varphi_1 = K(Z^{\dagger 3})$ and $\varphi(Z) = \varphi_0((Z^{\dagger 3})^2) + Z\varphi_1((Z^{\dagger 3})^2)$. Since φ_0 and φ_1 are $\mathbb{D}$ -bounded, φ is also $\mathbb{D}$ -bounded and in $L^{\infty}_{\mathbb{BC}}$. Now we aim to show that $K =$

$W_{\mathbb{BC}} S_{\mathbb{BC}}^{\varphi}$. Let $f = e_1 f_1 + e_2 f_2$ be an element of $L_{\mathbb{BC}}^{\infty}$. Then f can be written as

$$\begin{aligned} f(Z) &= f_0 \left((Z^{\dagger_3})^2 \right) + (Z^{\dagger_3}) f_1 \left((Z^{\dagger_3})^2 \right) \\ &= e_1 f_{0,1}(\bar{z}_1^2) + e_2 f_{0,2}(\bar{z}_2^2) + e_1 \bar{z}_1 f_{1,1}(\bar{z}_1^2) \\ &\quad + e_2 \bar{z}_2 f_{1,2}(\bar{z}_2^2) \\ &= e_1 \left(f_{0,1}(\bar{z}_1^2) + \bar{z}_1 f_{1,1}(\bar{z}_1^2) \right) \\ &\quad + e_2 \left(f_{0,2}(\bar{z}_2^2) + \bar{z}_2 f_{1,2}(\bar{z}_2^2) \right). \end{aligned}$$

From here, we get

$$\begin{aligned} W_{\mathbb{BC}} S_{\mathbb{BC}}^{\varphi} f &= W_{\mathbb{BC}} J_{\mathbb{BC}} M_{\varphi} f = W_{\mathbb{BC}} J_{\mathbb{BC}} (\varphi f) = \\ &= W_{\mathbb{BC}} \left(\varphi(Z^{\dagger_3}) f(Z^{\dagger_3}) \right) \\ &= W_{\mathbb{BC}} \left(\left(\varphi_0(Z^2) + Z^{\dagger_3} \varphi_1(Z^2) \right) (f_0(Z^2) \right. \\ &\quad \left. + Z f_1(Z^2)) \right) \\ &= W_{\mathbb{BC}} (\varphi_0(Z^2) f_0(Z^2) + \varphi_1(Z^2) f_1(Z^2)) \\ &= e_1 W_1 \left(\varphi_{0,1}(z_1^2) f_{0,1}(z_1^2) + \varphi_{1,1}(z_1^2) f_{1,1}(z_1^2) \right) \\ &\quad + e_2 W_2 \left(\varphi_{0,2}(z_2^2) f_{0,2}(z_2^2) \right. \\ &\quad \left. + \varphi_{1,2}(z_2^2) f_{1,2}(z_2^2) \right) \\ &= e_1 \left(\varphi_{0,1}(z_1) f_{0,1}(z_1) + \varphi_{1,1}(z_1) f_{1,1}(z_1) \right) \\ &\quad + e_2 \left(\varphi_{0,2}(z_2) f_{0,2}(z_2) \right. \\ &\quad \left. + \varphi_{1,2}(z_2) f_{1,2}(z_2) \right) \\ &= \varphi_0(Z) f_0(Z) + \varphi_1(Z) f_1(Z) \\ &= K f_0(Z) + f_1(Z) K(Z^{\dagger_3}) \\ &= K \left(f_0 \left((Z^{\dagger_3})^2 \right) \right) + K \left(Z^{\dagger_3} f_1 \left((Z^{\dagger_3})^2 \right) \right) \\ &= K \left(f_0 \left((Z^{\dagger_3})^2 \right) + Z^{\dagger_3} f_1 \left((Z^{\dagger_3})^2 \right) \right) \\ &= K(f) \end{aligned}$$

Hence K is a slant $\mathbb{BC}$ -Hankel operator.

Corollary 2. The set of all slant $\mathbb{BC}$ -Hankel operators is weakly $\mathbb{D}$ -closed and hence strongly $\mathbb{D}$ -closed.

Proof. Suppose that I is a directed set and K_{α} is a net of slant $\mathbb{BC}$ -Hankel operators for each $\alpha \in I$ and $K_{\alpha} \rightarrow K$ weakly. This means $\langle K_{\alpha} f, g \rangle_{\mathbb{BC}} \rightarrow \langle K f, g \rangle_{\mathbb{BC}}$ for all f, g in $L_{\mathbb{BC}}^2$. Therefore

$$\begin{aligned} \langle M_Z K_{\alpha} M_{Z^2} f, g \rangle_{\mathbb{BC}} &= \langle K_{\alpha} Z^2 f, Z^{\dagger_3} g \rangle_{\mathbb{BC}}, \\ \langle K_{\alpha} Z^2 f, Z^{\dagger_3} g \rangle_{\mathbb{BC}} &\rightarrow \langle K Z^2 f, Z^{\dagger_3} g \rangle_{\mathbb{BC}} \end{aligned}$$

and

$$\langle K Z^2 f, Z^{\dagger_3} g \rangle_{\mathbb{BC}} = \langle M_Z K M_{Z^2} f, g \rangle_{\mathbb{BC}}$$

can be written. Since K_{φ} is a slant $\mathbb{BC}$ -Hankel operator, we can get $M_Z K_{\alpha} M_{Z^2} = K_{\alpha}$ for each α . Thus $K = M_Z K M_{Z^2}$ and so K is slant $\mathbb{BC}$ -Hankel operator. This completes the proof.

4 Conclusion

In this study, we introduced bicomplex slant Hankel operators on the space $L_{\mathbb{BC}}^2(U_{\mathbb{BC}})$ as a natural extension of the classical slant Hankel operators, by employing the idempotent decomposition of bicomplex numbers, we showed that many operator-theoretic properties can be effectively reduced to corresponding results in the complex setting.

The boundedness results obtained in this work demonstrate that bicomplex slant Hankel operators behave consistently with their classical counterparts when viewed component-wise. An important implication of our results is that the bicomplex framework offers a unifying perspective for studying operator classes defined via non-standard index structures, such as slant operators. At the same time, the presence of zero divisors and the richer algebraic structure of bicomplex numbers suggest that new phenomena may arise beyond what is observed in the classical complex case.

Future work may focus on investigating spectrum and Fredholm properties, essential spectra, and invariant subspace structures of bicomplex slant Hankel operators.

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