

Neutrosophic Adjoint, Self-Adjoint, and Unitary Operators on Neutrosophic Hilbert Spaces

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Abstract: In this paper, we introduce new definitions of neutrosophic adjoint, self-adjoint, and unitary operators acting on a neutrosophic Hilbert space. Fundamental properties are established, and several key results are proved to characterize the behavior of these operators within the neutrosophic framework. It is further shown that operators on neutrosophic Hilbert spaces generalize their classical counterparts, with classical operators on Hilbert spaces recovered as a special case when the truth-membership degree is equal to 1 and both the indeterminacy-membership and falsity-membership degrees are equal to 0. Moreover, the paper presents a rigorous analytical study of the structural and functional properties of neutrosophic operators, thereby contributing to the advancement of operator theory on neutrosophic Hilbert spaces.

Key-Words: neutrosophic sets, t-norm, t-conorm, neutrosophic representable t-norm, neutrosophic Hilbert space (NHS), neutrosophic adjoint operator, neutrosophic self-adjoint operator, neutrosophic unitary operator.

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1 Introduction

The theory of fuzzy sets was first proposed by [1], marking a fundamental milestone in the modeling of uncertainty and vagueness. Since its inception, the fuzzy set theory has attracted considerable attention, leading to the publication of a vast body of research and the fuzzification of many classical mathematical theories. The concept of intuitionistic fuzzy sets (IFSs) was introduced by [2], in 1986 as a natural extension of this framework, and it includes a membership degree and a non-membership degree, providing a richer structure to deal with imprecise information. Later, [3], proposed the concept of neutrosophic sets (NSs) as a further generalization of the intuitionistic fuzzy sets. In contrast to the intuitionistic fuzzy sets, the total of the degrees of truth-membership, indeterminacy-membership and falsity-membership in neutrosophic sets can be less than, equal to, or greater than 1. Even in the special case where this sum equals one, neutrosophic aggregation operators yield results that differ from those obtained via intuitionistic fuzzy operators, since the latter typically neglect the role of indeterminacy. In contrast, neutrosophic theory treats indeterminacy on an equal footing with truth and falsity. Moreover, neutrosophic sets provide flexibility by accommodating not only independent components but also partially dependent and partially independent components, a feature not captured by intuitionistic fuzzy sets. A detailed comparative study

between neutrosophic logic, intuitionistic fuzzy logic, and their corresponding set theoretic frameworks was presented by [3], in 1999-2005.

In parallel with these advances, considerable attention has been directed toward extending fundamental notions of classical functional analysis to fuzzy and intuitionistic fuzzy frameworks. Early contributions in this direction were made by [4], as well as by [5], who initiated the systematic study of fuzzy inner product spaces and fuzzy norm functions. Subsequently, [6], revisited and refined Biswas's formulation, thereby strengthening the theoretical structure of these spaces. The alternative concept of fuzzy inner product spaces was amended by [7]. Fuzzy Hilbert spaces were formally established by [8]. During the same period, Goudarzi and S.Vaezpour proposed the concept of intuitionistic fuzzy inner product spaces (IFIP-spaces). This line of research was further advanced by [9], who established intuitionistic fuzzy normed linear spaces and provided a deeper theoretical development of IFIP-spaces in 2011. Further progress was achieved by [10], who introduced intuitionistic fuzzy Hilbert spaces (IFH-spaces) and examined several of their fundamental properties.

In recent years, neutrosophic theory has been progressively integrated into the fields of functional analysis and topology. [11], made a seminal contribution by introducing the concept of neutrosophic metric spaces (NMSs) and

examining various essential topological properties, such as openness, Hausdorffness, boundedness, compactness, and completeness, along with the neutrosophic equivalents of the Baire Category and Uniform Convergence Theorems. Later, [12], came up with the idea of neutrosophic soft linear spaces and then expanded on it to create neutrosophic soft normed linear spaces by [13]. They looked at ideas like neutrosophic norms, Cauchy sequences, convexity, and the metrics that these norms create in a systematic way. The concept of neutrosophic inner product spaces (NIPSSs) in [14], thereby expanding the analytical framework of neutrosophic mathematics. In [15], the concepts of neutrosophic inner product spaces and neutrosophic Hilbert spaces were presented, together with their fundamental properties such as convergence, continuity, completeness, orthogonality, and a neutrosophic version of the Riesz Representation Theorem.

Simultaneously, the study of operators on fuzzy and intuitionistic fuzzy structures garnered heightened interest. In [16], adjoint fuzzy linear operators and self-adjoint fuzzy linear operators were developed, and various important results and characteristics were established. Subsequently, [10], the theory of intuitionistic fuzzy adjoint and self-adjoint linear operators operating on intuitionistic fuzzy Hilbert spaces was formulated, offering a comprehensive analysis of their structural and analytical characteristics. While operator theory is well established in classical Hilbert spaces, its application to neutrosophic Hilbert spaces is not merely limited but remains largely unexplored and fundamentally new. The development of operator theory has traditionally been carried out in classical Hilbert spaces, where all notions are described with complete certainty through real or complex-valued inner products and norms. However, many practical and theoretical situations involve different levels of uncertainty that cannot be adequately represented within the classical framework. This limitation motivated the introduction of fuzzy Hilbert spaces, where uncertainty is incorporated through membership functions that describe the degree to which an element satisfies a given property.

Subsequently, intuitionistic fuzzy Hilbert spaces were introduced as a broader framework by considering both membership and non-membership degrees simultaneously, providing a richer representation of uncertainty than the fuzzy setting. Nevertheless, intuitionistic fuzzy theory still imposes certain restrictions on the treatment of indeterminacy, since the hesitation degree is implicitly determined by the membership and non-membership functions. In [17], the existence and uniqueness of solutions to hemi-equilibrium problems were investigated, and

an application of the obtained results was presented. Additionally, recent work, [18], studies Neutrosophic triangular norms and their derived residuated lattices.

Neutrosophic Hilbert spaces provide a further expansion by incorporating three independent elements that signify truth, indeterminacy, and falsity. This approach provides more flexibility for representing incomplete, inconsistent, and indeterminate information, serving as a natural generalization of intuitionistic fuzzy Hilbert spaces.

In the present work, the generalization of operators should be understood in the sense of extending operator theory to a neutrosophic Hilbert space environment. Since a neutrosophic Hilbert space constitutes a natural extension of an intuitionistic fuzzy Hilbert space, the associated operators are investigated within this broader mathematical framework. Accordingly, the generalization stems from the underlying space on which the operators act rather than from any modification of their fundamental algebraic nature. This approach enables the extension of classical, fuzzy, and intuitionistic fuzzy operator-theoretic concepts to the neutrosophic setting while preserving their essential structural properties.

Therefore, this paper develops and studies operator theory on neutrosophic Hilbert spaces, establishing neutrosophic analogues of several fundamental concepts, including adjoint, self-adjoint, and unitary operators, and thereby extending the scope of operator theory to a more general framework capable of accommodating truth, indeterminacy, and falsity simultaneously.

2 Preliminaries

Proposition 2.1. [18] Assume that $\mathbf{G}^* = \{\mu = (\mu_1, \mu_2, \mu_3) \mid \mu_1, \mu_2, \mu_3 \in [0, 1]\}$ and $\leq_{\mathbf{G}^*}$ operation on $\mathbf{G}^*$ defined by: $\mu \leq_{\mathbf{G}^*} \nu \Leftrightarrow \mu_1 \leq \nu_1, \mu_2 \geq \nu_2, \mu_3 \geq \nu_3$ for every $\mu, \nu \in \mathbf{G}^*$. Then $(\mathbf{G}^*, \leq_{\mathbf{G}^*})$ is a complete lattice.

Definition 2.1. [18] Let $\mathcal{N} \neq \emptyset$ and $E \subseteq \mathcal{N}$. A neutrosophic set (briefly NS) E in $\mathcal{N}$ is defined by three functions: truth-membership $\Phi_E(\mu)$, indeterminacy-membership $\Psi_E(\mu)$, and falsity-membership $\Upsilon_E(\mu)$. Then, a NS E can be described to be

$$E = \{(\mu, \Phi_E(\mu), \Psi_E(\mu), \Upsilon_E(\mu)) \mid \mu \in \mathcal{N}\} \quad (1)$$

where $\Phi_E(\mu), \Psi_E(\mu), \Upsilon_E(\mu) \in [0, 1]$ provided that

$$0 \leq \Phi_E(\mu) + \Psi_E(\mu) + \Upsilon_E(\mu) \leq 3 \quad (2)$$

Definition 2.2. [18] Assume that $E = \{(\mu, \Phi_E(\mu), \Psi_E(\mu), \Upsilon_E(\mu)) \mid \mu \in \mathcal{N}\}$ and $F = \{(\mu, \Phi_F(\mu), \Psi_F(\mu), \Upsilon_F(\mu)) \mid \mu \in \mathcal{N}\}$ are two NS in a set $\mathcal{N} \neq$

0. The initial type inclusion relation $\subseteq$ and its fundamental algebraic operations are delineated as follows:

- (i) $E \cup F = \{ \langle \mu, \max(\Phi_E(\mu), \Phi_F(\mu)), \min(\Psi_E(\mu), \Psi_F(\mu)), \min(\Upsilon_E(\mu), \Upsilon_F(\mu)) \rangle \mid \mu \in \mathcal{N} \}$;
- (ii) $E \cap F = \{ \langle \mu, \min(\Phi_E(\mu), \Phi_F(\mu)), \max(\Psi_E(\mu), \Psi_F(\mu)), \max(\Upsilon_E(\mu), \Upsilon_F(\mu)) \rangle \mid \mu \in \mathcal{N} \}$;
- (iii) $E \subseteq F \Leftrightarrow \Phi_E(\mu) \leq \Phi_F(\mu), \Psi_E(\mu) \geq \Psi_F(\mu), \Upsilon_E(\mu) \geq \Upsilon_F(\mu)$;
- (iv) $E^c = \{ \langle \mu, \Upsilon_E(\mu), 1 - \Upsilon_E(\mu), \Phi_E(\mu) \rangle \mid \mu \in \mathcal{N} \}$.

Definition 2.3. [18] Let $\star, \circ : [0, 1]^2 \rightarrow [0, 1]$ two binary operations on $[0, 1]$, then $\star$ is said to be a triangular norm (briefly, TN) if satisfies the following conditions:

- (i) commutative;
- (ii) associative;
- (iii) increasing;
- (iv) $a \star 1 = 1 \star a = a, \forall a \in [0, 1]$.

And $\circ$ is said to be a triangular co-norm (briefly, T-co-N) if satisfies the following conditions:

- (i) commutative;
- (ii) associative;
- (iii) increasing;
- (iv) $a \circ 0 = 0 \circ a = a, \forall a \in [0, 1]$.

Remark 2.1. Let $\mathcal{T}$ be a neutrosophic triangular norm (briefly, NTN) on $\mathbf{G}^*$. Then $\mathcal{T}$ is said to be a representable neutrosophic triangular norm (briefly, RNTN) if and only if there exist $\mathfrak{A}, \mathfrak{M}$ TN and T-co-N on interval $[0, 1]$ respectively, such that for any $\mathbf{a}, \mathbf{b} \in \mathbf{G}^*$

$$\mathcal{T}(\mathbf{a}, \mathbf{b}) = (\mathfrak{A}(\mathbf{a}_1, \mathbf{b}_1), \mathfrak{M}(\mathbf{a}_2, \mathbf{b}_2), \mathfrak{M}(\mathbf{a}_3, \mathbf{b}_3)).$$

Definition 2.4. [18] If a function $\mathcal{E} : \mathbf{G}^* \rightarrow \mathbf{G}^*$ fulfils the conditions stated below, it is hence known as a neutrosophic negator (briefly, N_{neg}):

1. $\mathcal{E}(0_{\mathbf{G}^*}) = 1_{\mathbf{G}^*}$;
2. $\mathcal{E}(1_{\mathbf{G}^*}) = 0_{\mathbf{G}^*}$;
3. $\mathcal{E}(\mathbf{a}) \geq_{\mathbf{G}^*} \mathcal{E}(\mathbf{b}), \forall \mathbf{a}, \mathbf{b} \in \mathbf{G}^*$ such that $\mathbf{a} \leq_{\mathbf{G}^*} \mathbf{b}$.

Furthermore, if $\mathcal{E}(\mathcal{E}(\mathbf{a})) = \mathbf{a}$ for all $\mathbf{a} \in \mathbf{G}^*$ then $\mathcal{E}$ is designated as an involutive N_{neg} .

Remark 2.2. [18] An involutive $N_{neg}, \mathcal{E}_s : \mathbf{G}^* \rightarrow \mathbf{G}^*$ is said to be a standard N_{neg} if $\mathcal{E}_s(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (\mathbf{a}_3, 1 - \mathbf{a}_2, \mathbf{a}_1)$ for all $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbf{G}^*$. Certainly, they $\mathcal{E}(\mathbf{a}) = (\mathbf{a}_3, 1 - \mathbf{a}_2, \mathbf{a}_1), \mathcal{E}(\mathbf{a}) = (\mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_1)$ are standard N_{negs} .

Definition 2.5. [15] Let $\Phi, \Psi, \Upsilon : \mathcal{N}^2 \times \mathbf{R} \rightarrow [0, 1]$ be fuzzy sets such that $\Phi(\mathbf{a}, \mathbf{b}, \varrho) + \Psi(\mathbf{a}, \mathbf{b}, \varrho) + \Upsilon(\mathbf{a}, \mathbf{b}, \varrho) \leq 3, \forall \mathbf{a}, \mathbf{b} \in \mathcal{N}$ and $\varrho > 0$. A neutrosophic inner product space (briefly, NIPS) is a triplet $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$, where $\mathcal{N}$ is a vector space over the real field, $\mathcal{T}$ is a continuous RNTN, and $\mathfrak{B}_{\Phi, \Psi, \Upsilon}$ is a neutrosophic set on $\mathcal{N}^2 \times \mathbf{R}$ fulfilling the following requirements for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{N}$ and $\varrho, \varsigma \in \mathbf{R}$:

(NIPS-1) $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, 0) = 0_{\mathbf{G}^*}$ and $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \varrho) >_{\mathbf{G}^*} 0_{\mathbf{G}^*}$ forevery $\varrho > 0$.

(NIPS-2) $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \varrho) = \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{b}, \mathbf{a}, \varrho)$.

(NIPS-3) $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \varrho) \neq \mathcal{V}(\varrho)$ for some $\varrho \in \mathbf{R} \Leftrightarrow \mathbf{a} \neq 0$, where $\mathcal{V}(\varrho) = \begin{cases} 1_{\mathbf{G}^*} & \varrho > 0 \\ 0_{\mathbf{G}^*} & \varrho \leq 0 \end{cases}$.

(NIPS-4) For any $\alpha \in \mathbf{R}$, $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\alpha \mathbf{a}, \mathbf{b}, \varrho) = \begin{cases} \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \frac{\varrho}{\alpha}) & \alpha > 0 \\ \mathcal{V}(\varrho) & \alpha = 0 \\ \mathcal{E}_s(\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \frac{\varrho}{\alpha})) & \alpha < 0 \end{cases}$

(NIPS-5) $\sup_{\varrho+\varsigma=\varrho} \{ \mathcal{T}(\mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathbf{a}, \mathbf{c}, \varrho), \mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathbf{b}, \mathbf{c}, \varsigma))) \} = \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a} + \mathbf{b}, \mathbf{c}, \varrho)$.

(NIPS-6) $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \cdot) : \mathbf{R} \rightarrow [0, 1]$ on $\mathbf{R} - \{0\}$, is continuous.

(NIPS-7) $\lim_{\varrho \rightarrow \infty} \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathbf{a}, \mathbf{b}, \varrho) = 1_{\mathbf{G}^*}$.

Remark 2.3. [15]

(i) The standard negator in this case $\mathcal{E}_s(\varrho) = 1 - \varrho, \forall \varrho \in [0, 1]$.

(ii) $0_{\mathbf{G}^*} = (0, 1, 1)$ and $1_{\mathbf{G}^*} = (1, 0, 0)$.

Definition 2.6. [15] Let $\{x_n\}$ be a sequence in probabilistic NIPS $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$. Then $\{x_n\}$ is called $\tau_{\mathfrak{B}}$ -convergent to $x \in \mathcal{N}$, If for any $\xi > 0$ and $0 < \varrho < 1, \exists N_0 = N_0(\xi, \varrho) \in \mathbf{Z}^+$ such that $\mathcal{B}_{\Phi, \Psi, \Upsilon}(x_n - x, \xi) >_{\mathbf{G}^*} \mathcal{E}_s(\varrho)$ whenever $n \geq N_0$.

Definition 2.7. [15] Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be probabilistic NIPS and let $\mathbb{T}$ be a continuous linear functional on $\mathcal{N}$. Then $\mathbb{T}$ is called $\tau_{\mathfrak{B}}$ -continuous if $x_n \xrightarrow{\tau_{\mathfrak{B}}} x \Rightarrow \mathbb{T}(x_n) \xrightarrow{\tau_{\mathfrak{B}}} \mathbb{T}(x)$ for any $\{x_n\}$ and $x \in \mathcal{N}$

Theorem 2.1. [15] Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a NIPS with a continuous RNTN $\mathcal{T}$ and for each $\mu, \nu \in \mathcal{N}$

$$\sup\{\varrho \in \mathbf{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\} < \infty.$$

and let $\langle \cdot, \cdot \rangle_{\Phi, \Psi, \Upsilon} : \mathcal{N} \times \mathcal{N} \longrightarrow R$ be defined as $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}$. Then $(\mathcal{N}, \langle \cdot, \cdot \rangle_{\Phi, \Psi, \Upsilon})$ is an inner product space, thereby rendering $(\mathcal{N}, \mathcal{P}_{\Phi, \Psi, \Upsilon})$ is a normed space, where $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = [\sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mu, \varrho) < 1\}]^{\frac{1}{2}}, \forall \mu \in \mathcal{N}$.

Definition 2.8. [15] Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a NIPS with a continuous RNTN $\mathcal{T}$ and for each $\mu, \nu \in \mathcal{N}$ $\sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\} < \infty$ such that $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = \left[\sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\} \right]^{\frac{1}{2}}$. We say that $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ is a neutrosophic Hilbert space (briefly, NHS) if $(\mathcal{N}, \mathcal{P}_{\Phi, \Psi, \Upsilon})$ is a complete normed space.

Theorem 2.2. [15] Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$. A sequence $\{x_n\}$ in $\mathcal{N}$ is $\tau_{\mathfrak{B}}$ -convergent to x ($x_n \xrightarrow{\tau_{\mathfrak{B}}} x$) if $x_n \xrightarrow{\mathcal{P}_{\Phi, \Psi, \Upsilon}} x$.

Theorem 2.3. [15] (Riesz's theorem) Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with $\tau_{\mathfrak{B}}$. If $\mathbb{T}$ is a linear functional on $\mathcal{N}$ that is continuous relative to $\tau_{\mathfrak{B}}$, then there exists exactly one element $\nu \in \mathcal{N}$ such that for all $\mu \in \mathcal{N}$, $\mathbb{T}(\mu) = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}$.

3 Neutrosophic Adjoint Operator, Neutrosophic Self-Adjoint Operator, and Their Properties

In this section, we introduce neutrosophic adjoint and neutrosophic self-adjoint linear operators on a neutrosophic Hilbert space. Their main definitions and fundamental properties are presented, establishing a framework that extends classical operator theory to the neutrosophic setting.

Theorem 3.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a NIPS with a continuous RNTN, $\mathcal{T}$ and $\sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\} < \infty$ for all $\mu, \nu \in \mathcal{N}$, then $\sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \mu) < 1\} = \sup\{\mu \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \mu) < 1\} + \sup\{\mu \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \mu) < 1\}, \forall \mu, \nu, \omega \in \mathcal{N}$.

Proof. By using definition (2.5) the condition (NIPS-5), for any $\gamma > 0$,

$$\begin{aligned} & \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \gamma) \\ & \geq \mathcal{T}\left(\mathfrak{B}_{\Phi, \Psi, \Upsilon}\left(\mu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \frac{\gamma}{2}\right), \right. \\ & \left. \mathfrak{B}_{\Phi, \Psi, \Upsilon}\left(\nu, \omega, \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \frac{\gamma}{2}\right)\right). \end{aligned} \quad (3)$$

This means that:

$$\begin{aligned} & \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \varrho) < 1\} \\ & \leq \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \varrho) < 1\} \\ & + \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \varrho) < 1\} + \gamma \end{aligned} \quad (4)$$

Since γ is arbitrary, it is an indication that

$$\begin{aligned} & \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \varrho) < 1\} \\ & \leq \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \varrho) < 1\} \\ & + \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \varrho) < 1\} \end{aligned} \quad (5)$$

Meanwhile, if we choose:

$$A = 1 - \mathcal{T}\left[\mathcal{E}_s(\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2})), \right. \quad (6)$$

$$\left. \mathcal{E}_s(\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2}))\right]$$

Using definition (2.5) the condition (NIPS-4) and (NIPS-5), we have

$$\begin{aligned} A &= 1 - \mathcal{T}\left[\mathfrak{B}_{\Phi, \Psi, \Upsilon}(-\mu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2}), \right. \\ & \left. \mathfrak{B}_{\Phi, \Psi, \Upsilon}(-\nu, \omega, \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2})\right] \\ & \geq \left(\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \gamma)\right) \end{aligned} \quad (7)$$

since $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2}) < 1$, $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \frac{\gamma}{2}) < 1$, Since $\mathcal{T}$ is RNTN and from definition (2.3), we have

$$\left(\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \langle \mu, \omega \rangle_{\Phi, \Psi, \Upsilon} + \langle \nu, \omega \rangle_{\Phi, \Psi, \Upsilon} - \gamma)\right) < 1. \quad (8)$$

And this indicates that

$$\begin{aligned} & \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \varrho) < 1\} \\ & \geq \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \varrho) < 1\} \\ & + \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \varrho) < 1\}. \end{aligned} \quad (9)$$

From equation (5) and equation (9) we get,

$$\begin{aligned} & \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu + \nu, \omega, \varrho) < 1\} \\ & = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \varrho) < 1\} \\ & + \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, \omega, \varrho) < 1\} \end{aligned} \quad (10)$$

□

Remark 3.1. The set of all neutrosophic continuous (bounded) linear operators on $\mathcal{N}$ is denoted by $NBO(\mathcal{N})$.

Definition 3.1. (Neutrosophic Adjoint Operator in NHS)

Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space and let $\mathcal{K} \in NBO(\mathcal{N})$. Then the neutrosophic adjoint operator $\mathcal{K}^* \in NBO(\mathcal{N})$ such that $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu, \nu \in \mathcal{N}$.

Theorem 3.2. (Existence)

The neutrosophic adjoint operator $\mathcal{K}^*$ of $\mathcal{K}$ in definition (3.1) is unique, exists, and is a bounded linear operator.

Proof. We choose $\nu \in \mathcal{N}$, define $T_\nu : \mathcal{N} \rightarrow \mathbb{R}$ by $T_\nu(\mu) = \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} \quad \forall \mu \in \mathcal{N}$ such that

$$T_\nu(\mu + \omega) = T_\nu(\mu) + T_\nu(\omega) \quad (11)$$

$T_\nu(\delta\mu) = \delta T_\nu(\mu), \forall \mu, \nu, \omega \in \mathcal{N}$ and α scalar in $\mathbb{R}$. Furthermore, T_ν is $\tau_{\mathfrak{B}}$ -continuous. Hence by Riesz theorem (2.3), there exists a unique $\omega_\nu \in \mathcal{N}$ such that $T_\nu(\mu) = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \omega, \varrho) < 1\}$. Define $\mathcal{K}^* : \mathcal{N} \rightarrow \mathbb{R}$ such that $\mathcal{K}_\nu^* = \omega_\nu, \quad \forall \nu \in \mathcal{N}$.

Thus, $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = T_\nu(\mu) = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}^*\nu, \varrho) < 1\}$.

Now we prove it $\mathcal{K}^*$ is a linear map on $\mathcal{N}$.

Let $\nu, \omega \in \mathcal{N}$ and δ, γ are scalars, then

$$\begin{aligned} \langle \mu, \mathcal{K}^*(\delta\nu + \gamma\omega) \rangle_{\Phi, \Psi, \Upsilon} &= \langle \mathcal{K}\mu, \delta\nu + \gamma\omega \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \delta\nu + \gamma\omega, \varrho) < 1\}. \end{aligned} \quad (12)$$

According theorem (3.1) and the condition (NIPS-4), we have

$$\begin{aligned} \langle \mu, \mathcal{K}^*(\delta\nu + \gamma\omega) \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \delta\nu, \varrho) < 1\} \\ &+ \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \gamma\omega, \varrho) < 1\}, \\ &= \delta \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} \\ &+ \gamma \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \omega, \varrho) < 1\}, \\ &= \delta \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} + \gamma \langle \mathcal{K}\mu, \omega \rangle_{\Phi, \Psi, \Upsilon}, \\ &= \delta \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} + \gamma \langle \mu, \mathcal{K}^*\omega \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (13)$$

Finally, we prove the uniqueness of $\mathcal{K}^*$, let $\mathcal{K}_1^*, \mathcal{K}_2^*$ be two neutrosophic adjoint operators for $\mathcal{K} \in NBO(\mathcal{N})$

$$\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}_1^*\nu \rangle_{\Phi, \Psi, \Upsilon} \quad (14)$$

$$\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}_2^*\nu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu, \nu \in \mathcal{N} \quad (15)$$

$$\implies \langle \mu, \mathcal{K}_1^*\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}_2^*\nu \rangle_{\Phi, \Psi, \Upsilon} \quad (16)$$

$$\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}_1^*\nu, \varrho) < 1\} \quad (17)$$

$$= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}_2^*\nu, \varrho) < 1\} \quad (18)$$

$$\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}_1^*\nu - \mathcal{K}_2^*\nu, \varrho) < 1\} = 0 \quad (19)$$

$$\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}_1^*\nu - \mathcal{K}_2^*\nu, \varrho) = 0 \iff \mathcal{K}_1^*\nu - \mathcal{K}_2^*\nu = 0. \quad (20)$$

$$(\mathcal{K}_1^* - \mathcal{K}_2^*)\nu = 0 \forall \nu \in \mathcal{N} \quad (21)$$

$$\mathcal{K}_1^* = \mathcal{K}_2^* \quad (22)$$

Therefore, $\mathcal{K}^*$ is unique. $\square$

Definition 3.2. (Neutrosophic Self-Adjoint Operator in NHS)

Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in NBO(\mathcal{N})$, then $\mathcal{K}$ is a neutrosophic self-adjoint operator, if $\mathcal{K} = \mathcal{K}^*$ where $\mathcal{K}^*$ is a neutrosophic adjoint operator of $\mathcal{K}$.

Theorem 3.3. Every self-adjoint operator is a neutrosophic self-adjoint operator, but the converse is not true in general.

Proof. Let $\mathcal{K}$ is self-adjoint operator with respect to $\langle \cdot, \cdot \rangle$.

Then, $\langle \mathcal{K}\mu, \nu \rangle = \langle \mu, \mathcal{K}\nu \rangle$.

$$\begin{aligned} \Phi(\mathcal{K}\mu, \nu, \varrho) &= \Phi(\mu, \mathcal{K}\nu, \varrho) = \begin{cases} 1 & \varrho > 0 \\ 0 & \varrho \leq 0 \end{cases}, \\ \Psi(\mathcal{K}\mu, \nu, \varrho) &= \Psi(\mu, \mathcal{K}\nu, \varrho) = \begin{cases} 0 & \varrho > 0 \\ 1 & \varrho \leq 0 \end{cases} \end{aligned} \quad (23)$$

And

$$\Upsilon(\mathcal{K}\mu, \nu, \varrho) = \Upsilon(\mu, \mathcal{K}\nu, \varrho) = \begin{cases} 0 & \varrho > 0 \\ 1 & \varrho \leq 0 \end{cases} \quad (24)$$

$$\begin{aligned} \implies \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) &= \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) \\ &= \begin{cases} 1 = (1, 0, 0), & \varrho > 0 \\ 0 = (0, 1, 1), & \varrho \leq 0 \end{cases} \end{aligned} \quad (25)$$

$$\begin{aligned} \implies \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\} = 0 \end{aligned} \quad (26)$$

Then $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon}$

Hence $\mathcal{K}$ is neutrosophic self-adjoint operator. $\square$

Example 3.1. Let $\mathcal{N} = \mathbb{R}^2$ and inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2, \quad \forall \mu = (x_1, y_1), \nu = (x_2, y_2) \in \mathcal{N}$. Define fuzzy set $\Phi, \Psi, \Upsilon : \mathcal{N} \times \mathcal{N} \times \mathbb{R} \rightarrow [0, 1]$ so that

$$\begin{aligned} \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) &= (\phi(\mu, \nu, \varrho), \psi(\mu, \nu, \varrho), \Upsilon(\mu, \nu, \varrho)) \\ &= \begin{cases} (\frac{\varrho^{1/2}}{\varrho^{1/2} + |\langle \mu, \nu \rangle|^{1/2}}, \frac{|\langle \mu, \nu \rangle|^{1/2}}{\varrho^{1/2} + |\langle \mu, \nu \rangle|^{1/2}}, \frac{|\langle \mu, \nu \rangle|^{1/2}}{\varrho^{1/2}}) & \varrho > 0 \\ 0 = (0, 1, 1), & \varrho \leq 0 \end{cases} \end{aligned} \quad (27)$$

$$\text{And } \mathcal{K} : \mathcal{N} \rightarrow \mathcal{N} \text{ by } \mathcal{K} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\mu = (1, 0), \nu = (0, 1)$$

Since $\langle \mathcal{K}\mu, \nu \rangle = -1$ and $\langle \mu, \mathcal{K}\nu \rangle = 1$

Thus, $\mathcal{K}$ is not self-adjoint.

We must show that $\mathcal{K}$ is neutrosophic self-adjoint operator for all $\varrho \in \mathbb{R}$.

Let $\mu = (x_1, y_1)$, $\nu = (x_2, y_2)$
 $\langle \mathcal{K}\mu, \nu \rangle = y_1x_2 - x_1y_2$ and $\langle \mu, \mathcal{K}\nu \rangle = x_1y_2 - y_1x_2$
 $= -(y_1x_2 - x_1y_2)$.
 $\Rightarrow |\langle \mathcal{K}\mu, \nu \rangle| = |\langle \mu, \mathcal{K}\nu \rangle| = m, \forall \mu, \nu \in \mathcal{N}$
 $\Rightarrow \Phi(\mathcal{K}\mu, \nu, \varrho) = \Phi(\mu, \mathcal{K}\nu, \varrho) = \frac{\sqrt{\varrho}}{\sqrt{\varrho+m}},$
 $\psi(\mathcal{K}\mu, \nu, \varrho) = \psi(\mu, \mathcal{K}\nu, \varrho) = \frac{m}{\sqrt{\varrho+m}},$ and
 $\psi(\mathcal{K}\mu, \nu, \varrho) = \psi(\mu, \mathcal{K}\nu, \varrho) = \frac{m}{\sqrt{\varrho}},$
for all $\varrho \in R$
 $\Rightarrow \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) = \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho),$
for all $\varrho \in R$.
 $\Rightarrow \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\}$
 $= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}.$
 $\Rightarrow \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon}.$
Hence $\mathcal{K}$ is neutrosophic self-adjoint.

A Comparative Structural Study of Classical and Neutrosophic Self-Adjoint Operators

(1) Nature of the Mathematical Space: Dimensionality and Measurement

- In the Classical Framework: When asserting the self-adjoint property $\langle \mathbf{T}\mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{p}, \mathbf{T}\mathbf{q} \rangle$, the evaluation relies on a single-valued scalar. The underlying symmetry is deterministic and binary; the condition holds if and only if the scalar values on both sides achieve exact numerical identity.

- In the Neutrosophic Framework: Symmetry transcends a solitary scalar representation, manifesting instead as a systemic state of stability evaluated across three continuous dimensions: truth-membership (Φ), indeterminacy-membership (Ψ), and falsity-membership (Υ). Consequently, even when algebraic symmetry is satisfied, the neutrosophic space characterizes the "degree of acceptance and validity" of this symmetry under the influence of the parametric variable ϱ .

(2) The Role of the Scalar Parameter (ϱ)

- In the Classical Framework: A self-adjoint operator remains invariant to external or environmental parameters, as the property is purely geometric and static within the traditional Hilbert space.

- In the Neutrosophic Framework: The integration of the parameter ϱ dictates that a self-adjoint operator must sustain its symmetry across all dynamic thresholds of ϱ . This implies that the operator is self-adjoint under varying levels of uncertainty or measurement scales, thereby presenting a more comprehensive and generalized paradigm than the rigid, static classical concept.

(3) Mathematical Interpretation of Non-Self-Adjointness

- Classically: If $\langle \mathcal{K}\mu, \nu \rangle \neq \langle \mu, \mathcal{K}\nu \rangle$, the geometric symmetry is immediately violated, classifying the operator strictly as non-self-adjoint without further structural qualification.

- Neutrosophically: If the exact algebraic symmetry

falters, the neutrosophic framework does not merely reject the operator. Instead, it quantifies the structural deviation. The operator provides a tripartite analytical characterization:

(i) Truth-degree of symmetry (Φ): Quantifies the proximity of the operator to achieving ideal self-adjointness.

(ii) Indeterminacy-degree of symmetry (Ψ): Measures the randomness, vagueness, or epistemic uncertainty induced by the absence of perfect symmetry at that specific point.

(iii) Falsity-degree of symmetry (Υ): Quantifies the explicit deviation from the idealized self-adjoint state.

Theorem 3.4. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in \mathcal{NBO}(\mathcal{N})$, then $\mathcal{K}$ is a neutrosophic self-adjoint operator if and only if $\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\}$

$= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}.$

Proof. Suppose that $\mathcal{K}$ is neutrosophic self-adjoint operator, this means that $\mathcal{K} = \mathcal{K}^*$.

$$\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} \quad (28)$$

$$= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}^*\nu, \varrho) < 1\} \quad (29)$$

(since $\mathcal{K} = \mathcal{K}^*$)

$$= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}. \quad (30)$$

Assume that

$$\begin{aligned} \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} \\ = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N} \end{aligned} \quad (31)$$

Therefore,

$$\begin{aligned} \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} \\ = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\} \\ = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N} \end{aligned} \quad (32)$$

$$\begin{aligned} \Rightarrow \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} \\ - \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mu, \nu, \varrho) < 1\} = 0 \quad (33) \\ \Rightarrow \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathcal{K} - \mathcal{K}^*)\mu, \nu, \varrho) < 1\} = 0 \end{aligned} \quad (34)$$

$$\begin{aligned} \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathcal{K} - \mathcal{K}^*)\mu, (\mathcal{K} - \mathcal{K}^*)\mu, \varrho) < 1\} \\ = 0 \end{aligned} \quad (35)$$

By using the condition (NIPS - 3), we get

$$\mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathcal{K} - \mathcal{K}^*)\mu, (\mathcal{K} - \mathcal{K}^*)\mu, \varrho) = 0, \forall \mu \in \mathcal{N}, \quad (36)$$

$$\Rightarrow (\mathcal{K} - \mathcal{K}^*)\mu = 0 \quad (37)$$

$$\Rightarrow \mathcal{K} = \mathcal{K}^*. \quad (38)$$

Hence, $\mathcal{K}$ is neutrosophic self-adjoint operator.

□

Corollary 3.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in NBO(\mathcal{N})$. If $\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$, then $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*, \varrho)$.

Proof. Since $\sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\} = \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}\nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$. From theorem (3.4), we get $\mathcal{K} = \mathcal{K}^*$ and $\mathcal{K}\mu = \mathcal{K}^*\mu, \forall \mu \in \mathcal{N}$.

Consequently,

$$\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\mu, \varrho) = \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mu, \mathcal{K}^*\mu, \varrho), \quad (39)$$

$$\Rightarrow \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\mu, \varrho) < 1\} \quad (40)$$

$$= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mu, \mathcal{K}^*\mu, \varrho) < 1\}, \quad (41)$$

$$\Rightarrow \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mathcal{K}^*\mu, \mathcal{K}^*\mu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu \in \mathcal{N}, \quad (42)$$

$$\text{Then } \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mu, \varrho), \forall \mu \in \mathcal{N}$$

Using definition of neutrosophic operator norm

$$\begin{aligned} \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) &= \sup_{\mu \neq 0} \left\{ \frac{\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho)}{\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)} \right\} \\ &= \sup_{\mu \in \mathcal{N}} \{ \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) : \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = 1 \} \end{aligned} \quad (43)$$

Taking supremum over all $\mu \in \mathcal{N}$ with $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = 1$, we obtain $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*, \varrho)$. □

Theorem 3.5. (Properties of neutrosophic adjoint operators)

Assume that $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ is a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$, and assume $\mathcal{K}^*, \mathcal{L}^*$, are the neutrosophic adjoint operators of $\mathcal{K}, \mathcal{L} \in NBO(\mathcal{N})$ respectively, then

$$(i) (\mathcal{K}^*)^* = \mathcal{K}.$$

$$(ii) (\delta \mathcal{K})^* = \delta \mathcal{K}^*.$$

$$(iii) (\delta \mathcal{K} + \gamma \mathcal{L})^* = \delta \mathcal{K}^* + \gamma \mathcal{L}^* \text{ Where } \delta, \gamma \text{ are scalars and } \mathcal{L} \in NBO(\mathcal{N}).$$

$$(iv) (\mathcal{K}\mathcal{L})^* = \mathcal{L}^*\mathcal{K}^*.$$

Proof. (i) By using the definition (2.5) the condition

(NIPS-2) and remark (3.1), we obtain

$$\begin{aligned} \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{K}^*\nu, \varrho) < 1\} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\nu, \mu, \varrho) < 1\} \\ &= \langle \mathcal{K}^*\nu, \mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \nu, (\mathcal{K}^*)^*\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\nu, (\mathcal{K}^*)^*\mu, \varrho) < 1\}, \\ &= \sup\{x \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}((\mathcal{K}^*)^*\mu, \nu, \varrho) < 1\}, \\ &= \langle (\mathcal{K}^*)^*\mu, \nu \rangle_{\Phi, \Psi, \Upsilon}, \\ &= \langle (\mathcal{K} - (\mathcal{K}^*)^*)\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \end{aligned} \quad (44)$$

$$\Rightarrow (\mathcal{K} - (\mathcal{K}^*)^*)\mu = 0, \forall \mu \in \mathcal{N} \quad (45)$$

$$\Rightarrow \mathcal{K} = (\mathcal{K}^*)^*. \quad (46)$$

(ii) From theorem (3.1), remark(3.1), definition (2.5) (NIPS-2, NIPS-4), and linearity of operators, we have

$$\begin{aligned} \langle \delta \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, (\delta \mathcal{K})^*\nu, \varrho) < 1\} \\ &= \langle \mu, (\delta \mathcal{K})^*\nu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (47)$$

And

$$\begin{aligned} \langle \delta \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\delta \mathcal{K}\mu, \nu, \varrho) < 1\} \\ &= \delta \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \nu, \varrho) < 1\}, \\ &= \delta \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \delta \sup\{\varrho \in \mathbb{R} : G(\mu, \mathcal{K}^*\nu, \varrho) < 1\}, \\ &= \delta \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (48)$$

From equation(47) and equation(48), we have

$$\begin{aligned} \langle \mu, (\delta \mathcal{K})^*\nu \rangle_{\Phi, \Psi, \Upsilon} - \delta \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} &= 0, \\ \langle \mu, ((\delta \mathcal{K})^* - \delta \mathcal{K}^*)\nu \rangle_{\Phi, \Psi, \Upsilon} &= 0, \quad \forall \mu, \nu \in \mathcal{N}, \\ ((\delta \mathcal{K})^* - \delta \mathcal{K}^*)\nu &= 0, \quad \forall \nu \in \mathcal{N}. \end{aligned}$$

$$(\delta \mathcal{K})^* = \delta \mathcal{K}^* \quad (49)$$

(iii) From theorem (3.1), remark (3.1), NIPS-2, NIPS-4 and linearity of operators, we have

$$\begin{aligned} \langle (\delta \mathcal{K} + \gamma \mathcal{L})\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, (\delta \mathcal{K} + \gamma \mathcal{L})^*\nu, \varrho) < 1\}, \\ &= \langle \mu, (\delta \mathcal{K} + \gamma \mathcal{L})^*\nu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (50)$$

And

$$\begin{aligned} \langle (\delta \mathcal{K} + \gamma \mathcal{L})\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}((\delta \mathcal{K} + \gamma \mathcal{L})\mu, \nu, \varrho) < 1\}, \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\delta \mathcal{K}\mu, \nu, \varrho) < 1\} \\ &\quad + \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\gamma \mathcal{L}\mu, \nu, \varrho) < 1\}, \\ &= \langle \delta \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} + \langle \gamma \mathcal{L}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon}, \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, (\delta \mathcal{K})^*\nu, \varrho) < 1\} \\ &\quad + \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, (\gamma \mathcal{L})^*\nu, \varrho) < 1\}, \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, ((\delta \mathcal{K})^* + (\gamma \mathcal{L})^*)\nu, \varrho) < 1\} \\ &= \langle \mu, ((\delta \mathcal{K})^* + (\gamma \mathcal{L})^*)\nu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (51)$$

From equation (50) and equation (51), we have

$$\langle \mu, (\delta\mathcal{K} + \gamma\mathcal{L})^* \rangle_{\Phi, \Psi, \Upsilon} - \langle \mu, ((\delta\mathcal{K})^* + (\gamma\mathcal{L})^*)\nu \rangle_{\Phi, \Psi} = 0 \quad (52)$$

$$\langle \mu, ((\delta\mathcal{K} + \gamma\mathcal{L})^* - ((\delta\mathcal{K})^* + (\gamma\mathcal{L})^*))\nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N} \quad (53)$$

$$((\delta\mathcal{K} + \gamma\mathcal{L})^* - ((\delta\mathcal{K})^* + (\gamma\mathcal{L})^*))\nu = 0, \forall \nu \in \mathcal{N} \quad (54)$$

$$(\delta\mathcal{K} + \gamma\mathcal{L})^* = (\delta\mathcal{K})^* + (\gamma\mathcal{L})^* \quad (55)$$

From equation (49) and equation (55) we have:

$$(\delta\mathcal{K} + \gamma\mathcal{L})^* = \delta\mathcal{K}^* + \gamma\mathcal{L}^* \quad (56)$$

(iv) By theorem (3.1), remark (3.1), NIPS-2 and linearity of operators, we get

$$\begin{aligned} \langle \mathcal{K}\mathcal{L}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, (\mathcal{K}\mathcal{L})^*\nu, \varrho) < 1\} \\ &= \langle \mu, (\mathcal{K}\mathcal{L})^*\nu \rangle_{\Phi, \Psi, \Upsilon} \end{aligned} \quad (57)$$

$$\begin{aligned} \langle \mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} &= \langle \mathcal{L}\mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{L}\mu, \mathcal{K}^*\nu, \varrho) < 1\} \\ &= \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mathcal{L}^*\mathcal{K}^*\nu, \varrho) < 1\} \\ &= \langle \mu, \mathcal{L}^*\mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} \end{aligned} \quad (58)$$

From equation (57) and equation (58), we have:

$$\langle \mu, (\mathcal{K}\mathcal{L})^*\nu \rangle_{\Phi, \Psi, \Upsilon} - \langle \mu, \mathcal{L}^*\mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon} = 0 \quad (59)$$

$$\langle \mu, ((\mathcal{K}\mathcal{L})^* - (\mathcal{L}^*\mathcal{K}^*))\nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N} \quad (60)$$

$$((\mathcal{K}\mathcal{L})^* - (\mathcal{L}^*\mathcal{K}^*))\nu = 0, \forall \mu, \nu \in \mathcal{N} \quad (61)$$

$$(\mathcal{K}\mathcal{L})^* = \mathcal{L}^*\mathcal{K}^*. \quad (62)$$

□

4 Neutrosophic Unitary Operator and its Properties

In this section, we explore the concept of neutrosophic unitary operators in neutrosophic Hilbert spaces. We present the fundamental definitions, theorems, and intrinsic properties of these operators.

Lemma 4.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$, and let $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K} = 0$ iff $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N}$.

Proof. Suppose that $\mathcal{K} = 0$ this means that $\mathcal{K}\mu = 0, \forall \mu \in \mathcal{N}$.

This implies $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle 0, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N}$.

Conversely, $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N}$ implies $\mathcal{K}\mu = 0, \forall \mu \in \mathcal{N}$.

$\mathcal{K} = 0$.

□

Remark 4.1. In what follows, we work on the complexification of the underlying real space and use the corresponding complex extension of the real inner product $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon}$ induced by the neutrosophic inner product space $\mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho)$. This setting will be assumed throughout the subsequent development.

Theorem 4.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K} = 0$ iff $\langle \mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu \in \mathcal{N}$.

Proof. Suppose that $\mathcal{K} = 0$, we get $\langle \mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu \in \mathcal{N}$.

Conversely, showing that $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu, \nu \in \mathcal{N}$, is sufficient. The following easily confirmed that identity is necessary for the confirmation of this.

$$\begin{aligned} \langle \mathcal{K}(\delta\mu + \gamma\nu), \delta\mu + \gamma\nu \rangle_{\Phi, \Psi, \Upsilon} - |\delta|^2 \langle \mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \\ - |\gamma|^2 \langle \mathcal{K}\nu, \nu \rangle_{\Phi, \Psi, \Upsilon} \\ = \delta\bar{\gamma} \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} + \bar{\delta}\gamma \langle \mathcal{K}\nu, \mu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (63)$$

According to the hypothesis, for any δ and γ , equation (63) holds and equals 0.

If we place $\delta = 1$ and $\gamma = 1$,

$$\text{then } \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}\nu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0. \quad (64)$$

And if we place $\delta = i$ and $\gamma = 1$,

$$\text{then } i\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} - i\langle \mathcal{K}\nu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0. \quad (65)$$

When we divide equation (65) by i and add the result to equation (64), we obtain

$$\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}\nu, \mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} - \langle \mathcal{K}\nu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0.$$

Therefore, $2\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0$, so $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = 0$. Using lemma (4.1), we obtain $\mathcal{K} = 0$.

□

Definition 4.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K}$ is a neutrosophic normal operator if its neutrosophic adjoint commutes with it; this means that $\mathcal{K}\mathcal{K}^* = \mathcal{K}^*\mathcal{K}$.

Definition 4.2. (Neutrosophic Unitary Operator in NHS)

Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K}$ is a neutrosophic unitary operator if it fulfills $\mathcal{K}\mathcal{K}^* = \mathcal{K}^*\mathcal{K} = I$.

Remark 4.2. (i) The arrangement $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ is composed of a real vector space $\mathcal{N}$, a neutrosophic set $\mathfrak{B}_{\Phi, \Psi, \Upsilon}$ on $\mathcal{N}^2 \times R$, and a representable neutrosophic t -norm $\mathcal{T}$ is a representable neutrosophic t -norm for neutrosophic value combination.

(i) All neutrosophic unitary operators are obviously neutrosophic normal.

(ii) A neutrosophic unitary operator can also be defined as follows: An isomorphism of a neutrosophic Hilbert space onto itself is termed a neutrosophic unitary operator.

Theorem 4.2. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K}$ is a neutrosophic unitary operator if and only if $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho), \forall \mu \in \mathcal{N}$ and $\mathcal{K}$ is surjective. In that case, $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho), \forall \mu \in \mathcal{N}$ and $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}, \varrho) = 1$.

Proof. Suppose that

$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$ and $\mathcal{K}$ is surjective. For $\mu \in \mathcal{N}$ we obtain,

$$\begin{aligned} & \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}\mu, \varrho) - \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) \\ &= \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} - \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon}, \\ &= \sup\{\varrho \in R : F(\mathcal{K}\mu, \mathcal{K}\mu, \varrho) < 1\} \\ &\quad - \sup\{\varrho \in R : F(\mu, \mu, \varrho) < 1\}, \\ &= \sup\{\varrho \in R : F(\mathcal{K}^*\mathcal{K}\mu, \mu, \varrho) < 1\} \\ &\quad - \sup\{\varrho \in R : F(\mu, \mu, \varrho) < 1\}, \\ &= \langle \mathcal{K}^*\mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} - \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon}, \\ &= \langle (\mathcal{K}^*\mathcal{K} - I)\mu, \mu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (66)$$

Since $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$, then $\langle (\mathcal{K}^*\mathcal{K} - I)\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0, \forall \mu \in \mathcal{N}$.

By using theorem (4.1), we have

$\mathcal{K}^*\mathcal{K} - I = 0$ which implies $\mathcal{K}^*\mathcal{K} = I$.

Therefore, $\mathcal{K}^*\mathcal{K} = I$ and $\mathcal{K}$ is surjective.

Hence, $\mathcal{K}^*\mathcal{K} = I$ and $\mathcal{K}$ is bijective

(i.e. $\mathcal{K}\mathcal{K}^{-1} = \mathcal{K}^{-1}\mathcal{K} = I$).

Now, $\mathcal{K}\mathcal{K}^* = (\mathcal{K}\mathcal{K}^*) \cdot I$

$= (\mathcal{K}\mathcal{K}^*) \cdot (\mathcal{K}\mathcal{K}^{-1})$

$= \mathcal{K} \cdot (\mathcal{K}^*\mathcal{K}) \cdot \mathcal{K}^{-1}$

$= \mathcal{K}\mathcal{K}^{-1} = I$.

Then $\mathcal{K}\mathcal{K}^* = \mathcal{K}^*\mathcal{K} = I$.

Hence $\mathcal{K}$ is neutrosophic unitary operator.

Conversely, $\mathcal{K}$ is a neutrosophic unitary operator.

Then, $\mathcal{K}\mathcal{K}^* = I$ and $\mathcal{K}^* = \mathcal{K}^{-1}$.

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}\mu, \varrho) = \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} \quad (67)$$

$$= \langle \mathcal{K}^*\mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \quad (68)$$

$$= \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho),$$

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho). \quad (69)$$

To prove $\mathcal{K}$ is surjective.

Let $\nu \in \mathcal{N}$ such that $\mu = \mathcal{K}^*\nu$.

$\mathcal{K}\mu = \mathcal{K}(\mathcal{K}^*\nu) = \mathcal{K}\mathcal{K}^*\nu = I\nu = \nu$.

Then, $\mathcal{K}$ is surjective.

It remains prove that $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$ and

$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}, \varrho) = 1, \forall \mu \in \mathcal{N}$.

Let $\nu = \mathcal{K}^{-1}\mu$, we have $\mu = \mathcal{K}\nu$.

Since $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho), \forall \mu \in \mathcal{N}$, we obtain

$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\nu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\nu, \varrho)$.

$\Rightarrow \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}\mu, \varrho)$.

Since $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$.

Using definition of neutrosophic operator norm

$$\begin{aligned} \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) &= \sup_{\mu \neq 0} \left\{ \frac{\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho)}{\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)} \right\} \\ &= \sup_{\mu \in \mathcal{N}} \{ \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) : \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = 1 \}. \end{aligned}$$

Taking supremum over all

$\mu \in \mathcal{N}$ with $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) = 1$, we obtain

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{-1}, \varrho) = 1. \quad (70)$$

□

Theorem 4.3. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space and $\mathcal{K}_1, \mathcal{K}_2$ two neutrosophic unitary operators on $\mathcal{N}$. Then $\mathcal{K}_1\mathcal{K}_2$ is a neutrosophic unitary operator.

Proof. Let $\mathcal{K}_1, \mathcal{K}_2$ are two neutrosophic unitary operators, we prove that $\mathcal{K}_1\mathcal{K}_2$. $(\mathcal{K}_1\mathcal{K}_2)^*\mathcal{K}_1\mathcal{K}_2$

$$= \mathcal{K}_2^*(\mathcal{K}_1^*\mathcal{K}_1)\mathcal{K}_2$$

$$= \mathcal{K}_2^*\mathcal{K}_2 = I.$$

Similarly, $\mathcal{K}_1\mathcal{K}_2(\mathcal{K}_1\mathcal{K}_2)^*$

$$= \mathcal{K}_1(\mathcal{K}_2\mathcal{K}_2^*)\mathcal{K}_1^*$$

$$= \mathcal{K}_1\mathcal{K}_1^*$$

$$= I.$$

Then $\mathcal{K}_1\mathcal{K}_2$ is a neutrosophic unitary operator. □

Theorem 4.4. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space and $\mathcal{K}_1, \mathcal{K}_2$ two neutrosophic unitary operators on $\mathcal{N}$. Then $\mathcal{K}_1 + \mathcal{K}_2$ is a neutrosophic unitary operator if and only if it is surjective and

$$Re\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} = -\frac{1}{2}, \forall \mu \in \mathcal{N}$$

$$\text{with } \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}_i, \varrho) = 1, \forall i = 1, 2.$$

Proof. Assume that $\mathcal{K}_1 + \mathcal{K}_2$ is surjective and $Re\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} = -\frac{1}{2}, \forall \mu \in \mathcal{N}$ with

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}_i, \varrho) = 1, \forall i = 1, 2.$$

From theorem (4.2), we prove that

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}((\mathcal{K}_1 + \mathcal{K}_2)\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) \quad (71)$$

$$\begin{aligned} & \mathcal{P}_{\Phi, \Psi, \Upsilon}^2((\mathcal{K}_1 + \mathcal{K}_2)\mu, \varrho) \\ &= \langle (\mathcal{K}_1 + \mathcal{K}_2)\mu, (\mathcal{K}_1 + \mathcal{K}_2)\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \langle \mathcal{K}_1\mu, \mathcal{K}_1\mu + \langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &+ \langle \mathcal{K}_2\mu, \mathcal{K}_1\mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}_2\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} \rangle \\ &= \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}_1\mu, \varrho) + \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}_2\mu, \varrho) \\ &+ 2\text{Re}\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &(\text{since } \mathcal{K}_1, \mathcal{K}_2 \text{ are two neutrosophic unitary}) \\ &= 1 + 1 + 2(-\frac{1}{2}) = 1. \end{aligned} \quad (72)$$

Since $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}_i, \varrho) = 1, \forall i = 1, 2$
and $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}_i\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho), \forall i = 1, 2$.
Therefore, $\mathcal{P}_{\Phi, \Psi, \Upsilon}((\mathcal{K}_1 + \mathcal{K}_2)\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$.
Then $\mathcal{K}_1 + \mathcal{K}_2$ is neutrosophic unitary operator.
Conversely, suppose that $\mathcal{K}_1 + \mathcal{K}_2$ is a neutrosophic unitary operator.
we have $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}_1 + \mathcal{K}_2)\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho),$
 $\forall \mu \in \mathcal{N}$ and $\mathcal{K}$ is surjective with $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho) = 1$.
It remains only to prove that $\text{Re}\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} = -\frac{1}{2}, \forall \mu \in \mathcal{N}$.

$$\begin{aligned} & \mathcal{P}_{\Phi, \Psi, \Upsilon}^2((\mathcal{K}_1 + \mathcal{K}_2)\mu, \varrho) \\ &= \langle (\mathcal{K}_1 + \mathcal{K}_2)\mu, (\mathcal{K}_1 + \mathcal{K}_2)\mu \rangle_{\Phi, \Psi, \Upsilon}, \quad (73) \\ & \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) = \langle \mathcal{K}_1\mu, \mathcal{K}_1\mu + \langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &+ \langle \mathcal{K}_2\mu, \mathcal{K}_1\mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}_2\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} \rangle, \quad (74) \\ & \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \quad (75) \\ &+ 2\text{Re}\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon}, (\text{since } \mathcal{K}_1, \mathcal{K}_2 \text{ neu. unitary}), \\ & \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 2\langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} + 2\text{Re}\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon}. \end{aligned} \quad (76)$$

$$\text{Then } \text{Re}\langle \mathcal{K}_1\mu, \mathcal{K}_2\mu \rangle_{\Phi, \Psi, \Upsilon} = -\frac{1}{2}, \forall \mu \in \mathcal{N}. \quad \square$$

Theorem 4.5. Let $\mathcal{K} \in NBO(\mathcal{N})$ is a neutrosophic unitary operator on $\mathcal{N}$, then the following conditions are all equivalent:

- (i) $\mathcal{K}\mathcal{K}^* = I$,
- (ii) $\langle \mathcal{K}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu, \nu \in \mathcal{N}$,
- (iii) $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho), \forall \mu \in \mathcal{N}$.

Proof. Let $\mathcal{K} \in NBO(\mathcal{N})$ is a neutrosophic unitary operator on $\mathcal{N}$.

Suppose that (i) is true.

$$\begin{aligned} & \text{Then } \langle \mathcal{K}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\nu, \varrho) < 1 \}, \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mathcal{K}\mu, \nu, \varrho) < 1 \}, \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1 \}, \quad (77) \\ & \langle \mathcal{K}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu, \nu \in \mathcal{N}. \end{aligned} \quad (78)$$

Thus (i) implies (ii).

Now we suppose that (ii) is true and by taking $\mu = \nu$, we have

$$\begin{aligned} & \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\mu, \varrho) < 1 \}, \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mathcal{K}\mu, \mu, \varrho) < 1 \}, \\ &= \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mu, \varrho) < 1 \}, \quad (79) \\ & \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \text{ for all } \mu \in \mathcal{N} \end{aligned} \quad (80)$$

Therefore $\mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho)$
and $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$.

Thus (ii) implies (iii).

Finally, we must prove that (iii) leads to (i).

Suppose that (iii) is true, then

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) \quad (81)$$

$$\Rightarrow \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) \quad (82)$$

$$\Rightarrow \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon}, \quad (83)$$

$$\Rightarrow \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\mu, \varrho) < 1 \} \\ = \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mu, \varrho) < 1 \}, \quad (84)$$

$$\Rightarrow \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^*\mathcal{K}\mu, \mu, \varrho) < 1 \} \\ = \sup \{ \varrho \in R \mid \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \mu, \varrho) < 1 \}, \quad (85)$$

$$\Rightarrow \langle \mathcal{K}^*\mathcal{K}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon}, \quad (86)$$

$$\Rightarrow \langle (\mathcal{K}^*\mathcal{K} - I)\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} = 0, \quad (87)$$

$$\Rightarrow \mathcal{K}^*\mathcal{K} - I = 0, \quad (88)$$

$$\Rightarrow \mathcal{K}^*\mathcal{K} = I. \quad (89)$$

Thus, (iii) $\Rightarrow$ (i).

Hence the conditions (i),(ii) and (iii) are equivalent. $\square$

Theorem 4.6. Every unitary operator is a neutrosophic unitary operator, but the converse is not true in general.

Proof. The proof of this theorem can be established by following the same argument used in the proof of theorem (3.3). $\square$

Example 4.1. Let $\mathcal{N} = R^2$ and inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2, \forall \mu = (x_1, y_1), \nu = (x_2, y_2) \in \mathcal{N}$. Define fuzzy set $\Phi, \Psi, \Upsilon : \mathcal{N} \times \mathcal{N} \times R \rightarrow [0, 1]$ so that

$$\begin{aligned} \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) &= (\phi(\mu, \nu, \varrho), \psi(\mu, \nu, \varrho), \Upsilon(\mu, \nu, \varrho)) \\ &= \begin{cases} \left(\frac{\varrho^{1/2}}{\varrho^{1/2} + |\langle \mu, \nu \rangle|^{1/2}}, \frac{|\langle \mu, \nu \rangle|^{1/2}}{\varrho^{1/2} + |\langle \mu, \nu \rangle|^{1/2}}, \frac{|\langle \mu, \nu \rangle|^{1/2}}{\varrho^{1/2}} \right) \varrho > 0 \\ 0 = (0, 1, 1), & \varrho \leq 0 \end{cases} \end{aligned} \quad (90)$$

$$\text{And } \mathcal{K} : \mathcal{N} \rightarrow \mathcal{N} \text{ by } \mathcal{K} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},$$

$$\mu = (1, 0)$$

$$\text{Since } \langle \mu, \mu \rangle = 1 \text{ and } \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle = 4$$

$$\Rightarrow \langle \mathcal{K}\mu, \mathcal{K}\mu \rangle \neq \langle \mu, \mu \rangle$$

Thus, $\mathcal{K}$ is not unitary operator.

We must show that $\mathcal{K}$ is neutrosophic self-adjoint operator for all $\varrho \in R$.

$$\text{Let } \mu = (x_1, y_1), \nu = (x_2, y_2)$$

$$\Rightarrow \langle \mu, \nu \rangle = x_1x_2 + y_1y_2 \text{ and } \langle \mathcal{K}\mu, \mathcal{K}\nu \rangle = 4(x_1x_2 + y_1y_2)$$

$$\Rightarrow |\langle \mathcal{K}\mu, \mathcal{K}\nu \rangle| = 4|\langle \mu, \nu \rangle|$$

$$\Rightarrow \Phi(\mathcal{K}\mu, \mathcal{K}\nu, \varrho) = \frac{\sqrt{\varrho}}{\sqrt{\varrho} + \sqrt{4|\langle \mathcal{K}\mu, \mathcal{K}\nu \rangle|}} =$$

$$\frac{\sqrt{\varrho}}{\sqrt{\varrho} + 2\sqrt{|\langle \mu, \nu \rangle|}} = \frac{\sqrt{\frac{\varrho}{4}}}{\sqrt{\frac{\varrho}{4}} + \sqrt{|\langle \mu, \nu \rangle|}} = \Phi(\mu, \nu, \eta), \quad \eta = \frac{\varrho}{4}$$

Similarly with respect to ψ, Υ .

$$\Rightarrow \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\nu, \varrho) = \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \eta),$$

for all $\varrho, \eta \in R$.

Since ϱ arbitrary in R , we have η arbitrary in R .

$$\Rightarrow \sup\{\varrho \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \mathcal{K}\nu, \varrho) < 1\} = \sup\{\eta \in \mathbb{R} : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \eta) < 1\}.$$

$$\Rightarrow \langle \mathcal{K}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon}.$$

Hence $\mathcal{K}$ is neutrosophic unitary operator.

Definition 4.3. Let $\mathcal{N}$ and $\mathcal{M}$ are neutrosophic normed linear spaces. A neutrosophic isometric isomorphism of $\mathcal{N}$ into $\mathcal{M}$ is a one-to-one linear transformation $\mathcal{K}$ of $\mathcal{N}$ into $\mathcal{M}$ such that $\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}\mu, \varrho) = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$ for all $\mu \in \mathcal{N}$.

Theorem 4.7. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in NBO(\mathcal{N})$. Then $\mathcal{K}$ is a neutrosophic unitary operator if and only if it is a neutrosophic isometric isomorphism of $\mathcal{N}$ onto itself.

Proof. Let $\mathcal{K}$ is a neutrosophic unitary operator; then it is onto.

By using theorem (4.2), it preserves norms.

Then $\mathcal{K}$ is a neutrosophic isometric isomorphism of $\mathcal{N}$ onto itself.

Conversely, let $\mathcal{K}$ is a neutrosophic isometric isomorphism of $\mathcal{N}$ onto itself, then $\mathcal{K}^{-1}$ exists. By using theorem (4.2), we have $\mathcal{K}^*\mathcal{K} = I$.

Now, multiplying on both sides by $\mathcal{K}^{-1}$

$$(\mathcal{K}^*\mathcal{K})\mathcal{K}^{-1} = I\mathcal{K}^{-1} \quad (91)$$

$$\mathcal{K}^* = \mathcal{K}^{-1} \quad (92)$$

$$\mathcal{K}\mathcal{K}^* = I = \mathcal{K}^*\mathcal{K} \quad (93)$$

Then $\mathcal{K}$ is neutrosophic unitary operator. $\square$

Remark 4.3. Let $\mathcal{K} \in NBO(\mathcal{N})$. Since $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon}, \forall \mu, \nu \in \mathcal{N}$, As we can see

(i) $\mathcal{K}$ is a neutrosophic normal operator if and only if $\langle \mathcal{K}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mathcal{K}^*\mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon}$.

(ii) $\mathcal{K}$ is a neutrosophic unitary operator if and only if $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mathcal{K}^*\mu, \mathcal{K}^*\nu \rangle_{\Phi, \Psi, \Upsilon}$.

(iii) $\mathcal{K}$ is a neutrosophic self-adjoint operator if and only if $\langle \mathcal{K}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon}$.

We admit that the Neutrosophic Inner Product $\langle \cdot, \cdot \rangle_{\Phi, \Psi, \Upsilon}$ on $\mathcal{N}$ describes the geometry of $\mathcal{N}$. Consequently, an operator $\mathcal{K}$ is characterized by a neutrosophic normal operator if $\mathcal{K}$ and $\mathcal{K}^*$ transforms the geometry $\mathcal{N}$ in the same way. It is a neutrosophic unitary operator if neither $\mathcal{K}$ nor $\mathcal{K}^*$ alters the geometry of the Hilbert space $\mathcal{N}$. Because of this, a neutrosophic unitary operator is referred to as a neutrosophic Hilbert space isomorphism.

Theorem 4.8. Let $\mathcal{N}$ be a vector space over the complex field $\mathbb{C}$. If $\mathcal{K} \in NBO(\mathcal{N})$, then there are unique neutrosophic self-adjoint operators $\mathcal{G}$ and $\mathcal{H}$ on $\mathcal{N}$ such that $\mathcal{K} = \mathcal{G} + i\mathcal{H}$. Moreover, $\mathcal{K}$ is a neutrosophic normal operator if and only if $\mathcal{G}\mathcal{H} = \mathcal{G}\mathcal{H}$, neutrosophic unitary if and only if $\mathcal{G}\mathcal{H} = \mathcal{G}\mathcal{H}$ and $\mathcal{G}^2 + \mathcal{H}^2 = I$ and $\mathcal{K}$ is a neutrosophic self-adjoint operator if and only if $\mathcal{H} = 0$.

$$\text{Proof. Let } \mathcal{G} = \frac{\mathcal{K} + \mathcal{K}^*}{2} \text{ and } \mathcal{H} = \frac{\mathcal{K} - \mathcal{K}^*}{2i}$$

Then $\mathcal{G}$ and $\mathcal{H}$ are neutrosophic self-adjoint operators and $\mathcal{K} = \mathcal{G} + i\mathcal{H}$.

We prove uniqueness, let $\mathcal{K} = \mathcal{G}_1 + i\mathcal{H}_2$ where $\mathcal{G}_1$ and $\mathcal{H}_2$ are neutrosophic self-adjoint operators, $\mathcal{K}^* = \mathcal{G}_1 - i\mathcal{H}_2$, so that $\mathcal{G}_1 = \frac{\mathcal{K} + \mathcal{K}^*}{2}$ and $\mathcal{H}_2 = \frac{\mathcal{K} - \mathcal{K}^*}{2i}$. Thus $\mathcal{G}_1$ and $\mathcal{H}_2$ are unique.

Now $\mathcal{K}$ is neutrosophic normal if and only if $(\mathcal{G} - i\mathcal{H})(\mathcal{G} + i\mathcal{H}) = \mathcal{K}^*\mathcal{K} = \mathcal{K}\mathcal{K}^* = (\mathcal{G} + i\mathcal{H})(\mathcal{G} - i\mathcal{H})$ $\mathcal{G}\mathcal{H} = \mathcal{G}\mathcal{H}$.

Similarly, $\mathcal{K}$ is neutrosophic unitary if and only if $\mathcal{K}\mathcal{K}^* = I = \mathcal{K}^*\mathcal{K}$

$$(\mathcal{G} + i\mathcal{H})(\mathcal{G} - i\mathcal{H}) = I = (\mathcal{G} - i\mathcal{H})(\mathcal{G} + i\mathcal{H}).$$

$$(\mathcal{G}^2 + \mathcal{H}^2) + i(\mathcal{H}\mathcal{G} - \mathcal{G}\mathcal{H}) = I = (\mathcal{G}^2 + \mathcal{H}^2) - i(\mathcal{H}\mathcal{G} - \mathcal{G}\mathcal{H}).$$

Therefore, $\mathcal{G}^2 + \mathcal{H}^2 = I$ and $(\mathcal{H}\mathcal{G} - \mathcal{G}\mathcal{H}) = 0$.

Finally, $\mathcal{K}$ is a neutrosophic self-adjoint operator if and only if $\mathcal{G} + i\mathcal{H} = \mathcal{G} - i\mathcal{H}$.

Then $\mathcal{H} = 0$. $\square$

5 An Application of Neutrosophic self-adjoint operators to the solution of linear operator equations via Fixed-Point Techniques

This section is devoted to the study of operator equations involving neutrosophic self-adjoint operators. We focus on the equation $(I + \mathcal{K}^p)\mu = \mathfrak{f}$ and derive conditions ensuring the existence and uniqueness of its solution, where $\mathcal{K} : \mathcal{N} \rightarrow \mathcal{N}$ is neutrosophic self-adjoint, $p \in \mathbb{N}$, and $\mathfrak{f} \in \mathcal{N}$. Clearly, If $\mathcal{K}$ is neutrosophic self-adjoint, then $I + \mathcal{K}^p$ for all $p \geq 1$.

Theorem 5.1. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}$, $\forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K} \in NBO(\mathcal{N})$ be a neutrosophic self-adjoint operator.

(i) The equation $\mu + \mathcal{K}^{2q}\mu = \mathfrak{f}$ has a unique solution in $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 1$).

(ii) If in addition A is positive, then the equation $\mu + \mathcal{K}^{2q+1}\mu = \mathfrak{f}$ has a unique solution in $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 0$).

Proof. (i) The equation $\mu + \mathcal{K}^{2q}\mu = \mathfrak{f}$ can be equivalently written as $(I + \mathcal{K}^{2q})\mu = \mathfrak{f}$ where I is the neutrosophic identity of $\mathcal{N}$.

We have $\mathcal{P}_{\Phi, \Psi, \Upsilon}((I + \mathcal{K}^{2q})\mu, \varrho) \leq \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) + \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{2q}\mu, \varrho) \leq (1 + \mathcal{P}_{\Phi, \Psi, \Upsilon}^{2k}(\mathcal{K}\mu, \varrho)) \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$. and

$$\langle (I + \mathcal{K}^{2q})\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \quad (94)$$

$$\begin{aligned} &= \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}^{2q}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) + \langle \mathcal{K}^q\mu, \mathcal{K}^q\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) + \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mathcal{K}^q\mu, \varrho) \\ &\geq \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho), \quad \text{for all } \mu \in \mathcal{N}. \end{aligned} \quad (95)$$

We take, $\alpha = \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}, \varrho)$ therefore, $\mathcal{P}_{\Phi, \Psi, \Upsilon}((I + \mathcal{K}^{2q})\mu, \varrho) \leq (1 + \alpha^{2q})\mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho)$ for all $\mu \in \mathcal{N}$. For $\gamma > 0$ we consider now the operator $S_\gamma : \mathcal{N} \rightarrow \mathcal{N}$ defined by $S_\gamma\mu = \mu - \gamma((I + \mathcal{K}^{2q})\mu - \mathfrak{f})$.

We obtain

$$\begin{aligned} &\mathcal{P}_{\Phi, \Psi, \Upsilon}^2(S_\gamma\mu - S_\gamma\nu, \varrho) \\ &= \langle S_\gamma\mu - S_\gamma\nu, S_\gamma\mu - S_\gamma\nu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \langle (\mu - \nu) - \gamma(I + \mathcal{K}^{2q})(\mu - \nu), (\mu - \nu) \\ &\quad - \gamma(I + \mathcal{K}^{2q})(\mu - \nu) \rangle_{\Phi, \Psi, \Upsilon} \\ &= \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu - \nu, \varrho) - 2\gamma \langle (I + \mathcal{K}^{2q})(\mu - \nu), \mu - \nu \rangle \\ &\quad + \gamma^2 \mathcal{P}_{\Phi, \Psi, \Upsilon}^2((I + \mathcal{K}^{2q})(\mu - \nu), \varrho) \\ &\leq (1 - 2\gamma + \gamma^2(1 + \alpha^{2q})^2) \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu - \nu, \varrho). \end{aligned} \quad (96)$$

Thus,

$$\mathcal{P}_{\Phi, \Psi, \Upsilon}(S_\gamma\mu - S_\gamma\nu, \varrho) \leq \sqrt{1 - 2\gamma + \gamma^2(1 + \alpha^{2q})^2} \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu - \nu, \varrho) \text{ for all } \mu, \nu \in \mathcal{N}$$

As we note, if $\gamma \in (0, \frac{2}{(1 + \alpha^{2q})^2})$, then S_γ is a contraction and, due to the Banach fixed point theorem, it results that S_γ has a unique fixed point $z \in \mathcal{N}$. Now z is the unique solution of the equation $\mu + \mathcal{K}^{2q}\mu = \mathfrak{f}$.

To prove (ii) we use that

$$\begin{aligned} &\mathcal{P}_{\Phi, \Psi, \Upsilon}((I + \mathcal{K}^{2q+1})\mu, \varrho) \\ &\leq \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho) + \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mathcal{K}^{2q+1}\mu, \varrho) \\ &\leq (1 + \mathcal{P}_{\Phi, \Psi, \Upsilon}^{2q+1}(\mathcal{K}\mu, \varrho)) \mathcal{P}_{\Phi, \Psi, \Upsilon}(\mu, \varrho). \end{aligned} \quad (97)$$

and

$$\begin{aligned} &\langle (I + \mathcal{K}^{2q+1})\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \langle \mu, \mu \rangle_{\Phi, \Psi, \Upsilon} + \langle \mathcal{K}^{2q+1}\mu, \mu \rangle_{\Phi, \Psi, \Upsilon} \\ &= \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho) + \langle \mathcal{K}(\mathcal{K}^q\mu), \mathcal{K}^q\mu \rangle_{\Phi, \Psi, \Upsilon} \\ &\geq \mathcal{P}_{\Phi, \Psi, \Upsilon}^2(\mu, \varrho), \end{aligned} \quad (98)$$

Because of the positivity of $\mathcal{K}$, Now the operator S_γ is defined as:

$$S_\gamma\mu = \mu - \gamma((I + \mathcal{K}^{2q+1})\mu - \mathfrak{f}), \quad (99)$$

the contraction constant is

$$\sqrt{1 - 2\gamma + \gamma^2(1 + \alpha^{2q+1})^2} \quad (100)$$

with $\gamma \in (0, \frac{2}{(1 + \alpha^{2q+1})^2})$, and we employ the same type of logic as in (i). Thus, the theorem proof is finished. $\square$

Theorem 5.2. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}$, $\forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K}, \mathcal{M} \in NBO(\mathcal{N})$ be two neutrosophic self-adjoint operators.

- (i) The equation $\mu + (\mathcal{K} + \mathcal{M})^{2q}\mu = \mathfrak{f}$ has a unique solution in $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 1$).
- (ii) If in addition $\mathcal{K} + \mathcal{M}$ is positive, then the equation $\mu + (\mathcal{K} + \mathcal{M})^{2q+1}\mu = \mathfrak{f}$ has a unique solution in $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 0$).

Proof. Clearly $\mathcal{K} + \mathcal{M}$ is neutrosophic self-adjoint, and it is sufficient to apply theorem (5.1). $\square$

Theorem 5.3. Let $(\mathcal{N}, \mathfrak{B}_{\Phi, \Psi, \Upsilon}, \mathcal{T})$ be a neutrosophic Hilbert space with IP: $\langle \mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \sup\{\varrho \in R : \mathfrak{B}_{\Phi, \Psi, \Upsilon}(\mu, \nu, \varrho) < 1\}, \forall \mu, \nu \in \mathcal{N}$ and let $\mathcal{K}, \mathcal{M} \in NBO(\mathcal{N})$ be two neutrosophic self-adjoint operators with $\mathcal{K}\mathcal{M} = \mathcal{M}\mathcal{K}$.

- (i) The equation $\mu + (\mathcal{K}\mathcal{M})^{2q}\mu = \mathfrak{f}$ has a unique solution in $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 1$).
- (ii) If in addition $\mathcal{K}\mathcal{M}$ is positive, then the equation $\mu + (\mathcal{K}\mathcal{M})^{2q+1}\mu = \mathfrak{f}$ has a unique solution $\mathcal{N}$ for all $\mathfrak{f} \in \mathcal{N}$ ($q \geq 0$).

Proof. It is sufficient to demonstrate that $\mathcal{K}\mathcal{M}$ is a neutrosophic self-adjoint operator, according to theorem (5.1). In fact, we have $\langle \mathcal{K}\mathcal{M}\mu, \nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mathcal{M}\mu, \mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{M}\mathcal{K}\nu \rangle_{\Phi, \Psi, \Upsilon} = \langle \mu, \mathcal{K}\mathcal{M}\nu \rangle_{\Phi, \Psi, \Upsilon}$, for all $\mu, \nu \in \mathcal{N}$. $\square$

6 Conclusion

In this paper, we have presented a comprehensive structural study of linear operators within the newly introduced framework of neutrosophic Hilbert spaces (NHS). This framework successfully generalizes the classical theory of linear operators, which is shown to be a restrictive, idealized limiting case that occurs only when indeterminacy and falsity are entirely eliminated and the parametric variable ϱ is held static. By transitioning from traditional single-valued scalar evaluations and deterministic geometric relations to a multi-dimensional neutrosophic environment, the characterization of general linear operators has been fundamentally elevated. Under this new paradigm, the behavior, boundedness, and structural properties of an operator are evaluated simultaneously across three continuous dimensions: truth-membership (Φ), indeterminacy-membership (Ψ), and falsity-membership (Υ). This structural approach provides a "degree of validity and certainty" inherent to the space, allowing the operator's characteristics to be dynamically assessed across varying thresholds of the scalar parameter ϱ . Furthermore, this framework offers a deeper mathematical interpretation when operational identities or algebraic relations falter. Rather than strictly rejecting an operator's relevance due to a

lack of exact numerical equality—as is typical in the binary classical framework—the neutrosophic framework provides a robust, tripartite analytical characterization that quantifies the nature of the deviation. It explicitly measures the proximity to ideal algebraic or geometric conditions through the truth-degree (Φ), evaluates the randomness or epistemic uncertainty induced within the system via the indeterminacy-degree (Ψ), and captures the explicit operational error using the falsity-degree (Υ). Specifically, as investigated throughout this paper, we have focused on the structural behavior of specific classes of operators, including adjoint, self-adjoint, and unitary operators within the neutrosophic context. We have rigorously established and proven their foundational algebraic and geometric structural properties, demonstrating how their classical definitions extend harmoniously into the neutrosophic domain while preserving their core identities under uncertainty. In addition to these theoretical developments, we have provided a practical mathematical application by utilizing the neutrosophic adjoint operator to solve systems of linear equations. This was achieved by developing a dedicated fixed-point iteration methodology adapted specifically to the metric and topological structures of the neutrosophic Hilbert space. Ultimately, analyzing operators within this framework guarantees balanced stability across the system's constituent dimensions (Φ, Ψ, Υ), opening new avenues for modeling real-world physical and mathematical transformations governed by inherent indeterminacy.

It is worth emphasizing the relationship between the present work and our previous studies. In earlier work, we established the foundational framework of neutrosophic inner product spaces and neutrosophic Hilbert spaces. In particular, we introduced the notions of convergence, completeness, continuity, and orthogonality, and established a Riesz representation theorem in the setting of neutrosophic Hilbert spaces. Building upon that foundation, the current paper focuses on operator theory in the neutrosophic setting by introducing neutrosophic adjoint, self-adjoint, and unitary operators and investigating their fundamental algebraic and structural properties. Thus, the present results represent a natural continuation and further development of the previously established neutrosophic Hilbert space framework.

References:

- [1] L. Zadeh. Fuzzy sets, *Information and control*, vol. 8, no. 3, pp. 338-353, 1965. DOI:[http://dx.doi.org/10.1016/S0019-9958\(65\)90241-X](http://dx.doi.org/10.1016/S0019-9958(65)90241-X).

- [2] K. T. Atanassov, Intuitionistic fuzzy sets, *Intuitionistic fuzzy sets: theory and applications*, vol. 20, no. 1, pp. 87-96, 1986. DOI: [http://dx.doi.org/10.1016/S0165-0114\(86\)80034-3](http://dx.doi.org/10.1016/S0165-0114(86)80034-3).
- [3] F. Smarandache, A unifying field in logics: neutrosophic logic, neutrosophy, neutrosophic set, neutrosophic probability and statistics, *Ann Arbor, Michigan, USA*, ISBN. 9781879585638, American Research Press, 1998. <https://fs.unm.edu/eBook-Neutrosophics6.pdf>. [Access Date:01/01/2026].
- [4] R. Biswas, Fuzzy inner product spaces and fuzzy norm functions. *Information Sciences*, vol. 53, no.(1-2), pp. 185-190, 1991. DOI: [https://doi.org/10.1016/0020-0255\(91\)90063-Z](https://doi.org/10.1016/0020-0255(91)90063-Z).
- [5] S. Chang and J. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, *Bulletin of the american mathematical society*, vol. 86, pp. 429-436, 1994. DOI: <https://www.sid.ir/paper/543370/en>. [Access Date:01/01/2026].
- [6] J. K. Kohli and R. Kumar, On fuzzy inner product spaces and fuzzy co-inner product spaces, *Fuzzy sets and systems*, vol. 53, no.2, pp.227-232, 1993. [https://doi.org/10.1016/0165-0114\(93\)90177-J](https://doi.org/10.1016/0165-0114(93)90177-J).
- [7] P. Majumdar and S.K. Samanta, On fuzzy inner product spaces, *Journal of Fuzzy Mathematics*, vol. 16, no. 2, pp. 377, 2008. https://www.researchgate.net/publication/230886068_On_Fuzzy_Inner_Product_Spaces. [Access Date:01/01/2026].
- [8] M. Goudarzi and S.M. Vaezpour, On the definition of fuzzy Hilbert spaces and its application, *Journal of Nonlinear Sciences and its Applications*, vol. 2, no. 1, pp. 46-59, 2009. DOI: <http://dx.doi.org/10.22436/jnsa.002.01.07>.
- [9] P. Majumdar and S.K. Samanta, On intuitionistic fuzzy inner product spaces, *Journal of fuzzy Mathematics*, vol. 19, no. 1, pp.115-124, 2011. https://www.researchgate.net/publication/230886070_Intuitionistic_Fuzzy_Inner_Product_Spaces. [Access Date:01/01/2026].
- [10] A Radharamani, S Maheswari, and A Brindha, Intuitionistic fuzzy hilbert space and some properties, *IOSR Journal of Engineering (IOSRJEN)*, vol. 8, no. 9, pp. 15-21, 2018. https://iosrjen.org/Papers/vol8_issue9/Version-1/C0809011521.pdf. [Access Date:01/01/2026].
- [11] M. Kirisci and Ne. simsek, Neutrosophic normed spaces and statistical convergence, *The Journal of Analysis*, vol. 28, no. 4, pp. 1059-1073, 2020. DOI: <https://doi.org/10.1007/s41478-020-00234-0>.
- [12] T. Bera and N. K. Mahapatra, On neutrosophic soft linear spaces, *Fuzzy Information and Engineering*, vol. 9, no. 3, pp. 299-324, 2017. DOI: <https://doi.org/10.1016/j.fiae.2017.09.004>.
- [13] T. Bera and N. K. Mahapatra, Neutrosophic soft normed linear spaces, *Neutrosophic Sets and Systems: An International Book Series in Information Science and Engineering*, vol. 23, no. 1 page. 52, 2018. https://digitalrepository.unm.edu/nss_journal/vol23/iss1/6. [Access Date:01/01/2026].
- [14] S. Omran and A. Elrawy, Generalized t-norms, t-co-norms, and neutrosophic inner product spaces: A comprehensive overview, *SVU-International Journal of Basic Sciences*, vol. 2, no. 1, pp. 1-9, 2025. DOI: <https://doi.org/10.21608/svuijbs.2024.315222.1000>.
- [15] K. B. Tarish and H. A. Hassan, Neutrosophic inner product space and the construction of neutrosophic Hilbert space with application, Accepted in *Nonlinear Function Analysis and Applications*, vol. 31, no. 3, pp.905-924, 2026. <http://nfaa.kyungnam.ac.kr/journal-nfaa/index.php/NFAA/article/view/2495/1540>. [Access Date:31/08/2026].
- [16] S. M. Rasheed, Self-adjoint fuzzy operator in fuzzy hilbert space and its properties, *Sulaimani Journal for Pure and Applied Sciences*, vol. 19, no. 1, pp.233-238, 2017. DOI: <https://doi.org/10.17656/jzs.10601>.
- [17] E. A. Huzamm, A. A. Alwan, W. K. Ghafil, A. E. Hashoos, Existence and uniqueness of solutions to hemi equilibrium problem with application, *Journal of Advanced Research in Dynamical and Control Systems Open source preview*, vol. 11, (1 special issue), pp. 1754-1758, 2019. <https://www.researchgate.net/publication/334001130>. [Access Date:01/01/2026].
- [18] Q. Hu and X. Zhang, Neutrosophic triangular norms and their derived residuated lattices, *Symmetry*, vol. 11, no. 6, pp. 817, 2019. DOI: <https://doi.org/10.3390/sym11060817>.

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