

Numerical Simulation of Fractional Duffing Problem Using the Ujlayan-Dixit Operator and Adomian Decomposition Method

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Abstract: This paper proposes a generalized computational framework for solving the fractional Duffing equation based on the Adomian Decomposition Method combined with the Ujlayan–Dixit operator. A novel generalized formulation for repeated fractional differentiation is derived, leading to an efficient iterative solution scheme. The proposed approach is applied to different cases of the Duffing system, and its accuracy and convergence behavior are systematically evaluated through comparison with the Runge–Kutta fourth-order method. Furthermore, the influence of the fractional order on the system dynamics is analyzed. The results demonstrate that the proposed method provides accurate, stable, and computationally efficient solutions, highlighting its potential for solving nonlinear fractional differential equations.

Key- Words: Fractional calculus, Fractional Duffing Equation, Adomian Decomposition Method (ADM), Ujlayan-Dixit operator, Nonlinear Differential Equation, Numerical Simulation.

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1 Introduction

Fractional calculus has emerged as a powerful tool for modeling complex systems characterized by memory and hereditary effects, where classical integer-order models often fail to capture realistic system behavior. Such characteristics appear in various applications including viscoelastic materials, fluid dynamics, and biological systems, [1], [2], [3], [4].

Among nonlinear fractional models, the fractional Duffing equation represents a challenging problem due to the combined effects of nonlinearity and fractional-order dynamics. Analytical and numerical methods developed for these equations often suffer from difficulties related to convergence, stability, and the treatment of initial conditions, [5], [6], [7], [8].

The ADM has proven to be an efficient semi-analytical technique for solving nonlinear

differential equations without requiring linearization or discretization. However, its performance in fractional settings depends strongly on the choice of the fractional operator. In this context, the UD operator has been recently introduced as a flexible and computationally efficient alternative to classical definitions, offering a simpler structure and improved handling of initial conditions, [9], [10].

In this paper, we propose a generalized computational framework that combines ADM with the UD operator for solving fractional Duffing systems. A generalized formulation for repeated fractional differentiation is derived, leading to a systematic iterative solution procedure. The proposed method is applied to representative cases of the Duffing equation, and its accuracy and convergence behavior are assessed through numerical comparisons, [9], [10],

[11], [12], [13]. The main contributions of this work include the development of a generalized formulation based on the UD operator, the construction of an efficient ADM-based solution scheme, and the numerical validation of the proposed approach in solving nonlinear fractional systems. It is worth noting that, unlike non-local operators such as Riemann-Liouville or Caputo, the UD operator is constructed as a local convex combination of a function and its first derivative; its adoption in this work is therefore motivated by computational simplicity and analytical tractability rather than an attempt to model non-local memory effects.

2 Preliminaries

In this section, we review important definitions and key properties of fractional calculus that underpin the main results. Throughout this manuscript, the notation D^α refers specifically to the Ujlayan-Dixit operator, and should not be conflated with standard non-local fractional derivative definitions such as Riemann-Liouville or Caputo.

Definition 1. [10]. For a given function $f: [0, \infty) \rightarrow \mathbb{R}$ and $\alpha \in [0, 1]$, the UD fractional derivative of order α is defined as

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{e^{\epsilon(1-\alpha)} f(xe^{\frac{\epsilon}{x}}) - f(x)}{\epsilon}, \quad (1)$$

if this limit exists, then $D^\alpha f(x)$ is called the UD derivative of f for $\alpha \in [0, 1]$, with the understanding that

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}.$$

Also, if f is UD differentiable on the interval $(0, x)$, for $x > 0$, and $\alpha \in [0, 1]$ such that $\lim_{x \rightarrow 0^+} D^\alpha f(x)$ exists, then we have

$$f^\alpha(0) = \lim_{x \rightarrow 0^+} f^\alpha(x).$$

Theorem 1. [10], [14]. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function and $\alpha \in [0, 1]$. Then, f is UD differentiable, and the derivative of UD is given by

$$D^\alpha f(x) = (1 - \alpha)f(x) + \alpha f'(x). \quad (2)$$

Furthermore, for specific values of α , we obtain

- If $\alpha = 0$, then $D^0 f(x) = f(x)$.
- If $\alpha = 1$, then $D^1 f(x) = f'(x)$.

Remark 1. [10]. The UD derivatives of order α , $\alpha \in [0, 1]$, of some elementary real-valued UD differentiable functions in $[0, \infty)$, can be given as follows:

- (i) $D^\alpha(\lambda) = (1 - \alpha)\lambda$, for any constant $\lambda \in \mathbb{R}$,
- (ii) $D^\alpha((ax+b)^n) = (1 - \alpha)(ax+b)^n + \alpha n(ax+b)^{n-1}$, for all $a, b \in \mathbb{R}$,
- (iii) $D^\alpha(e^{ax+b}) = ((1 - \alpha) + \alpha a)e^{ax+b}$, for all $a, b \in \mathbb{R}$.

Definition 2. [10]. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be an n times differentiable function. Then, the UD derivative of order α , $\alpha \in (n, n+1]$, is defined as

$$D^\alpha f(x) = \lim_{\epsilon \rightarrow 0} \frac{e^{\epsilon(1-\alpha)} f^{[\alpha]-1}(xe^{\frac{\epsilon}{x}}) - f^{[\alpha]-1}(x)}{\epsilon},$$

where $[\alpha]$ represents the smallest integer greater than or equal to α .

Definition 3 (UD Integral). [10]. Let f be a continuous function on $[a, b]$. The UD integral, $I_a^\alpha f$, is defined as follows:

$$I_a^\alpha f(x) = \frac{1}{\alpha} \int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f(t) dt, \quad \text{where } \alpha \in (0, 1]. \quad (3)$$

Theorem 2 (Pascal's Identity). [15] For all integers $n \geq 0$, and $1 \leq k \leq n-1$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \quad (4)$$

3 Main Results

The Duffing equation is a well-known nonlinear model used to describe complex oscillatory behavior, including phenomena such as resonance and chaotic dynamics. Extending this model to the fractional domain allows for a more accurate representation of systems that exhibit memory effects.

The ADM provides a practical semi-analytical approach for handling nonlinear differential equations without the need for linearization or discretization, which makes it particularly suitable for fractional problems.

In this section, we apply ADM to the fractional Duffing equation formulated using the UD operator. The required theoretical results are first established, followed by the derivation of analytical solutions for selected cases.

3.1 Some Necessary Results

Lemma 1. If $x(t)$ is a twice differentiable function, then the UD-derivative of order 2α , $\alpha \in (0, 1]$, can be expressed as

$$D^{2\alpha}x(t) = (1-\alpha)^2 x(t) + 2\alpha(1-\alpha) x'(t) + \alpha^2 x''(t). \quad (5)$$

Proof. Let $y(t) = D^\alpha x(t)$. Then by definition of the UD-operator, we obtain

$$D^\alpha x(t) = y(t) = (1-\alpha)x(t) + \alpha x'(t).$$

By applying UD-operator once again, we get

$$\begin{aligned} D^{2\alpha}x(t) &= D^\alpha y(t) \\ &= (1-\alpha)y(t) + \alpha y'(t) \\ &= (1-\alpha)((1-\alpha)x(t) + \alpha x'(t)) \\ &\quad + \alpha((1-\alpha)x'(t) + \alpha x''(t)) \\ &= (1-\alpha)^2 x(t) + \alpha(1-\alpha)x'(t) \\ &\quad + \alpha(1-\alpha)x'(t) + \alpha^2 x''(t), \end{aligned}$$

which implies

$$D^{2\alpha}x(t) = (1-\alpha)^2 x(t) + 2\alpha(1-\alpha) x'(t) + \alpha^2 x''(t).$$

□

Remark 2. From the previous lemma, we can infer

1. When $\alpha = 1$, we obtain

$$D^2x(t) = x''(t).$$

2. When $\alpha = 0$, we obtain

$$D^0x(t) = x(t).$$

Theorem 3. Let $x(t)$ be a continuous and differentiable function on $[0, \infty)$ n -times. Then, the UD-derivative of order $n\alpha$ is given by

$$D^{n\alpha}x(t) = \sum_{k=0}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \frac{d^k}{dt^k} x(t), \quad (6)$$

for $n \geq 1$ and $\alpha \in (0, 1]$.

Proof. To prove this result, we use mathematical induction. To do so, we have the following cases:

Case 1: For $n = 1$, the L.H.S of (6) is

$$D^\alpha x(t) = (1-\alpha)x(t) + \alpha x'(t). \quad (7)$$

Also, the R.H.S of the same equation is

$$\begin{aligned} &\sum_{k=0}^1 \binom{1}{k} \alpha^k (1-\alpha)^{1-k} \frac{d^k}{dt^k} x(t) \\ &= \binom{1}{0} \alpha^0 (1-\alpha)^1 x(t) + \binom{1}{1} \alpha (1-\alpha)^0 x'(t) \\ &= (1-\alpha)x(t) + \alpha x'(t) \end{aligned} \quad (8)$$

By (7) and (8), we observe that the statement is true for $n = 1$.

Case 2: For $n = m$, we assume that the statement is true for $n = m$, i.e., the statement

$$D^{m\alpha}x(t) = \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \frac{d^k}{dt^k} x(t),$$

is true. We need to show that the statement is true for $n = m + 1$, i.e., we need to show that

$$\begin{aligned} &D^{(m+1)\alpha}x(t) \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} \alpha^k (1-\alpha)^{m+1-k} \frac{d^k}{dt^k} x(t) \end{aligned}$$

is true. To do so, we observe

$$D^{(m+1)\alpha}x(t) = D^\alpha(D^{m\alpha}x(t)).$$

Let $y(t) = D^{m\alpha}x(t)$, then

$$\begin{aligned} &D^{(m+1)\alpha}x(t) = D^\alpha(D^{m\alpha}x(t)) = D^\alpha y(t) \\ &= (1-\alpha)y(t) + \alpha y'(t) \\ &= (1-\alpha) \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \frac{d^k}{dt^k} x(t) \\ &\quad + \alpha \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \frac{d^{k+1}}{dt^{k+1}} x(t), \end{aligned}$$

i.e.,

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k+1} \frac{d^k}{dt^k} x(t) \\ &\quad + \sum_{k=0}^m \binom{m}{k} \alpha^{k+1} (1-\alpha)^{m-k} \frac{d^{k+1}}{dt^{k+1}} x(t). \end{aligned} \quad (9)$$

Now, let $j = k + 1$ in the second term, then as $k = 0$ and $k = m$, we have $j = 1$ and $j = m + 1$, respectively. So, (9) becomes

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k+1} \frac{d^k}{dt^k} x(t) \\ &\quad + \sum_{j=1}^{m+1} \binom{m}{j-1} \alpha^j (1-\alpha)^{m-j+1} \frac{d^j}{dt^j} x(t). \end{aligned}$$

To combine the above two sums, we separate the first sum at $k = 0$ and the second sum at $j = m + 1$ to obtain

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= \binom{m}{0}\alpha^0(1-\alpha)^{m-0+1}x(t) \\ &+ \sum_{k=1}^m \binom{m}{k}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t) \\ &+ \binom{m}{m+1-1}\alpha^{m+1}(1-\alpha)^{m-m-1+1}\frac{d^{m+1}}{dt^{m+1}}x(t) \\ &+ \sum_{j=1}^m \binom{m}{j-1}\alpha^j(1-\alpha)^{m-j+1}\frac{d^j}{dt^j}x(t). \end{aligned}$$

Then, we get

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= (1-\alpha)^{m+1}x(t) \\ &+ \sum_{k=1}^m \binom{m}{k}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t) \\ &+ \alpha^{m+1}\frac{d^{m+1}}{dt^{m+1}}x(t) \\ &+ \sum_{k=1}^m \binom{m}{k-1}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t). \end{aligned}$$

Then, we obtain

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= (1-\alpha)^{m+1}x(t) \\ &+ \sum_{k=1}^m \left[\binom{m}{k} + \binom{m}{k-1} \right] \alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t) \\ &+ \alpha^{m+1}\frac{d^{m+1}}{dt^{m+1}}x(t). \end{aligned}$$

By (4), we have

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= (1-\alpha)^{m+1}x(t) \\ &+ \sum_{k=1}^m \binom{m+1}{k}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t) \\ &+ \alpha^{m+1}\frac{d^{m+1}}{dt^{m+1}}x(t), \end{aligned}$$

or

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= \binom{m+1}{0}\alpha^0(1-\alpha)^{m-0+1}\frac{d^0}{dt^0}x(t) \\ &+ \sum_{k=1}^m \binom{m+1}{k}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t) \\ &+ \binom{m+1}{m+1}\alpha^{m+1}(1-\alpha)^{m-m-1+1}\frac{d^{m+1}}{dt^{m+1}}x(t). \end{aligned}$$

This implies

$$\begin{aligned} D^{(m+1)\alpha}x(t) &= \\ &\sum_{k=0}^{m+1} \binom{m+1}{k}\alpha^k(1-\alpha)^{m-k+1}\frac{d^k}{dt^k}x(t). \end{aligned}$$

Therefore, the statement is true for $n = m + 1$, which completes the proof of this result. $\square$

3.2 Handling UD-Fractional Duffing Problem

In this subsection, two forms of the fractional Duffing equation defined under the UD operator are analyzed using the ADM.

The first model considers the unforced case with fractional damping, representing dissipative systems with memory effects, while the second model corresponds to the unforced and undamped case, describing conservative nonlinear dynamics. These two cases are examined to highlight the differences between dissipative and conservative fractional systems and to demonstrate the effectiveness of ADM in constructing analytical approximations.

3.2.1 Unforced UD-Fractional Duffing Problem

Consider the equation

$$D^{2\alpha}x(t) + \delta D^\alpha x(t) + kx(t) + \gamma x^3(t) = 0, \quad (10)$$

where $0 < \alpha \leq 1$, with initial conditions

$$x(0) = A \quad \text{and} \quad D^\alpha x(0) = C. \quad (11)$$

To solve the above problem, we recall that

$$D^{2\alpha}x(t) = (1-\alpha)^2 x(t) + 2\alpha(1-\alpha) x'(t) + \alpha^2 x''(t) \quad (12)$$

and

$$D^\alpha x(t) = (1-\alpha)x(t) + \alpha x'(t). \quad (13)$$

By substituting (12) and (13) in (10), we get

$$\begin{aligned} &\left[(1-\alpha)^2 x(t) + 2\alpha(1-\alpha) x'(t) + \alpha^2 x''(t) \right] \\ &+ \delta \left[(1-\alpha)x(t) + \alpha x'(t) \right] + kx(t) + \gamma x^3(t) = 0. \end{aligned}$$

By simplification, we obtain

$$\begin{aligned} &\alpha^2 x''(t) + \left(2\alpha(1-\alpha) + \delta\alpha \right) x'(t) \\ &+ \left((1-\alpha)^2 + \delta(1-\alpha) + k \right) x(t) + \gamma x^3(t) = 0, \end{aligned}$$

which implies

$$\begin{aligned} x''(t) &= - \left(\frac{2\alpha(1-\alpha) + \delta\alpha}{\alpha^2} \right) x'(t) \\ &- \left(\frac{(1-\alpha)^2 + \delta(1-\alpha) + k}{\alpha^2} \right) x(t) - \frac{\gamma}{\alpha^2} x^3(t). \end{aligned} \quad (14)$$

Now, we let

$$a = \left(\frac{2\alpha(1-\alpha) + \delta\alpha}{\alpha^2} \right),$$

$$b = \left(\frac{(1-\alpha)^2 + \delta(1-\alpha) + k}{\alpha^2} \right), \text{ and } c = \frac{\gamma}{\alpha^2}.$$

Then Eq(14) becomes

$$x''(t) = -ax'(t) - bx(t) - cx^3(t). \quad (15)$$

Now, by using the initial conditions given in (11), we obtain

$$D^\alpha x(0) =$$

$$(1-\alpha)x(0) + \alpha x'(0) = (1-\alpha)A + \alpha x'(0) = C,$$

which implies

$$\alpha x'(0) = C - (1-\alpha)A.$$

This yields

$$x'(0) = \frac{C - (1-\alpha)A}{\alpha} = B.$$

Thus, the classical initial conditions are

$$x(0) = A \quad \text{and} \quad x'(0) = B.$$

By using the ADM with considering

$$L = \frac{d^2}{dt^2},$$

equation (15) becomes

$$L(x(t)) = -ax'(t) - bx(t) - cx^3(t).$$

By applying the inverse operator $(L^{-1}(\cdot) = \int_0^t \int_0^\tau (\cdot) ds d\tau)$, we obtain

$$x(t) - x(0) - tx'(0)$$

$$= L^{-1}(-ax'(t) - bx(t) - cx^3(t)).$$

This gives

$$x(t) = x(0) + tx'(0)$$

$$+ L^{-1}(-ax'(t) - bx(t) - cx^3(t)).$$

This implies

$$x(t) = A + Bt - aL^{-1}(x'(t))$$

$$- bL^{-1}(x(t)) - cL^{-1}(x^3(t)). \quad (16)$$

Now, in the sense of the ADM, the solution can be defined as

$$x(t) = \sum_{n=0}^{\infty} x_n(t).$$

Also; the nonlinear term $x^3(t)$ can be expressed with the use of Adomian polynomials as follows:

$$x^3(t) = \sum_{n=0}^{\infty} A_n(t),$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\left(\sum_{k=0}^{\infty} \lambda^k x_k \right)^3 \right]_{\lambda=0}.$$

By using this form, we can obtain the first three polynomials, which has the following forms:

$$A_0 = x_0^3(t)$$

$$A_1 = 3x_0^2(t)x_1(t)$$

$$A_2 = 3x_0^2(t)x_2(t) + 3x_0(t)x_1^2(t).$$

Now, from the previous discussion, (16) can be rewritten in a form of recurrence relation as follows:

$$x_0(t) = A + Bt,$$

$$x_{n+1}(t) = -aL^{-1}(x'_n(t)) - bL^{-1}(x_n(t))$$

$$- cL^{-1}(A_n(t)), \quad \forall n \geq 0. \quad (17)$$

Based on (17), we can have

$$x_0(t) = A + Bt \quad \text{then} \quad x'_0(t) = B,$$

where

$$B = \frac{C - (1-\alpha)A}{\alpha}, \quad A = x(0).$$

Similarly, we can compute $x_1(t)$ as follows:

$$x_1(t) = -aL^{-1}(x'_0(t)) - bL^{-1}(x_0(t)) - cL^{-1}(A_0(t)),$$

which implies

$$x_1(t) = -aL^{-1}(B) - bL^{-1}(A+Bt) - cL^{-1}(x_0^3(t))$$

$$= -a \int_0^t \int_0^\tau B ds d\tau - b \int_0^t \int_0^\tau (A + Bs) ds d\tau$$

$$- cL^{-1}((A+Bt)^3)$$

Then we obtain

$$x_1(t) = -\left(\frac{aB + bA + cA^3}{2} \right) t^2$$

$$- \left(\frac{bB + 3cA^2B}{6} \right) t^3 - \frac{cAB^2}{4} t^4 - \frac{cB^3}{20} t^5. \quad (18)$$

Similarly, we can compute $x_2(t)$ as follows:

$$x_2(t) = -a L^{-1}(x_1'(t)) - b L^{-1}(x_1(t)) - c L^{-1}(A_1(t)). \quad (19)$$

Referring to (18), we assume that

$$p_0 = \left(\frac{aB + bA + cA^3}{2} \right), \quad p_1 = \left(\frac{bB + 3cA^2B}{6} \right), \\ p_2 = \frac{cAB^2}{4}, \quad p_3 = \frac{cB^3}{20}.$$

Then (18) becomes

$$x_1(t) = -p_0 t^2 - p_1 t^3 - p_2 t^4 - p_3 t^5. \quad (20)$$

By differentiating (18), we obtain

$$x_1'(t) = -\left(aB + bA + cA^3 \right) t - \left(\frac{bB + 3cA^2B}{2} \right) t^2 \\ - \left(cAB^2 \right) t^3 - \left(\frac{cB^3}{4} \right) t^4. \quad (21)$$

Let

$$q_0 = \left(aB + bA + cA^3 \right), \quad q_1 = \left(\frac{bB + 3cA^2B}{2} \right), \\ q_2 = (cAB^2), \quad q_3 = \frac{cB^3}{4}.$$

Then (21) becomes

$$x_1'(t) = -q_0 t - q_1 t^2 - q_2 t^3 - q_3 t^4, \quad (22)$$

and the corresponding Adomian polynomial is given by

$$A_1(t) = 3x_0^2(t) x_1(t),$$

Then we obtain

$$A_1(t) = \left[-\left(\frac{3aA^2B + 3bA^3 + 3cA^5}{2} \right) t^2 \right. \\ - \left(\frac{6aAB^2 + 7bA^2B + 9cA^4B}{2} \right) t^3 \\ - \left(\frac{6aB^3 + 10bAB^2 + 21cA^3B^2}{4} \right) t^4 \\ - \left(\frac{10bB^3 + 63cA^2B^3}{20} \right) t^5 \\ \left. - \left(\frac{21cAB^4}{20} \right) t^6 - \left(\frac{3cB^5}{20} \right) t^7 \right]. \quad (23)$$

Let

$$r_0 = \left(\frac{3aA^2B + 3bA^3 + 3cA^5}{2} \right), \\ r_1 = \left(\frac{6aAB^2 + 7bA^2B + 9cA^4B}{2} \right), \\ r_2 = \left(\frac{6aB^3 + 10bAB^2 + 21cA^3B^2}{4} \right), \\ r_3 = \left(\frac{10bB^3 + 63cA^2B^3}{20} \right), \\ r_4 = \left(\frac{21cAB^4}{20} \right), \quad r_5 = \left(\frac{3cB^5}{20} \right).$$

Then (23) becomes

$$A_1(t) = -r_0 t^2 - r_1 t^3 - r_2 t^4 - r_3 t^5 - r_4 t^6 - r_5 t^7. \quad (24)$$

By substituting (20), (22), (24) into (19), we get

$$x_2(t) = -a L^{-1} \left(-q_0 t - q_1 t^2 - q_2 t^3 - q_3 t^4 \right) \\ - b L^{-1} \left(-p_0 t^2 - p_1 t^3 - p_2 t^4 - p_3 t^5 \right) \\ - c L^{-1} \left(-r_0 t^2 - r_1 t^3 - r_2 t^4 \right. \\ \left. - r_3 t^5 - r_4 t^6 - r_5 t^7 \right).$$

Then the final expression of $x_2(t)$ is obtained as

$$x_2(t) = \left(\frac{a^2B + abA + caA^3}{6} \right) t^3 \\ + \left(\frac{2abB + 6acA^2B + 4cbA^3 + b^2A + 3c^2A^5}{24} \right) t^4 \\ + \left(\frac{24acAB^2 + b^2B + 24cbA^2B + 27c^2A^4B}{120} \right) t^5 \\ + \left(\frac{7acB^3 + 11cbAB^2 + 21c^2A^3B^2}{120} \right) t^6 \\ + \left(\frac{11cbB^3 + 63c^2A^2B^3}{840} \right) t^7 \\ + \left(\frac{3c^2AB^4}{160} \right) t^8 + \left(\frac{c^2B^5}{480} \right) t^9. \quad (25)$$

Consequently, the solution of three terms is given by

$$x(t) \approx x_0(t) + x_1(t) + x_2(t).$$

i.e.,

$$\begin{aligned} x(t) \approx & A + Bt - \frac{aB + bA + cA^3}{2}t^2 \\ & + \frac{a^2B + abA + caA^3 - bB - 3cA^2B}{6}t^3 \\ & + \frac{2abB + 6acA^2B + 4cbA^3 + b^2A + 3c^2A^5 - 6cAB^2}{24}t^4 \\ & + \frac{24acAB^2 + b^2B + 24cbA^2B + 27c^2A^4B - 6cB^3}{120}t^5 \\ & + \frac{7acB^3 + 11cbAB^2 + 21c^2A^3B^2}{120}t^6 \\ & + \frac{11cbB^3 + 63c^2A^2B^3}{840}t^7 + \frac{3c^2AB^4}{160}t^8 + \frac{c^2B^5}{480}t^9. \end{aligned}$$

Where

$$B = \frac{C - (1 - \alpha)A}{\alpha}, \quad a = \left(\frac{2\alpha(1 - \alpha) + \delta\alpha}{\alpha^2} \right),$$

$$b = \left(\frac{(1 - \alpha)^2 + \delta(1 - \alpha) + k}{\alpha^2} \right), \quad \text{and} \quad c = \frac{\gamma}{\alpha^2}.$$

Consequently, when $\alpha = 1$, we get $B = C$, $a = \delta$, $b = k$, and $c = \gamma$. Thus, the solution becomes

$$\begin{aligned} x(t) \approx & A + Ct - \frac{\delta C + kA + \gamma A^3}{2}t^2 \\ & + \frac{\delta^2 C + \delta kA + \delta \gamma A^3 - kC - 3\gamma A^2 C}{6}t^3 \\ & + \frac{2\delta kC + 6\delta \gamma A^2 C + 4\gamma kA^3}{24}t^4 \\ & + \frac{24\delta \gamma AC^2 + k^2 C + 24\gamma kA^2 C}{120}t^5 \\ & + \frac{7\delta \gamma C^3 + 11\gamma kAC^2 + 21\gamma^2 A^3 C^2}{120}t^6 \\ & + \frac{11\gamma kC^3 + 63\gamma^2 A^2 C^3}{840}t^7 + \frac{3\gamma^2 AC^4}{160}t^8 + \frac{\gamma^2 C^5}{480}t^9. \end{aligned} \quad (26)$$

Which matches the ADM's solution of the classical unforced duffing problem.

3.2.2 Unforced and Undamped UD-Fractional Duffing Problem

Consider the following unforced and undamped Duffing problem formulated in the sense of the UD-fractional derivative operator:

$$D^{2\alpha}x(t) + kx(t) + \gamma x^3(t) = 0, \quad (27)$$

where $0 < \alpha \leq 1$, with initial conditions

$$x(0) = A \quad \text{and} \quad D^\alpha x(0) = C. \quad (28)$$

To solve the above problem, we recall that

$$D^{2\alpha}x(t) = (1 - \alpha)^2 x(t) + 2\alpha(1 - \alpha) x'(t) + \alpha^2 x''(t). \quad (29)$$

By substituting (29) into (27), we get

$$\begin{aligned} & \left[(1 - \alpha)^2 x(t) + 2\alpha(1 - \alpha) x'(t) + \alpha^2 x''(t) \right] \\ & + kx(t) + \gamma x^3(t) = 0. \end{aligned}$$

By simplification, we obtain

$$\begin{aligned} & \alpha^2 x''(t) + 2\alpha(1 - \alpha) x'(t) \\ & + \left((1 - \alpha)^2 + k \right) x(t) + \gamma x^3(t) = 0, \end{aligned}$$

which implies

$$\begin{aligned} x''(t) = & -\frac{2\alpha(1 - \alpha)}{\alpha^2} x'(t) - \left(\frac{(1 - \alpha)^2 + k}{\alpha^2} \right) x(t) \\ & - \frac{\gamma}{\alpha^2} x^3(t). \end{aligned} \quad (30)$$

Now, we let

$$\begin{aligned} a = & \frac{2\alpha(1 - \alpha)}{\alpha^2}, \quad b = \left(\frac{(1 - \alpha)^2 + k}{\alpha^2} \right), \\ c = & \frac{\gamma}{\alpha^2}. \end{aligned}$$

Then (30) becomes

$$x''(t) = -ax'(t) - bx(t) - cx^3(t). \quad (31)$$

The unforced and undamped UD-Fractional duffing problem in this case varies from the previous (Unforced UD-Fractional Duffing Problem) case only in the constant coefficients multiplying each term. Consequently, the Adomian polynomials and the solution remain identical. Therefore, the derivation steps are omitted here to avoid repetition. As a results, the solution of three terms for problem (27)-(28) is given by

$$x(t) \approx x_0(t) + x_1(t) + x_2(t).$$

In other words, we obtain

$$\begin{aligned} x(t) \approx & A + Bt - \left(\frac{aB + bA + cA^3}{2} \right) t^2 \\ & + \left(\frac{a^2B + abA + caA^3 - bB - 3cA^2B}{6} \right) t^3 \\ & + \left(\frac{2abB + 6acA^2B + 4cbA^3 + b^2A + 3c^2A^5 - 6cAB^2}{24} \right) t^4 \\ & + \left(\frac{24acAB^2 + b^2B + 24cbA^2B + 27c^2A^4B - 6cB^3}{120} \right) t^5 \\ & + \left(\frac{7acB^3 + 11cbAB^2 + 21c^2A^3B^2}{120} \right) t^6 \\ & + \left(\frac{11cbB^3 + 63c^2A^2B^3}{840} \right) t^7 \\ & + \left(\frac{3c^2AB^4}{160} \right) t^8 + \left(\frac{c^2B^5}{480} \right) t^9. \end{aligned}$$

Where

$$B = \frac{C - (1 - \alpha)A}{\alpha}, \quad a = \frac{2\alpha(1 - \alpha)}{\alpha^2},$$

$$\text{and } b = \frac{(1 - \alpha)^2 + k}{\alpha^2}, \quad c = \frac{\gamma}{\alpha^2}.$$

Consequently, when $\alpha = 1$, we get $B = C$, $a = 0$, $b = k$, and $c = \gamma$. Thus, the solution becomes

$$\begin{aligned} x(t) \approx & A + Ct - \left(\frac{kA + \gamma A^3}{2}\right)t^2 - \left(\frac{kC + 3\gamma A^2 C}{6}\right)t^3 \\ & + \left(\frac{4\gamma kA^3 + k^2 A + 3\gamma^2 A^5 - 6\gamma AC^2}{24}\right)t^4 \\ & + \left(\frac{k^2 C + 24\gamma kA^2 C + 27\gamma^2 A^4 C - 6\gamma C^3}{120}\right)t^5 \\ & + \left(\frac{11\gamma kAC^2 + 21\gamma^2 A^3 C^2}{120}\right)t^6 + \left(\frac{11\gamma kC^3 + 63\gamma^2 A^2 C^3}{840}\right)t^7 \\ & + \left(\frac{3\gamma^2 AC^4}{160}\right)t^8 + \left(\frac{\gamma^2 C^5}{480}\right)t^9. \end{aligned} \quad (32)$$

which matches the ADM's solution of the classical unforced and undamped duffing problem.

4 Numerical Simulation

Example 1. Consider the following unforced and undamped fractional Duffing problem:

$$D^{2\alpha}x(t) + x(t) + x^3(t) = 0, \quad (33)$$

with fractional initial conditions:

$$x(0) = 0 \quad \text{and} \quad D^\alpha x(0) = 1. \quad (34)$$

To solve this problem, we need to convert it into the classical form, then we have

$$\begin{aligned} x''(t) = & -\frac{2(1 - \alpha)}{\alpha}x'(t) - \left(\frac{(1 - \alpha)^2 + 1}{\alpha^2}\right)x(t) - \frac{1}{\alpha^2}x^3(t). \end{aligned}$$

Now, assume

$$a = \frac{2(1 - \alpha)}{\alpha}, \quad b = \left(\frac{(1 - \alpha)^2 + 1}{\alpha^2}\right),$$

$$\text{and } c = \frac{1}{\alpha^2}. \quad (35)$$

This implies

$$x''(t) = -ax'(t) - bx(t) - cx^3(t),$$

with fractional initial conditions

$$x(0) = 0 \quad \text{and} \quad D^\alpha x(0) = 1.$$

The second initial condition implies

$$1 = D^\alpha x(0) = (1 - \alpha)x(0) + \alpha x'(0).$$

This gives

$$x'(0) = \frac{1}{\alpha},$$

So, the main problem (33) and (34) becomes

$$x''(t) = -ax'(t) - bx(t) - cx^3(t), \quad (36)$$

with initial conditions

$$x(0) = 0 \quad \text{and} \quad x'(0) = \frac{1}{\alpha}, \quad (37)$$

where a , b , and c are defined in (35). Now, applying $(L^{-1}(\cdot) = \int_0^t \int_0^\tau (\cdot) ds d\tau)$ on (36) yields

$$x(t) - x(0) - t x'(0) = L^{-1}(-a x'(t) - b x(t) - c x^3(t)).$$

This gives

$$\begin{aligned} x(t) = & \frac{t}{\alpha} - a L^{-1}(x'(t)) \\ & - b L^{-1}(x(t)) - c L^{-1}(x^3(t)). \end{aligned}$$

Then we obtain the following recurrence relation:

$$\begin{aligned} x_0(t) = & \frac{t}{\alpha} \\ x_{n+1}(t) = & -a L^{-1}(x'_n(t)) - b L^{-1}(x_n(t)) \\ & - c L^{-1}(A_n(t)), \quad \forall n \geq 0. \end{aligned} \quad (38)$$

Using the recurrence relation (38), the first iterations $x_0(t)$, $x_1(t)$, and $x_2(t)$ are obtained. Hence the approximate solution is given by

$$x(t) \approx x_0(t) + x_1(t) + x_2(t).$$

This implies

$$\begin{aligned} x(t) \approx & \frac{t}{\alpha} - \frac{at^2}{2\alpha} + \left(\frac{a^2 - b}{6\alpha}\right)t^3 + \frac{abt^4}{12\alpha} \\ & + \left(\frac{b^2}{120\alpha} - \frac{c}{20\alpha^3}\right)t^5 + \frac{7cat^6}{120\alpha^3} + \frac{11cbt^7}{840\alpha^3} + \frac{c^2t^9}{480\alpha^5}. \end{aligned} \quad (39)$$

To depict the solution of problem (33)-(34) generated from the ADM, we plot this solution in Figure 1 and compare it with the Runge-Kutta (RK4) solution. From this figure, we can observe that the two solutions are very close to each other, confirming the validity of our proposed scheme.

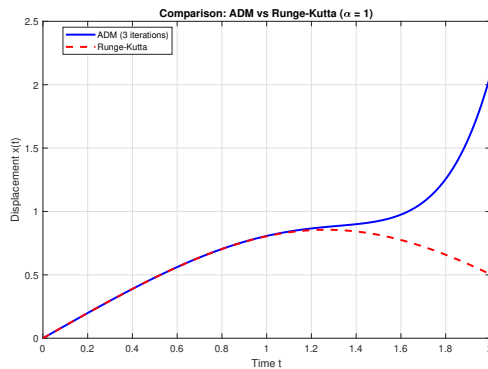


Figure 1: Comparison between the ADM and RK4 solutions when $\alpha = 1$.

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In the same regard, we present Figure 2 and Table 1 (Appendix), which illustrate the absolute error between the two solutions: the ADM solution and the Fourth-order Runge-Kutta solution RK4 when $\alpha = 1$. One might observe, from these results, that the error remains extremely small for most of the interval and starts to increase only near the end, indicating that the ADM maintains high accuracy for small and moderate values of t .

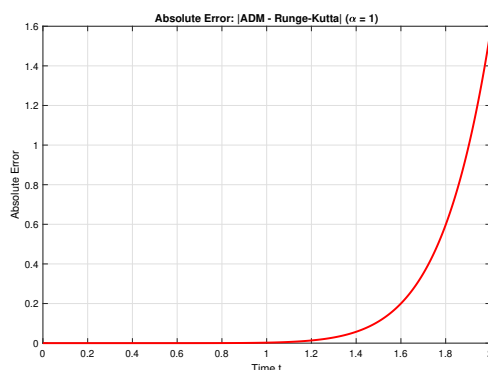


Figure 2: The absolute error between the ADM and RK4 solutions for $\alpha = 1$.

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Furthermore, Figure 3 demonstrates the effect of different fractional orders α on the system's displacement. As shown in this figure, decreasing the value of α leads to a faster growth in the solution, with the response exhibiting a sharper increase as the fractional order becomes smaller. On the other hand, values of α closer to 1 yield a more stable and smoother behavior. This indicates that the fractional order plays a significant role in shaping the dynamics of the problem, and highlights the strong memory effect introduced by the fractional operator.

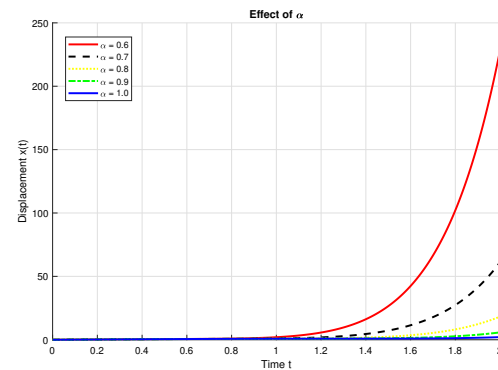


Figure 3: Effect of different fractional orders α on the displacement $x(t)$.

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To further examine the convergence behavior of the Adomian series beyond the 3-term truncation used in the main results, we computed the absolute error at $t = 1.00$ for truncations ranging from 1 to 6 terms for Example 1 (Figure 4). This example is selected as a representative case, since both examples share the same underlying nonlinear structure and are expected to exhibit qualitatively similar convergence behavior. As shown in Figure 4, the absolute error decreases monotonically from approximately 10^{-1} at 1 term to approximately 10^{-6} at 6 terms, confirming a consistent and stable convergence trend as the number of Adomian terms increases. This behavior provides numerical evidence supporting the reliability of the 3-term truncation adopted throughout the manuscript, and further theoretical characterization of the convergence rate is identified as a direction for future work.

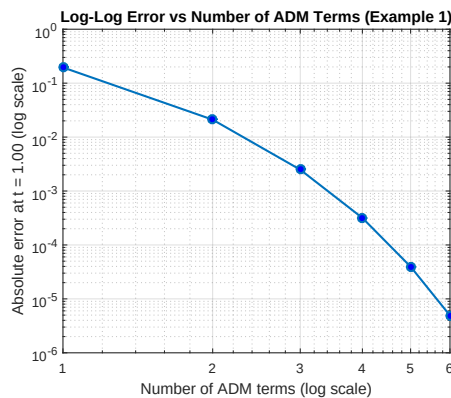


Figure 4: Log-log plot of the absolute error at $t = 1.00$ versus the number of Adomian terms (1 to 6) for Example 1, illustrating the convergence trend of the proposed scheme.

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Example 2. Consider the following fractional Duffing problem

$$D^{2\alpha}x(t) + 0.8D^\alpha x(t) + 1.5x(t) + 0.3x^3(t) = 0, \quad (40)$$

with fractional initial conditions:

$$x(0) = 2 \quad \text{and} \quad D^\alpha x(0) = -1. \quad (41)$$

To solve problem (40)-(41), we need to convert it into its classical form, i.e., (40) becomes

$$x''(t) = -\left(\frac{2\alpha(1-\alpha) + 0.8\alpha}{\alpha^2}\right)x'(t) - \left(\frac{(1-\alpha)^2 + 0.8(1-\alpha) + 1.5}{\alpha^2}\right)x(t) - \frac{0.3}{\alpha^2}x^3(t). \quad (42)$$

Now, we let

$$a = \left(\frac{2\alpha(1-\alpha) + 0.8\alpha}{\alpha^2}\right), \quad b = \left(\frac{(1-\alpha)^2 + 0.8(1-\alpha) + 1.5}{\alpha^2}\right), \quad \text{and} \quad c = \frac{0.3}{\alpha^2}.$$

Then (42) becomes

$$x''(t) = -ax'(t) - bx(t) - cx^3(t).$$

Now, for the fractional initial conditions, we have

$$x(0) = 2 \quad \text{and} \quad D^\alpha x(0) = -1.$$

The second initial condition implies

$$-1 = D^\alpha x(0) = (1-\alpha)x(0) + \alpha x'(0).$$

This yields

$$x'(0) = \frac{-1 - 2(1-\alpha)}{\alpha} = B.$$

Accordingly, the fractional problem (40)-(41) is converted into its corresponding classical problem that has the following form:

$$x''(t) = -ax'(t) - bx(t) - cx^3(t), \quad (43)$$

with initial conditions

$$x(0) = 2 \quad \text{and} \quad x'(0) = \frac{-1 - 2(1-\alpha)}{\alpha} = B. \quad (44)$$

Now, applying $(L^{-1}(\cdot) = \int_0^t \int_0^\tau (\cdot) ds d\tau)$ on (43) yields

$$x(t) = 2 + Bt - aL^{-1}(x'(t)) - bL^{-1}(x(t)) - cL^{-1}(x^3(t)). \quad (45)$$

Then we obtain the following recurrence relation:

$$\begin{aligned} x_0(t) &= 2 + Bt \\ x_{n+1}(t) &= -aL^{-1}(x'_n(t)) - bL^{-1}(x_n(t)) - cL^{-1}(x_n^3(t)), \quad \forall n \geq 0. \end{aligned} \quad (46)$$

Using the recurrence relation (46), the first iterations $x_0(t)$, $x_1(t)$, and $x_2(t)$ are obtained. Hence the approximate solution is given by

$$x(t) \approx x_0(t) + x_1(t) + x_2(t).$$

This gives

$$\begin{aligned} x(t) \approx & 2 + Bt - \left(\frac{aB + 2b + 8c}{2}\right)t^2 \\ & + \left(\frac{a^2B + 2ab + 8ca - bB - 12cB}{6}\right)t^3 \\ & + \left(\frac{abB + 12acB + 16cb + b^2 + 48c^2 - 6cB^2}{12}\right)t^4 \\ & + \left(\frac{48acB^2 + b^2B + 96cbB + 432c^2B - 6cB^3}{120}\right)t^5 \\ & + \left(\frac{7acB^3 + 22cbB^2 + 168c^2B^2}{120}\right)t^6 \\ & + \left(\frac{11cbB^3 + 252c^2B^3}{840}\right)t^7 \\ & + \left(\frac{3c^2B^4}{80}\right)t^8 + \left(\frac{c^2B^5}{480}\right)t^9. \end{aligned} \quad (47)$$

Figure 5 presents a comparison between the ADM approximation and the RK4 method for the unforced fractional Duffing equation when $\alpha = 1$. The two curves match closely throughout most of the interval, indicating that the ADM provides a reliable approximation of the true solution. The slight deviation observed near the end of the interval arises because only the first three ADM components were used in constructing the series solution, which limits the accuracy as the solution grows more rapidly.

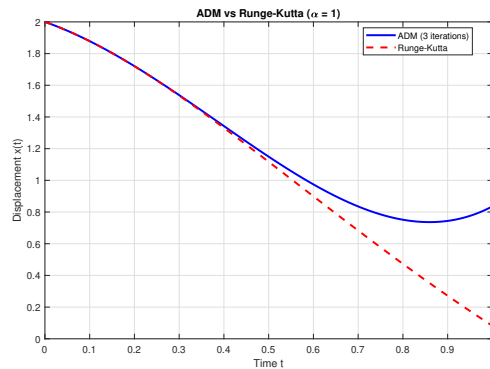


Figure 5: Comparison between the ADM and RK4 solutions when $\alpha = 1$
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The absolute and relative errors between the ADM and RK4 solutions are summarized in Figure 6 and Table 2 (Appendix). As shown in these results, the error remains extremely small for the early part of the interval, demonstrating the high accuracy of the ADM for small and moderate values of t . However, both the absolute and relative errors increase substantially as t grows larger. This behavior is expected, since truncating the ADM series to only three components becomes insufficient once the solution enters a region of faster growth. Nevertheless, the agreement in the earlier phase confirms that ADM captures the qualitative behavior of the system effectively.

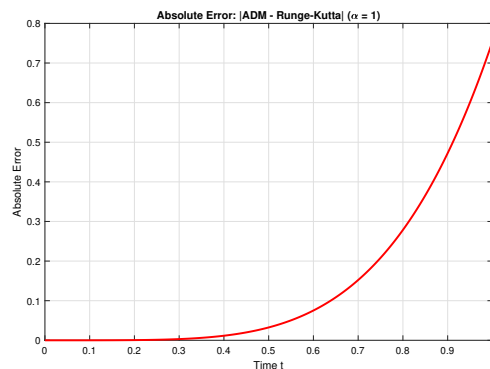


Figure 6: The absolute error between the ADM and RK4 solutions for $\alpha = 1$.
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Figure 7 illustrates the effect of different fractional orders α on the displacement $x(t)$ in the unforced case. The results show that decreasing the fractional order significantly accelerates the system's growth, leading to a much larger displacement over time. In contrast, values of α close to 1 produce a slower and more stable

response. This behavior highlights the strong role of the fractional order in controlling the memory and damping characteristics of the system smaller values of α reduce the effective damping, causing the solution to grow rapidly even in the absence of forcing.

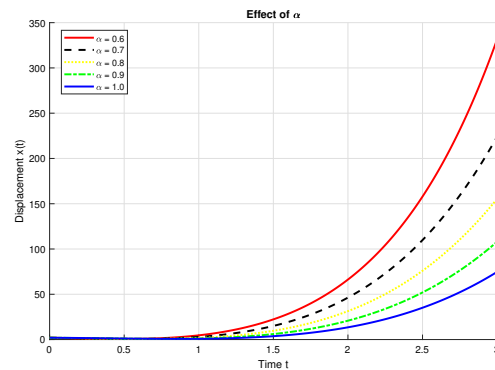


Figure 7: Effect of different fractional orders α on the displacement $x(t)$.
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5 Conclusion

This study presented an approach for solving the fractional Duffing equation using the ADM with the UD operator. The results confirmed the method's accuracy and computational efficiency in capturing the nonlinear dynamics of the system, while its local operator structure motivates its use as a computationally tractable alternative to non-local fractional formulations. Several limitations should be acknowledged, including the locally convergent nature of the truncated series and the absence of a general convergence guarantee. Future work may focus on extending the approach to forced and stochastic fractional Duffing systems, incorporating acceleration techniques such as Padé approximants to extend the domain of validity, and developing adaptive algorithms for automatically determining the optimal number of ADM terms.

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Appendix

Table 1: Absolute error between the ADM and RK4 solutions when $\alpha = 1$.

Time	ADM Solution	RK4 Solution	Absolute Error	Relative Error
0.0	0	0	0	0
0.2	0.19865350	0.19865354	3.98e-08	0
0.4	0.38892867	0.38892799	6.75e-07	0.0002
0.6	0.56114758	0.56112286	2.47e-05	0.0044
0.8	0.70403922	0.70371694	3.22e-04	0.0458
1.0	0.80684524	0.80435845	2.49e-03	0.3092
1.2	0.86599215	0.85243240	1.36e-02	1.5907
1.4	0.89965867	0.84224195	5.74e-02	6.8171
1.6	0.97511486	0.77503056	2.00e-01	25.8163
1.8	1.25564523	0.65852304	5.97e-01	90.6760
2.0	2.07619048	0.50429669	1.57e+00	311.7002

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Table 2: Comparison between ADM and RK4 solutions

Time	ADM Solution	RK4 Solution	Absolute Error	Relative Error
0.0	2.00000000	2.00000000	0	0
0.1	1.87852923	1.87850419	2.50e-05	0.0013
0.2	1.72076347	1.72025258	5.11e-04	0.0297
0.3	1.53787220	1.53480498	3.07e-03	0.1998
0.4	1.34263893	1.33107983	1.16e-02	0.8684
0.5	1.14947917	1.11700276	3.25e-02	2.9075
0.6	0.97445840	0.89924036	7.52e-02	8.3646
0.7	0.83531013	0.68314541	1.52e-01	22.2741
0.8	0.75145387	0.47285927	2.79e-01	58.9170
0.9	0.74401310	0.27159658	4.72e-01	173.9405
1.0	0.83583333	0.08183656	7.54e-01	921.3446

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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