

Steiner conics and quadratic Bézier curves with control mass points

LIONEL GARNIER¹, LUCIE DRUOTON¹, JEAN-PAUL BÉCAR²

¹L.I.B., University of Burgundy, B.P. 47870,
21078 Dijon Cedex,
FRANCE

²U.P.H.F. - Campus Mont Houy – 5
9313 Valenciennes Cedex 9,
FRANCE

Abstract: The authors propose a new approach of the Steiner problem. The Steiner problem consists in constructing a conic tangent to three sides of a triangle—possibly extended sides-. The method in use is based on Bézier curves with control mass points. Bézier curves with control mass points generalize classical rational Bézier curves by introducing control vectors in addition to control points. Here, any quadratic rational Bézier curves with three control mass points model conic arcs, semi-ellipses, and hyperbola branches while preserving rational parameterizations with any homographic parameter change. This method simplifies the construction and avoids implicit conic equations, solving systems of five linear equations in five unknowns, or reducing the implicit form of a conic. All conic features can be obtained by reparametrizations.

Key-Words: Triangle, Steiner conic, Bézier curves, mass points, homographic parameter change.

Received: May 5, 2026. **Revised:** June 6, 2026. **Accepted:** July 8, 2026. **Published:** September 30, 2026.

1 Introduction

Polynomial Bézier curves are a fundamental tool in geometric modeling and computer-aided design (CAD). Introduced by Pierre Bézier and Paul de Faget de Casteljau in the 1960s, they provide an intuitive way to define parametric shapes using a set of control points [1],[2],[3],[4],[5],[6],[7],[8],[9],[10]. These curves are widely used in various fields, including graphic design, robotics, and simulation. When these curves are of degree 2, they represent bounded parabolic arcs [11],[12].

Rational Bézier curves of degree 2 extend the concept of quadratic polynomial Bézier curves by introducing weights associated with the control points [13],[14],[15],[16]. These curves can be viewed as loci of barycenters of weighted points [17], where the weights are the product of a nonzero real number with a Bernstein polynomial of degree 2. These Bézier curves allow for accurate modeling of complex shapes such as conic arcs [13],[14],[15],[18],[19],[20],[21]. However, modeling semi-ellipses or hyperbola branches is not possible using this model [15].

To generalize the notion of barycenter, the set of mass points—points with nonzero scalar weights or zero-weight vectors—was introduced in [22],[23],[24],[25]. This space is an affine space associated with a vector space whose addition and scalar multiplication are defined in Appendix A. This model enables the modeling of semi-ellipses and branches of hyperbolas.

Let ABC be a non-degenerate triangle. An extended side of the triangle is the line defined by one of its sides. Steiner problem [26],[27] consists in constructing a conic tangent to three sides of triangle ABC , where some of the sides may be extended, leading to:

- an ellipse if the conic is tangent to all three sides of triangle ABC . In the standard case, the ellipse is tangent to the triangle at the midpoints of its sides, whereas in the generalized case, the conic is tangent to the three sides at points that are not known in advance, see Figure 3(a) in the Appendix, 3(b) in the Appendix and 3(c) in the Appendix;
- a parabola if it is tangent to one side and to the two extended sides of triangle ABC , see Figure 4(a) in the Appendix;
- a hyperbola if it is tangent to one side of triangle ABC and the two other extended sides are asymptotes of the hyperbola, see Figure 4(b) in the Appendix.

Regarding Steiner ellipse $\mathcal{E}$, the construction of five points on $\mathcal{E}$ is relatively simple. The next steps are:

- 1) determining the implicit equation of $\mathcal{E}$,
- 2) reducing this implicit equation,
- 3) computing the characteristic elements of ellipse $\mathcal{E}$.

A simple and direct definition of an arc of ellipse $\mathcal{E}$ is allowed by our method, using a rational Bézier curve with weights 1, $\frac{1}{2}$, and 1.

The construction of Steiner parabola $\mathcal{P}$ uses a point F on the circumcircle of triangle ABC , along with its reflections across the side of the triangle. These three reflected points are collinear and lie on line $\mathcal{D}$, known as Steiner's line. The parabola $\mathcal{P}$ is then defined by the focus F and the directrix $\mathcal{D}$. Unlike our method, which uses polynomial Bézier curves, this approach does not permit for control over the point of tangency between parabola $\mathcal{P}$ and the triangle's side.

In the case of the hyperbola, four of its points can be easily constructed using symmetry properties. The fifth point is more difficult to determine. As with the ellipse, the steps involving the implicit equation and the calculation of characteristic elements require long and unstable computations. The mass point control structure can be directly obtained by our method.

In all three cases of proper conics, the characteristic elements of the conic are determined using homographic reparametrizations [28],[?]. Moreover, using a quadratic parameter change, the modeling of any conic section by a quartic Bézier curve with mass point controls is possible [29].

In Section 2, standard constructions of Steiner conics are recalled. A review of quadratic Bézier curves is provided in Section 3. The construction of Steiner conics using quadratic Bézier curves is detailed in Section 4. Following the conclusion and future directions, Appendix A recalls the definition of the space $\vec{\mathcal{P}}$ of mass points based on the usual Euclidean affine plane $\mathcal{P}$ with associated vector space $\vec{\mathcal{P}}$. Appendix B provides a review of homographic parameter changes. Appendices C, D, E, and F contain the proofs of the theorems and corollaries given in Section 4.

2 Usual constructions of Steiner conics

Let ABC be a non-degenerate triangle. The segment $[XY]$, which is one side of the triangle ABC , defines an extended side, which is the line (XY) .

2.1 Steiner Ellipse

Let A_1 be the midpoint of $[BC]$, B_1 the midpoint of $[AC]$, and C_1 the midpoint of $[AB]$.

The Steiner ellipse $\mathcal{E}$ is the ellipse passing through A_1 , B_1 , and C_1 , and tangent at these points to the lines (BC) , (AC) , and (AB) , respectively. The center Ω of the ellipse $\mathcal{E}$ is the centroid of the points A , B , and C .

Let A_2 and B_2 be the symmetric points of A_1 and B_1 with respect to Ω , respectively. The ellipse $\mathcal{E}$ is the conic passing through the points A_1 , B_1 , C_1 , A_2 , and B_2 .

2.2 Generalized Steiner Ellipse

It is possible to generalize the problem of determining an ellipse tangent to the three sides of a non-degenerate triangle: the lines joining each vertex to a point on the opposite side must be concurrent, see https://fr.wikipedia.org/wiki/Ellipse_de_Steiner.

Theorem 1 (Generalization of the Steiner Ellipse)
Let ABC be a non-degenerate triangle. Let:

$$A_1 \in]BC[\quad B_1 \in]AC[\quad C_1 \in]AB[$$

The ellipse passing through A_1 , tangent to line (BC) , through B_1 , tangent to (AC) , and through C_1 , tangent to (AB) , exists if and only if the lines (AA_1) , (BB_1) , and (CC_1) are concurrent.

2.3 Steiner Parabola

Let F be a point on the circumcircle $\mathcal{C}$ of triangle ABC . Let A_1 , B_1 , and C_1 be the reflections of F with respect to the extended sides (BC) , (AC) , and (AB) , respectively. The points A_1 , B_1 , and C_1 are collinear, and this line, denoted $\mathcal{D}$, is called the Steiner line. Note that the orthocenter H of triangle ABC also lies on the line $\mathcal{D}$.

The arc $\widehat{AB}$ is the arc of $\mathcal{C}$ with endpoints A and B not containing the point C . The arcs $\widehat{AC}$ and $\widehat{BC}$ are defined similarly.

The parabola $\mathcal{P}$, with focus F and directrix $\mathcal{D}$, is a parabola tangent to one side and two extended sides. The non-extended side of triangle ABC is defined by the arc that contains point F . Thus, if

$F \in \widehat{AB}$, then the non-extended side is $[AB]$.

Unfortunately, controlling the point of tangency between the parabola $\mathcal{P}$ and the non-extended side of triangle ABC is very difficult.

2.4 Steiner Hyperbola

The extended sides of triangle ABC are $[AB]$ and $[AC]$. Let D_0 be the midpoint of the side $[BC]$.

- 1) Let Δ_F be the interior angle bisector of lines (AB) and (AC) . Let D_1 be the reflection of D_0 across the line Δ_F .
- 2) Let D_2 (resp. D_3) be the reflection of D_0 (resp. D_1) with respect to the center of the hyperbola A .

- 3) The points D_0, D_1, D_2 , and D_3 are four points on the desired hyperbola.

One point of the hyperbola $\mathcal{H}$ is still missing.

- 1) Let $\mathcal{D}_1$ be the line through B parallel to (AC) .
- 2) Let $\mathcal{D}_2$ be the line through D_0 parallel to (AB) .
- 3) Let E be the intersection of lines (AC) and $\mathcal{D}_2$.
- 4) Let F be the midpoint of $[AE]$.
- 5) Let $\mathcal{D}_3$ be the line through F parallel to (AB) .
- 6) Let D_4 be the intersection of $\mathcal{D}_1$ and $\mathcal{D}_3$.

The point D_4 is the fifth point of the hyperbola, which allows us to construct the hyperbola $\mathcal{H}$.

3 Rational quadratic Bézier curves

The degree 2 Bernstein polynomials are:

$$\begin{aligned} B_0(t) &= (1-t)^2 \\ B_1(t) &= 2t(1-t) \\ B_2(t) &= t^2 \end{aligned} \quad (1)$$

Definition 1 (Quadratic Rational Bézier Curve)

Let $(P_i)_{0 \leq i \leq 2}$ be three non-collinear control points and $(\omega_i)_{0 \leq i \leq 2}$ be three non-zero weights satisfying the condition:

$$\forall t \in [0, 1], \quad \sum_{i=0}^2 \omega_i B_i(t) \neq 0 \quad (2)$$

A point $M(t)$, for $t \in [0, 1]$, belongs to the quadratic rational Bézier curve defined by the control points $(P_i)_{0 \leq i \leq 2}$ and weights $(\omega_i)_{0 \leq i \leq 2}$ if, for a point O in $\mathcal{P}$, $\overrightarrow{OM}(t)$ satisfies the equation:

$$\overrightarrow{OM}(t) = \frac{1}{\sum_{i=0}^2 \omega_i B_i(t)} \sum_{i=0}^2 \omega_i B_i(t) \overrightarrow{OP_i} \quad (3)$$

Condition (2) ensures that the quadratic rational Bézier curve remains within the affine plane $\mathcal{P}$. Geometrically, for a fixed t in $[0, 1]$, $M(t)$ is the barycenter of the weighted points:

$$\left(P_i, \frac{\omega_i B_i(t)}{\sum_{i=0}^2 \omega_i B_i(t)} \right)_{i \in \{0,1,2\}}$$

Definition 2 (Bézier curve in standard form)

Under the hypotheses of Definition 1, if $\omega_0 = \omega_2 = 1$, the curve is said to be in standard form and the weight ω_1 is denoted by ω .

This model cannot represent half-ellipses or hyperbola branches; the definition of Bézier curves with control mass points is provided in Appendix A.

Definition 3 (Bézier curve with mass points)

Let $(P_i, \omega_i)_{i \in \{0,1,2\}}$ be three control mass points. Let I (resp. J) be the set of indices with non-zero (resp. zero) weights. Define the function ω_f on $[0, 1]$ by:

$$\omega_f(t) = \sum_{i \in I} \omega_i B_i(t) \quad (4)$$

A mass point (M, ω) or vector $(\vec{u}, 0)$ belongs to the Bézier curve γ if there exists a real $t_0 \in [0, 1]$ such that:

- If $\omega_f(t_0) \neq 0$, then:

$$\begin{cases} \overrightarrow{OM} = \frac{1}{\omega_f(t_0)} \sum_{i \in I} \omega_i B_i(t_0) \overrightarrow{OP_i} \\ \quad + \frac{1}{\omega_f(t_0)} \sum_{i \in J} B_i(t_0) \vec{P_i} \\ \omega = \omega_f(t_0) \end{cases} \quad (5)$$

- If $\omega_f(t_0) = 0$, then:

$$\vec{u} = \sum_{i \in I} \omega_i B_i(t_0) \overrightarrow{OP_i} + \sum_{i \in J} B_i(t_0) \vec{P_i} \quad (6)$$

If $J = \emptyset$, Definition 1 is recovered. In equations (5) and (6), one can distinguish an affine part defined from points P_i via $\overrightarrow{OP_i}$, and a vector part using $\vec{P_i}$. In Equation (5), the affine part corresponds to the barycenter of the weighted points $(P_i, \omega_i B_i(t_0))_{i \in I}$, translated by the vector $\sum_{i \in J} B_i(t_0) \vec{P_i}$.

Let P_1 be a weighted point with non-zero weight ω_1 , and for $k \in \{0, 2\}$, let $\vec{P_k}$ be a control vector, and (P_{2-k}, ω_{2-k}) a mass point. The line through P_1 with direction vector $\vec{P_k}$ is an asymptote to the Bézier curve defined by the mass points $(\vec{P_k}, 0)$, (P_1, ω_1) , and (P_{2-k}, ω_{2-k}) .

For quick implementation, without coding the addition $\oplus$ and scalar multiplication $\odot$, Bézier curves with control mass points can be plotted using the following simplification:

Proposition 1 (Computational Bézier Curve)

Let the function:

$$\chi : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^* \\ 0 & \longmapsto & 1 \\ \omega \neq 0 & \longmapsto & \omega \end{array} \quad (7)$$

Let $\widetilde{P}_k = \overrightarrow{OP_k}$ if $\omega_k \neq 0$ and $\widetilde{P}_k = \overrightarrow{P_k}$ if $\omega_k = 0$. Then Equation (5) becomes:

$$\widetilde{M} = \frac{1}{\sum_{k=0}^2 \omega_k B_k(t)} \sum_{k=0}^2 \chi(\omega_k) B_k(t) \widetilde{P}_k \quad (8)$$

where $\widetilde{M}$ is a mass point: $\widetilde{M} = \overrightarrow{OM}$ if $\sum_{k=0}^2 \omega_k B_k(t) \neq 0$ and $\widetilde{M} = \overrightarrow{M}$ otherwise.

4 Steiner problems and Bézier curves

4.1 Steiner ellipse

Let $P_0 = A_1$ with weight 1, $P_1 = B$ with weight ω , and $P_2 = C_1$ with weight 1. Let γ_P be the standard-form quadratic rational Bézier curve with control points $(P_0, 1)$, (P_1, ω) , and $(P_2, 1)$. The goal is to find ω such that the Bézier curve γ_P is an arc of the Steiner ellipse $\mathcal{E}$.

Let $w_f(t)$ be defined by:

$$w_f(t) = B_0(t) + \omega B_1(t) + B_2(t) \quad (9)$$

and for any point O , a point $\gamma_P(t)$ of the Bézier curve is defined by:

$$\overrightarrow{O\gamma_P(t)} = \frac{B_0(t)}{w_f(t)} \overrightarrow{OP_0} + \frac{B_1(t)}{w_f(t)} \overrightarrow{OP_1} + \frac{B_2(t)}{w_f(t)} \overrightarrow{OP_2} \quad (10)$$

so that:

$$\overrightarrow{B_1\gamma_P(t)} = \frac{B_0(t)}{w_f(t)} \overrightarrow{B_1P_0} + \frac{B_1(t)}{w_f(t)} \overrightarrow{B_1P_1} + \frac{B_2(t)}{w_f(t)} \overrightarrow{B_1P_2} \quad (11)$$

and the value of t is computed such that $\overrightarrow{B_1\gamma_P(t)} = \overrightarrow{O}$. Now, Theorem 2 can be stated.

Theorem 2 (Steiner inellipse)

Let ABC be a non-degenerate triangle. Let A_1 be the midpoint of $[BC]$, B_1 the midpoint of $[AC]$, and C_1 the midpoint of $[AB]$.

The Steiner inellipse $\mathcal{E}$ is the ellipse passing through A_1 , B_1 , and C_1 , and tangent at these points to the lines (BC) , (AC) , and (AB) , and

modeled by the standard-form quadratic rational Bézier curves γ_+ and γ_- with control points $(A_1, 1)$, $(B, \frac{1}{2})$, and $(C_1, 1)$ on one hand, and $(A_1, 1)$, $(B, -\frac{1}{2})$, and $(C_1, 1)$ on the other.

Proof: See Appendix C.

Let $A(1, 1)$, $B(3, 5)$, and $C(5, 3)$. Figure 1 shows the Steiner inellipse $\mathcal{E}$ of triangle ABC using Theorem 2.

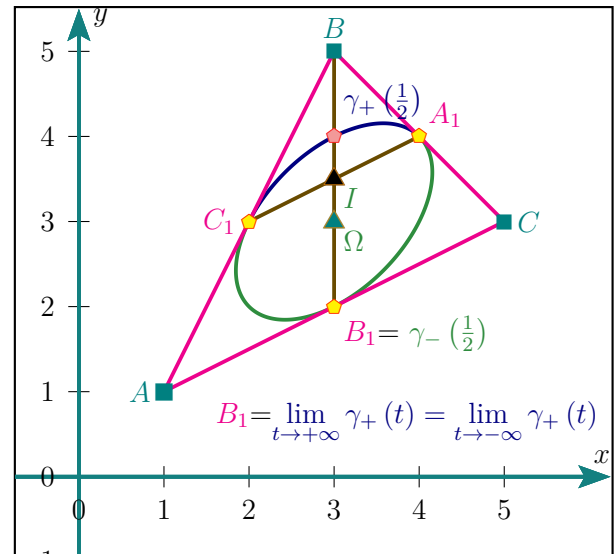


Figure 1: Steiner inellipse of a triangle ABC constructed using two quadratic rational Bézier curves γ_+ and γ_- . Source: created by the authors.

Steiner inellipse of a triangle ABC constructed using two quadratic rational Bézier curves.

Note that if γ_P is a quadratic rational Bézier curve with control points P_0 with weight 1, P_1 with weight ω , and P_2 with weight 1. Let I be the midpoint of the segment $[P_0P_2]$. If $\omega \in \mathbb{R} \setminus \{-1, 1\}$, then:

- 1) the conic generated by the curve γ_P has a center, denoted by Ω ;
- 2) the points P_1 , $\gamma_P(\frac{1}{2})$, I , Ω , and

$$M_\infty = \lim_{t \rightarrow +\infty} \gamma_P(t) = \lim_{t \rightarrow -\infty} \gamma_P(t)$$

lie on the median of triangle $P_0P_1P_2$ issued from vertex P_1 .

4.2 Generalized Steiner Ellipse

Theorem 3 (Generalized Steiner Ellipse)

Let ABC be a non-degenerate triangle. Let A_1 belong to $[BC]$, B_1 to $[AC]$, and C_1 to $[AB]$, such that the lines (AA_1) , (BB_1) , and (CC_1) are concurrent.

Let:

$$\begin{cases} \omega_+ = \frac{\sqrt{\det(\overrightarrow{AC_1}, \overrightarrow{AC}) \det(\overrightarrow{AA_1}, \overrightarrow{AC})}}{|\det(\overrightarrow{AB}, \overrightarrow{AC})|} \\ \omega_- = -\frac{\sqrt{\det(\overrightarrow{AC_1}, \overrightarrow{AC}) \det(\overrightarrow{AA_1}, \overrightarrow{AC})}}{|\det(\overrightarrow{AB}, \overrightarrow{AC})|} \end{cases}$$

The generalized Steiner ellipse $\mathcal{E}$ is the ellipse passing through A_1 , B_1 , and C_1 , tangent at these points respectively to (BC) , (AC) , and (AB) , and modeled by the quadratic rational B  zier curves in standard form γ_+ and γ_- with respective control points $(A_1, 1)$, (B, ω_+) , $(C_1, 1)$ on the one hand, and $(A_1, 1)$, (B, ω_-) , $(C_1, 1)$ on the other.

Let:

$$t_+ = -\frac{\omega_+ \det(\overrightarrow{AB}, \overrightarrow{AC}) - \det(\overrightarrow{AA_1}, \overrightarrow{AC})}{2(\det(\overrightarrow{AA_2}, \overrightarrow{AC}) - \det(\overrightarrow{AB}, \overrightarrow{AC})\omega_+)}$$

and:

$$t_- = -\frac{\omega_- \det(\overrightarrow{AB}, \overrightarrow{AC}) - \det(\overrightarrow{AA_1}, \overrightarrow{AC})}{2(\det(\overrightarrow{AA_2}, \overrightarrow{AC}) - \det(\overrightarrow{AB}, \overrightarrow{AC})\omega_-)}$$

which leads to $\gamma_+(t_+) = \gamma_-(t_-) = B_1 \in [AC]$.

Proof: See Appendix D.

■

Let $A(1, 1)$, $B(3, 5)$, and $C(5, 3)$ on the one hand, and $A_1(4, 4)$, $B_1(\frac{7}{3}, \frac{5}{3})$, $C_1(\frac{5}{3}, \frac{7}{3})$ on the other, and $A_2(\frac{17}{6}, \frac{19}{6})$.

The weight is:

$$\omega_- = -\frac{\sqrt{6}}{6}$$

and:

$$t_1 = 3 - \sqrt{6}$$

Figure 2 shows the generalized Steiner ellipse $\mathcal{E}$ of a triangle ABC using Theorem 3.

4.3 Steiner parabola

Let ABC be a non-degenerate triangle.

Let $t_A \in]0, 1[$ and define P_0 by:

$$\overrightarrow{OP_0} = (1 - t_A)\overrightarrow{OB} + t_A\overrightarrow{OC}$$

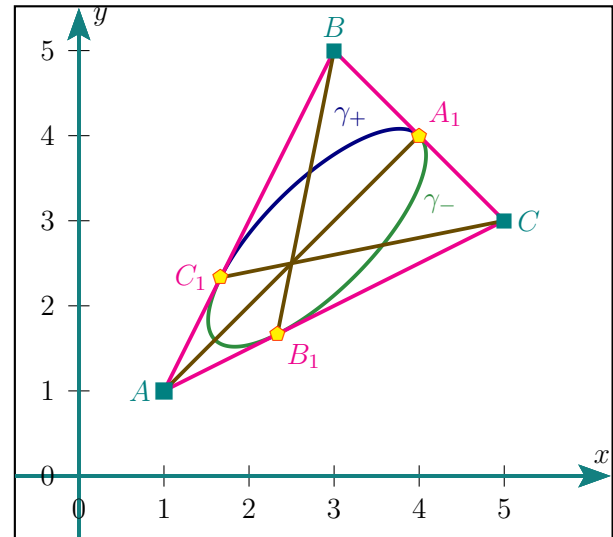


Figure 2: Generalized Steiner ellipse of a triangle ABC constructed using two rational quadratic B  zier curves γ_+ and γ_- . Source: created by the authors.

and $P_1 = B$. Let P_2 be a point on $[AB]$, that is:

$$\overrightarrow{OP_2} = \overrightarrow{OB} + k_2\overrightarrow{AB}$$

and let γ_P be the polynomial B  zier curve with control points P_0 , P_1 , and P_2 , that is:

$$\overrightarrow{O\gamma_P(t)} = (1 - t)^2\overrightarrow{OP_0} + 2t(1 - t)\overrightarrow{OP_1} + t^2\overrightarrow{OP_2}$$

in other words:

$$\begin{aligned} \overrightarrow{B\gamma_P(t)} &= (1 - t)^2\overrightarrow{BP_0} + 2t(1 - t)\overrightarrow{BP_1} + t^2\overrightarrow{BP_2} \\ &= (1 - t)^2t_A\overrightarrow{BC} + t^2k_2\overrightarrow{AB} \end{aligned}$$

The point $\gamma_P(t)$ is required to lie on the line (AC) , which allows the statement of Theorem 4.

Theorem 4 (Steiner Parabola)

Let ABC be a non-degenerate triangle. Let t_A be a real number in $]0, 1[$ and define P_0 by:

$$\overrightarrow{OP_0} = (1 - t_A)\overrightarrow{OB} + t_A\overrightarrow{OC}$$

Let $P_1 = B$. Let:

$$k_2 = \frac{-t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})}$$

and let P_2 be the point on the ray $[AB]$ defined by $\overrightarrow{BP_2} = k_2\overrightarrow{AB}$. Define:

$$t_0 = \frac{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}$$

Then the polynomial Bézier curve γ_P with control points P_0 , P_1 , and P_2 is an arc of the Steiner parabola $\mathcal{P}_S$ tangent to the lines (AB) at P_2 , (BC) at P_0 , and (AC) at $\gamma_P(t_0)$.

Proof: Appendix E.

■

Corollary 1 (Steiner Parabola)

Let γ_P be the Bézier curve from Theorem 4. Then:

- 1) The polynomial Bézier curve γ_Q with control points $Q_0 = P_0$, $Q_1 = C$, and $Q_2 = \gamma_P(t_0)$ is the arc of the Steiner parabola $\mathcal{P}_S$ whose first (resp. last) point lies on the line (BC) (resp. (AC)).
- 2) The polynomial Bézier curve γ_R with control points $R_0 = P_2$, $R_1 = A$, and $R_2 = Q_2$ is the arc of the Steiner parabola $\mathcal{P}_S$ whose first (resp. last) point lies on the line (AB) (resp. (AC)).
- 3) The rational Bézier curve γ_S with weighted control points $(R_0, 1)$, $(R_1, -1)$, and $(R_2, 1)$ is the arc of the Steiner parabola $\mathcal{P}_S$ complementary to γ_R .

Proof: Appendix E.

- 1) It suffices to apply the homographic parameter transformation with $a_h = 0$, $b_h = t_0$ and $c_h = d_h = 1$ to γ_P .
- 2) It suffices to apply the homographic parameter transformation with $a_h = t_0$, $b_h = 1$, and $c_h = d_h = 1$ to γ_P .
- 3) It suffices to apply the homographic parameter transformation with $a_h = 0$, $b_h = 1$, $c_h = -1$, and $d_h = 1$ to γ_P .

■

Figure 5 in the Appendix illustrates the construction of the Steiner parabola with $A(3, 4)$, $B(1, 2)$, and $C(5, 1)$ using $t_A = \frac{1}{3}$, which gives $k_2 = \frac{1}{2}$ and $t_0 = -2$.

Figure 6 in the Appendix shows the construction with $t_A = \frac{2}{3}$, which gives $k_2 = 2$ and $t_0 = -\frac{1}{2}$.

4.4 Steiner hyperbola

Let ABC be a non-degenerate triangle. Construct a hyperbola $\mathcal{H}$, tangent to the segment $[BC]$, with asymptotes given by the lines (AB) and (AC) .

4.4.1 General Case

The point A is the center of the hyperbola $\mathcal{H}$. Theorem 5 can be stated.

Theorem 5 (Steiner Hyperbola $\mathcal{H}$)

Let $u_0 \in]0, 1[$. The control mass points of the Bézier curve γ_P are:

$$\begin{cases} (\vec{P}_0, 0) &= \left(\frac{u_0}{1-u_0} \vec{AB}, 0 \right) \\ (P_1, 1) &= (A, 1) \\ (\vec{P}_2, 0) &= \left(\frac{1-u_0}{u_0} \vec{AC}, 0 \right) \end{cases} \quad (12)$$

and $\gamma_P(u_0)$ is the midpoint of the segment $[BC]$, and the line (BC) is tangent to γ_P at $\gamma_P(u_0)$. The lines (AB) and (AC) are the asymptotes of the Steiner hyperbola $\mathcal{H}$.

Proof: Appendix F.

■

The characteristic elements of the hyperbola $\mathcal{H}$ can be easily obtained using homographic parameter transformations [28],[30].

Let $A(2, 4)$, $B(0, 2)$ and $C(4, 1)$. Figure 7 in the Appendix shows the construction for $u_0 = \frac{1}{2}$, yielding $\vec{P}_0 = \vec{AB}$ and $\vec{P}_2 = \vec{AC}$, which leads to $\|\vec{P}_0\| = 2\sqrt{2}$ and $\|\vec{P}_2\| = \sqrt{13}$.

4.4.2 Vectors $\vec{P}_0$ and $\vec{P}_2$ with equal norm
Finding u_0 for

$$\|\vec{P}_0\| = \|\vec{P}_2\|$$

in Theorem 5 leads to Corollary 2 which allows immediate determination of the hyperbola parameters [28],[30].

Corollary 2 (Extremal vectors with equal norm)

From Theorem 5, if $\|\vec{AC}\| = \|\vec{AB}\|$, then $u_0 = \frac{1}{2}$ else

$$u_0 = \frac{\|\vec{AC}\| - \sqrt{\|\vec{AB}\| \|\vec{AC}\|}}{\|\vec{AC}\| - \|\vec{AB}\|} \quad (13)$$

Thus:

$$\|\vec{P}_0\| = \|\vec{P}_2\|$$

Proof: Appendix F.

Figure 8 in the Appendix shows the construction for:

$$u_0 = \frac{\sqrt{13} - \sqrt{2\sqrt{26}}}{\sqrt{13} - 2\sqrt{2}} \simeq 0.530$$

(see Formula (13)). Then:

$$1 - u_0 = \frac{\sqrt{2\sqrt{26}} - 2\sqrt{2}}{\sqrt{13} - 2\sqrt{2}}$$

which gives:

$$\begin{cases} \frac{u_0}{1 - u_0} = \frac{\sqrt{\sqrt{26}}}{2} \\ \frac{1 - u_0}{u_0} = \frac{\sqrt{\sqrt{26}}^3}{13} \end{cases}$$

which leads to:

$$\vec{P}_0 = \frac{\sqrt{\sqrt{26}}}{2} \vec{AB}$$

and

$$\vec{P}_2 = \frac{\sqrt{\sqrt{26}}^3}{13} \vec{AC}$$

which leads to $\|\vec{P}_0\| = \sqrt{2\sqrt{26}}$ and $\|\vec{P}_2\| = \sqrt{2\sqrt{26}}$.

One can note the following property: Let Ω be the center of a hyperbola and let Δ_0 be a tangent to this hyperbola intersecting the first asymptote at M_1 and the second asymptote at M_2 . No matter the position of the line Δ_0 , the area of triangle $\Omega M_1 M_2$ remains constant.

5 Conclusion and future works

We propose a resolution of Steiner problems using mass points. We show that the constructed conic can be model by Bézier curve with control mass points : a vertex and two midpoints with coefficients 1 and $\frac{1}{2}$. Our method offers reduced computational time and complexity, finding directly the parameters of the Steiner conics unlike usual resolutions: for instance, in the case of the ellipse, the usual method (or the method on the litterature) involves determining an affine transformation that maps the given triangle to an equilateral one.

In the near future, the use of rational Bézier curves with complex weights will be considered to study, among other topics, cusp points, inflection points, and the modeling of conic inversions, while maintaining a degree 2 curve—even though such transformations would require degree 4 when using real weights.

Display on a computer can be seen as set of points in a finite plane. The Bézier curve could be used in this background as finite curves. In particular, elliptic curves used in cryptography in finite fields could be modeled with Bézier curves. This domain had never been explored by the authors . It gives new opportunities to open a huge domain of new applications.

In considering Bézier curves with control mass points, we intend to model handwriting by additionally incorporating the stroke kinematics, employing the loops of the Descartes Folium or the Bernoulli Lemniscate...

Furthermore, we envisage applying this approach to the modeling of Dupin cyclides within the Minkowski–Lorentz space.

References:

- [1] Pierre Bézier. Méthode de construction des courbes et surfaces coordonnées polynomi-ales. *Automatisme*, 11:625–632, 1962. Foundational work introducing Bézier curves.
- [2] Pierre Bézier. Définition numérique des courbes et surfaces. *Automatisme*, 11(12):625–632, 1966.
- [3] Pierre Bézier. L'ordinateur et les fabrications mécaniques. *Automatisme*, 14(5):189–196, 1969.
- [4] Paul de Casteljaeu. Outillage méthodique pour l'étude des fonctions équations algébriques. Report at Citroën, 1959. Original internal report introducing the De Casteljaeu algorithm.
- [5] P. De Casteljaeu. *Mathématiques et CAO. Volume 2 : formes à pôles*. Hermes, 1985.
- [6] P. Bézier. *Courbe et surface*, volume 4. Hermès, Paris, 2ème edition, Octobre 1986.
- [7] Pierre Bézier. *L'Utilisation des courbes et surfaces en CAO*. Technologies de pointe. Hermes Science Publications, Paris, 1988.
- [8] Paul de Casteljaeu. *Surfaces à pôles*. Rapport technique interne, Société Anonyme André Citroën, Paris, 1963. Enregistré à l'INPI (Institut National de la Propriété Industrielle).

- [9] Paul de Casteljau. Courbes à pôles. Rapport technique interne, Société Anonyme André Citroën, Paris, 1959. Enregistré à l'INPI (Institut National de la Propriété Industrielle).
- [10] Gerald Farin. Chapter 1 - a history of curves and surfaces in cagd. In Gerald Farin, Josef Hoschek, and Myung-Soo Kim, editors, *Handbook of Computer Aided Geometric Design*, pages 1–21. North-Holland, Amsterdam, 2002.
- [11] J. W. Downs. *Practical Conic Sections: The Geometric Properties of Ellipses, Parabolas and Hyperbolas*. Dover Publications, Mineola, New York, 2012.
- [12] Christer Bergsten. The parabola and historical metamorphoses of meanings. Linköping University Electronic Press, 2010. Presented at the 6th European Summer University on History and Epistemology in Mathematics Education.
- [13] G. Farin. *NURBS from Projective Geometry to Practical Use*. A K Peters, Ltd, 2 edition, 1999. ISBN 1-56881-084-9.
- [14] Gerald Farin. *Curves and Surfaces for CAGD: A Practical Guide*. Morgan Kaufmann, 5th edition, 2002. Comprehensive introduction to curves and surfaces, including Bézier and B-splines.
- [15] L. A. Piegl and W. Tiller. *The NURBS book*. Monographs in visual communication. Springer, 1995.
- [16] G. Demengel and J. P. Pouget. *Mathématiques des Courbes et des Surfaces. Modèles de Bézier, des B-Splines et des NURBS*. Ellipse, 1998.
- [17] M. Berger and S. Levy. *Geometry I. Geometry /Marcel Berger* ; Translated from the French by M. Cole Abd S. Leavy. Springer, 1987.
- [18] M. Berger, M. Cole, and S. Levy. *Geometry II. Number vol. 2 in Electronic Workshops in Computing*. Springer, 1987.
- [19] Gerald Farin. From conics to nurbs: A tutorial and survey. *IEEE Comput. Graph. Appl.*, 12(5):78–86, September 1992.
- [20] E.T.Y. Lee. The rational Bézier representation for conics. In G. Farin (ed.), editor, *In Geometric Modeling, Algorithms and New Trends*, SIAM, pages 3–19, Philadelphia, 1985.
- [21] Gudrun Albrecht. Mathematical methods for curves and surfaces. chapter Determination of Geometrical Invariants of Rationally Parametrized Conic Sections, pages 15–24. Vanderbilt University, Nashville, TN, USA, 2001.
- [22] J. C. Fiorot and P. Jeannin. *Courbes et surfaces rationnelles*, volume RMA 12. Masson, 1989.
- [23] J. C. Fiorot and P. Jeannin. *Courbes splines rationnelles, applications à la CAO*, volume RMA 24. Masson, 1992.
- [24] Jean-Charles Fiorot, Pierre Jeannin, and Salim Taleb. Rational cubic curves as br-curves. *Computer Aided Geometric Design*, 12(1):67–88, 1995.
- [25] Jean-Charles Fiorot, Pierre Jeannin, and Salim Taleb. New control massic polygon of a b-rational curve resulting from a homographic change of parameter. *Numerical Algorithms*, 6(3–4):379–418, 1994.
- [26] Giuseppina Anatriello and Vincenzi Giovanni. Cubics, hyperbolas and steiner ellipses. *Journal of Geometry*, 112, 12 2021.
- [27] D. Minda and S. Phelps. Triangles, ellipses, and cubic polynomials. *The American Mathematical Monthly*, 115(8):679–689, 2008.
- [28] J. P. Bécar. *Forme (BR) des coniques et de leurs faisceaux*. PhD thesis, Université de Valenciennes et de Hainaut-Cambrésis, LIMAV, Décembre 1997.
- [29] Lionel Garnier, Jean-Paul Bécar, and Laurent Fuchs. How null vector performs in a rational Bézier curve with mass points. *Geometry*, 2(1), 2025.
- [30] Lionel Garnier and Jean-Paul Bécar. Mass points, Bézier curves and conics: a survey. In *Eleventh International Workshop on Automated Deduction in Geometry, Proceedings of ADG 2016*, pages 97–116, Strasbourg, France, June 2016.

Appendices

A Space of mass points

Definition 4 (Space of mass points $\tilde{\mathcal{P}}$)

The space of mass points is:

$$\tilde{\mathcal{P}} = (\mathcal{P} \times \mathbb{R}^*) \cup (\vec{\mathcal{P}} \times \{0\}) \quad (14)$$

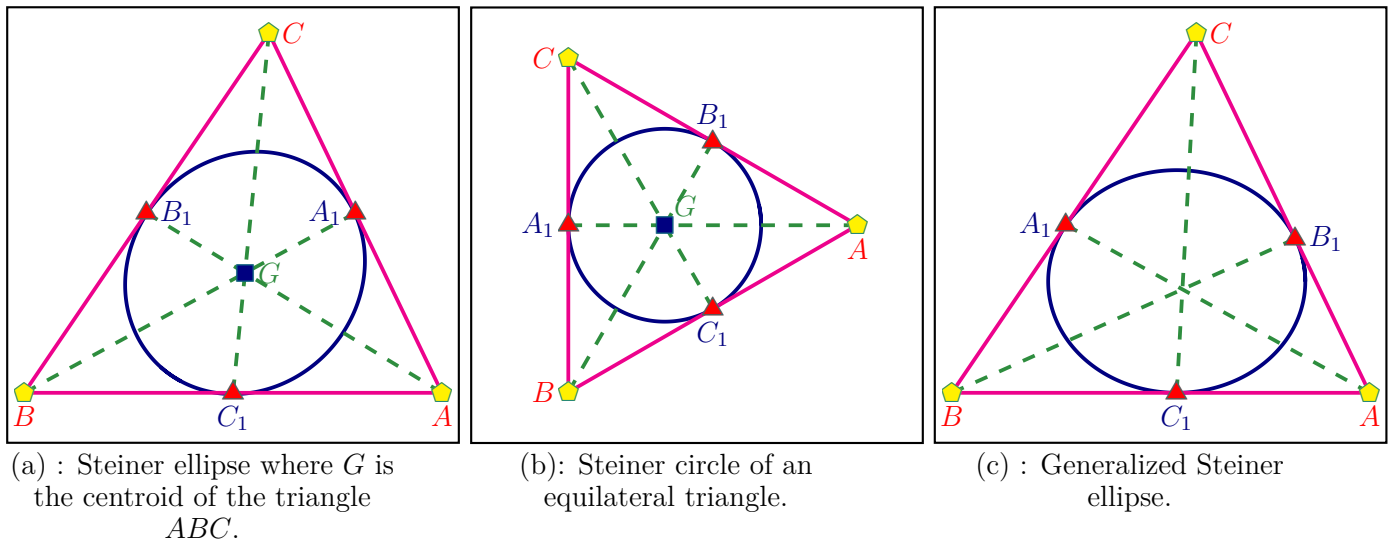


Figure 3: Steiner problems: cases of an ellipse. In the Figures(a) and (c), the ellipse is computed from the same triangle ABC . Source: created by the authors.

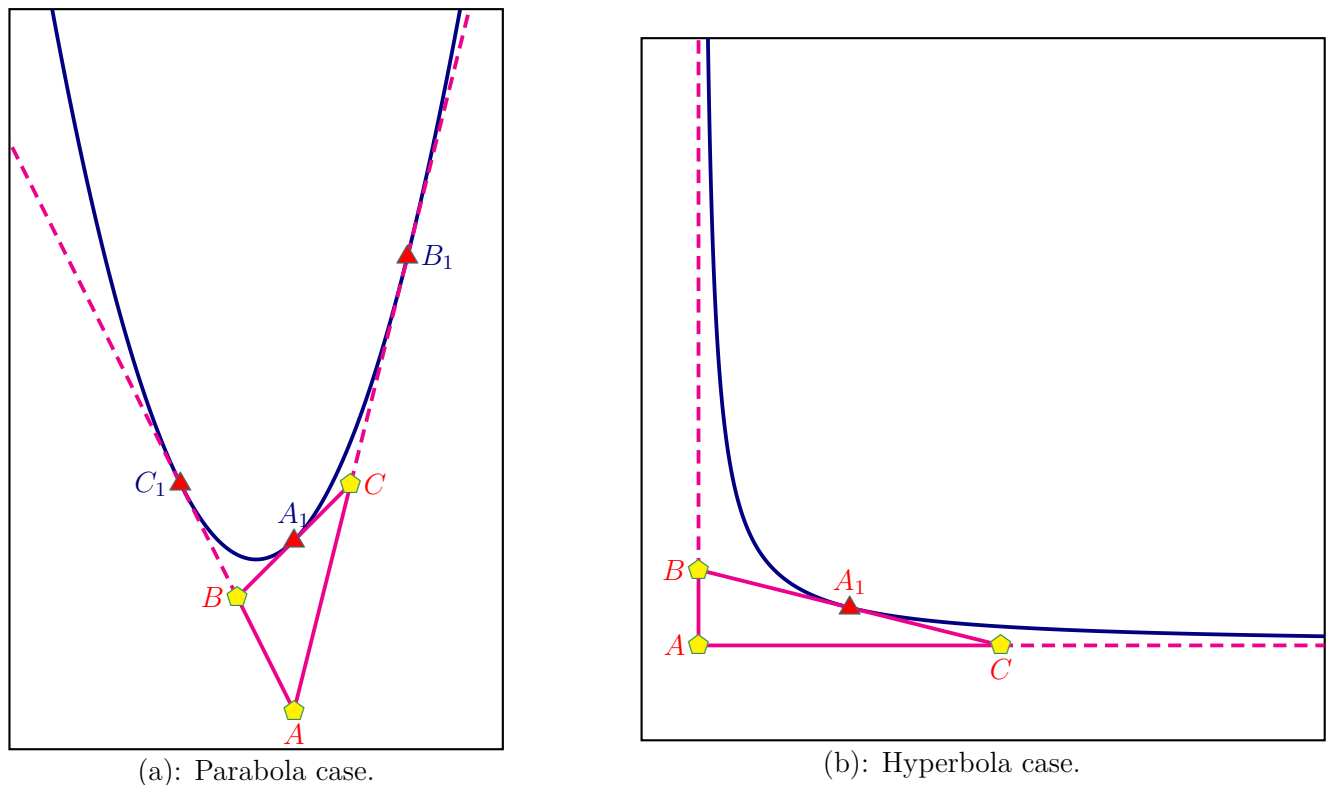


Figure 4: Steiner parabola and Steiner hyperbola problems. Source: created by the authors.

On the space $\tilde{\mathcal{P}}$, an addition $\oplus$ and a scalar multiplication $\odot$ are defined and then, $(\tilde{\mathcal{P}}, \oplus, \odot)$ is a vector space [22],[23].

Addition is defined as follows:

Definition 5 (Addition $\oplus$ in $(\tilde{\mathcal{P}}, \oplus, \odot)$)

Let (M, ω) and (N, μ) be two weighted points with non-zero weights, and a vector $(\vec{u}, 0)$.

• $\omega\mu(\omega + \mu) \neq 0$:

$$(M, \omega) \oplus (N, \mu) = \left(\text{bar} \left\{ (M, \omega), (N, \mu) \right\}, \omega + \mu \right)$$

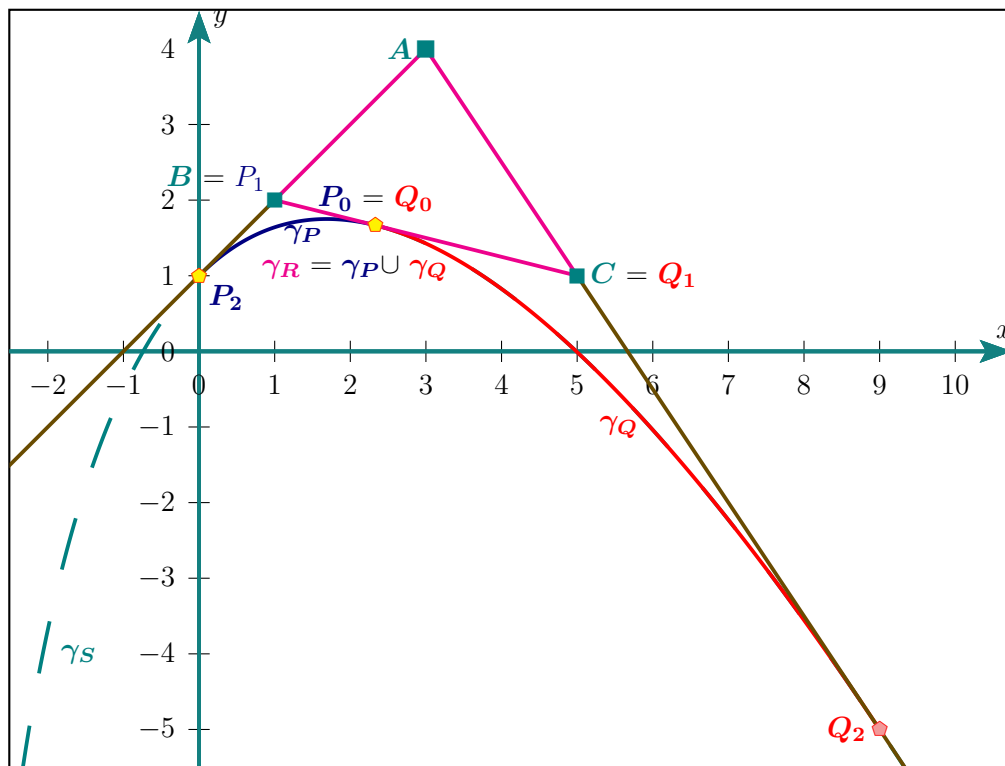


Figure 5: Steiner parabola of a triangle ABC constructed using rational quadratic Bézier curves with $t_A = \frac{1}{3}$. Source: created by the authors.

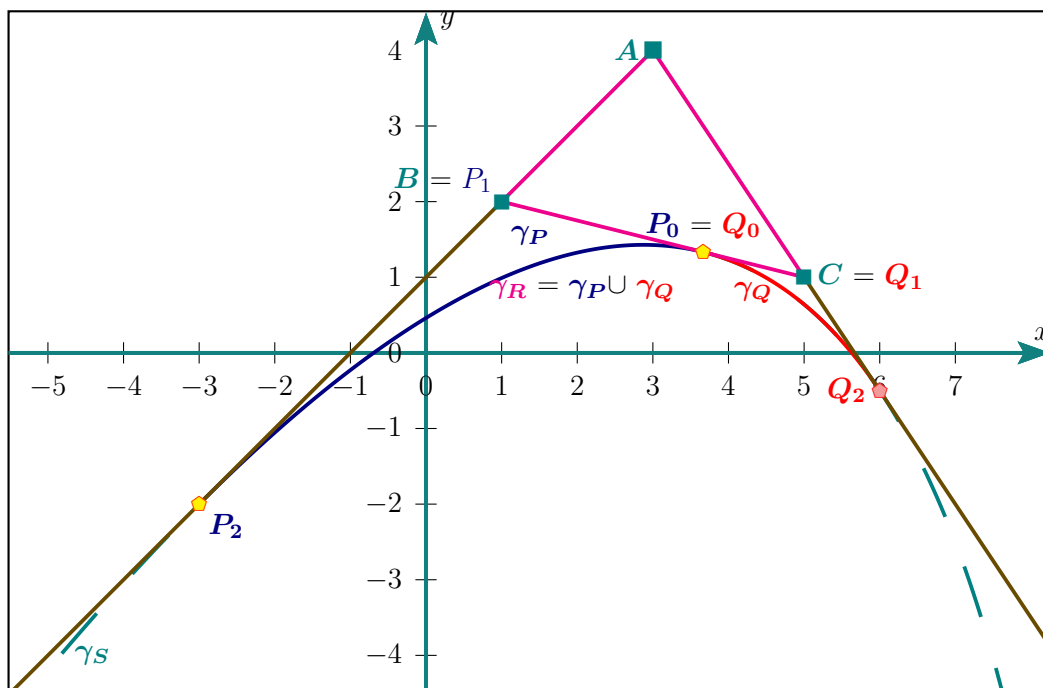


Figure 6: Steiner parabola of a triangle ABC constructed using rational quadratic Bézier curves with $t_A = \frac{2}{3}$. Source: created by the authors.

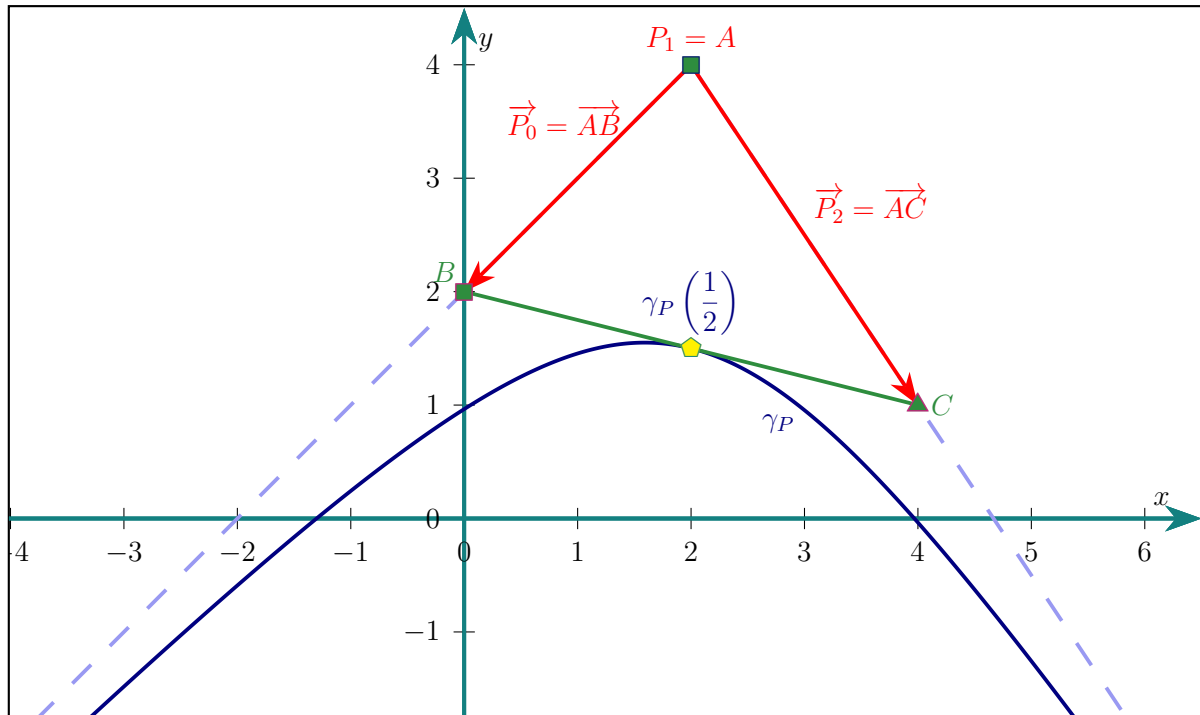


Figure 7: Steiner hyperbola branch defined from constraints on the triangle ABC with $u_0 = \frac{1}{2}$. Source: created by the authors.

- $\omega \neq 0$:

$$(M, \omega) \oplus (N, -\omega) = (\omega \overrightarrow{NM}, 0)$$

- $\omega \neq 0$:

$$(M, \omega) \oplus (\vec{u}, 0) = (M', \omega) = \left(\mathcal{T}_{\frac{1}{\omega} \vec{u}}(M), \omega \right)$$

where $\mathcal{T}_{\vec{v}}$ denotes the translation by vector $\vec{v}$, that is

$$\overrightarrow{MM'} = \frac{1}{\omega} \vec{u}$$

- $(\vec{u}, 0) \oplus (\vec{v}, 0) = (\vec{u} + \vec{v}, 0)$.

In particular:

$$\begin{aligned} (M, -1) \oplus (N, 1) &= (N, 1) \oplus (M, -1) \\ &= (\overrightarrow{MN}, 0) \end{aligned}$$

Scalar multiplication is defined as follows:

Definition 6 (Scalar multiplication $\odot$ in $(\tilde{\mathcal{P}}, \oplus, \odot)$)

Let (M, ω) be a weighted point with non-zero weight and a vector $(\vec{u}, 0)$, and let α be a non-zero scalar.

- $\alpha \times \omega \neq 0 \implies \alpha \odot (M, \omega) = (M, \alpha\omega)$,

- $\alpha \odot (\vec{u}, 0) = (\alpha \vec{u}, 0)$,

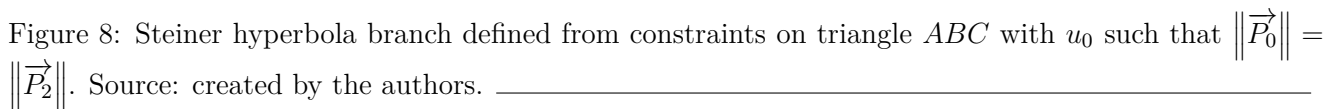
- $0 \odot (M, \omega) = (\vec{0}, 0)$,

- $0 \odot (\vec{u}, 0) = (\vec{0}, 0)$.

For a weighted point, the weight is scaled by the non-zero scalar α , while a vector is directly multiplied by α . Let (M, α) , with $\alpha \in \mathbb{R} \setminus \{0\}$, be a weighted point in $\tilde{\mathcal{P}}$. The point M in $\mathcal{P}$ is identified with the weighted point $(M, 1)$. Thus:

$$\begin{aligned} (P, \omega) \oplus \overrightarrow{MN} &= (P, \omega) \oplus (N, 1) \oplus (M, -1) \\ &= \omega \odot (P, 1) \oplus \omega \odot (N, \frac{1}{\omega}) \oplus \omega \odot (M, -\frac{1}{\omega}) \\ &= \omega \odot ((P, 1) \oplus (N, \frac{1}{\omega}) \oplus (M, -\frac{1}{\omega})) \\ &= \omega \odot ((P, 1) \oplus \frac{1}{\omega} \overrightarrow{MN}) \\ &= \omega \odot (P', 1) \end{aligned}$$

where the point P' is the image of point P by the translation $\mathcal{T}_{\frac{1}{\omega} \overrightarrow{MN}}$ of vector $\frac{1}{\omega} \overrightarrow{MN}$, written


$$(P, \omega) \oplus \overrightarrow{MN} = (P', \omega) = \left(\mathcal{T}_{\frac{1}{\omega} \overrightarrow{MN}}(P), \omega \right) \quad (15)$$
$$\overrightarrow{PP'} = \overrightarrow{P\mathcal{T}_{\frac{1}{\omega}} \overrightarrow{MN}}(P) = \frac{1}{\omega} \overrightarrow{MN} \quad (16)$$
$$\begin{aligned}
& (M, \omega) \oplus (N, \mu) \\
= & (M, \omega) \oplus \underbrace{(M, \mu) \oplus (M, -\mu)}_{(\vec{0}, 0)} \oplus (N, \mu) \\
= & (M, \omega + \mu) \oplus (M, -\mu) \oplus (N, \mu) \\
= & (M, \omega + \mu) \oplus (\mu \overrightarrow{MN}, 0) \\
= & \left(\mathcal{T}_{\frac{\mu}{\omega + \mu} \overrightarrow{MN}}(M), \omega + \mu \right)
\end{aligned} \tag{17}$$
$$\begin{vmatrix} a_h & b_h \\ c_h & d_h \end{vmatrix} \neq 0 \quad (18)$$
$$h(u) = \frac{a_h(1-u) + b_h u}{c_h(1-u) + d_h u} \quad (19)$$

To replace one arc of a curve by another arc of the same curve, Table 1 gives the values of a_h , b_h , c_h and d_h depending on h ($[0, 1[$).

$h ([0, 1[)$	Expression	a_h	b_h	c_h	d_h
$] -\infty, \alpha_1[, h(0) = \alpha_1$	$\frac{\alpha_1(1-u) - u}{1-u}$	α_1	-1	1	0
$] -\infty, \alpha_1[, h(1) = \alpha_1$	$\frac{(1-u) - \alpha_1 u}{-u}$	1	$-\alpha_1$	0	-1
$] \alpha_i, \alpha_{i+1}[$	$\alpha_i(1-u) + \alpha_{i+1}u$	α_i	α_{i+1}	1	1
$] \alpha_q, +\infty[, h(0) = \alpha_q$	$\frac{\alpha_q(1-u) + u}{1-u}$	α_q	1	1	0
$] \alpha_q, \alpha_1[, h\left(\frac{1}{2}\right) = +\infty$	$\frac{-\alpha_q(1-u) + \alpha_1 u}{-1+2u}$	$-\alpha_q$	α_1	-1	1
$+\infty \in h ([\alpha_q, \alpha_1])$	$u_0 \frac{-\alpha_q(1-u) + \alpha_1 u}{u - u_0}$	$-\alpha_q$	α_1	-1	$\frac{1-u_0}{u_0}$

Table 1: Intervals and values of a_h , b_h , c_h , and d_h in Formula (19). Source: created by the authors.

B.2 Degree 2 Case

Theorem 7 (Homographic Parameter Change)

Let γ be a B  zier curve with control mass points (P_0, ω_0) , (P_1, ω_1) , and (P_2, ω_2) supported on the conic C.

Let h be defined by Formula (19).

Then $\gamma \circ h$ is the B  zier curve, supported on the conic C, with control mass points (Q_0, ϖ_0) , (Q_1, ϖ_1) , and (Q_2, ϖ_2) given in Table 2.

Proof: [28]



C Proof of Theorem 2

$$\overrightarrow{B_1 \gamma_P(t)} = \overrightarrow{0}$$

$$\begin{aligned} &\iff (1-t)^2 \overrightarrow{B_1 P_0} + 2\omega t(1-t) \overrightarrow{B_1 P_1} \\ &+ t^2 \overrightarrow{B_1 P_2} = \overrightarrow{0} \\ &\iff \left(\overrightarrow{B_1 P_0} - 2\omega \overrightarrow{B_1 P_1} + \overrightarrow{B_1 P_2} \right) t^2 \\ &+ \left(-2\overrightarrow{B_1 P_0} + 2\omega \overrightarrow{B_1 P_1} \right) t + \overrightarrow{B_1 P_0} = \overrightarrow{0} \\ &\iff \left(\overrightarrow{B_1 A_1} - 2\omega \overrightarrow{B_1 B} + \overrightarrow{B_1 C_1} \right) t^2 \\ &+ \left(-2\overrightarrow{B_1 A_1} + 2\omega \overrightarrow{B_1 B} \right) t + \overrightarrow{B_1 A_1} = \overrightarrow{0} \\ &\iff \left(\overrightarrow{B_1 B} - 2\omega \overrightarrow{B_1 B} \right) t^2 + \overrightarrow{B_1 A_1} \\ &+ \left(-2\overrightarrow{B_1 A_1} + 2\omega \overrightarrow{B_1 B} \right) t = \overrightarrow{0} \end{aligned}$$

$$\begin{aligned} &\iff (1-2\omega) \overrightarrow{B_1 B} t^2 + \overrightarrow{B_1 A_1} \\ &+ (-2\overrightarrow{B_1 A_1} + 2\omega \overrightarrow{B_1 B}) t = \overrightarrow{0} \end{aligned}$$

which leads to the system:

$$\begin{cases} (1-2\omega) \overrightarrow{B_1 B} t^2 + 2\omega \overrightarrow{B_1 B} t = \overrightarrow{0} \\ -2\overrightarrow{B_1 A_1} t + \overrightarrow{B_1 A_1} = \overrightarrow{0} \end{cases}$$

which is equivalent to:

$$\begin{cases} ((1-2\omega)t + 2\omega) \overrightarrow{B_1 B} t = \overrightarrow{0} \\ (-2t+1) \overrightarrow{B_1 A_1} = \overrightarrow{0} \end{cases}$$

The second equation gives $t = \frac{1}{2}$. The first equation becomes:

$$(1-2\omega) \frac{1}{2} + 2\omega = 0$$

that is:

$$\frac{1}{2} - \omega + 2\omega = 0$$

which gives:

$$\omega = -\frac{1}{2}$$

To show that the line (AC) is tangent to the ellipse generated by γ_P , a homographic parameter change with $a_h = 0$, $b_h = \frac{1}{2}$ and $c_h = d_h = 1$ is applied. Table 2 yields the control points:

$$(Q_0, \varpi_0) = 1^2 \odot (P_0, \omega_0) = (P_0, 1) = (A_1, 1)$$

and:

$$\left\{ \begin{array}{l} (Q_0, \varpi_0) = (c_h - a_h)^2 \odot (P_0, \omega_0) \oplus 2a_h (c_h - a_h) \odot (P_1, \omega_1) \oplus a_h^2 \odot (P_2, \omega_2) \\ (Q_1, \varpi_1) = (c_h - a_h) (d_h - b_h) \odot (P_0, \omega_0) \\ \quad \oplus (b_h c_h - 2a_h b_h + a_h d_h) \odot (P_1, \omega_1) \oplus a_h b_h \odot (P_2, \omega_2) \\ (Q_2, \varpi_2) = (d_h - b_h)^2 \odot (P_0, \omega_0) \oplus 2b_h (d_h - b_h) \odot (P_1, \omega_1) \oplus b_h^2 \odot (P_2, \omega_2) \end{array} \right. \quad (20)$$

Table 2: Control mass points of Theorem 7. Source:[28].

$$\begin{aligned} (Q_1, \varpi_1) &= \frac{1}{2} \odot (P_0, \omega_0) \oplus \frac{1}{2} \odot (P_1, \omega_1) \\ &= \frac{1}{2} \odot (A_1, 1) \oplus \frac{1}{2} \odot \left(B, -\frac{1}{2}\right) \\ &= \left(A_1, \frac{1}{2}\right) \oplus \left(B, -\frac{1}{4}\right) \\ &= \left(\text{bar} \left\{ \left(A_1, \frac{1}{2}\right), \left(B, -\frac{1}{4}\right) \right\}, \frac{1}{4}\right) \\ &= \left(\text{bar} \left\{ (A_1, 2), (B, -1) \right\}, \frac{1}{4}\right) \end{aligned}$$

that is:

$$\overrightarrow{A_1 Q_1} = \frac{-1}{1} \overrightarrow{A_1 B}$$

which shows that $Q_1 = C$ (since A_1 is the midpoint of $[BC]$). Let I_B be the midpoint of the segment $[A_1 C_1]$. Finally:

$$\begin{aligned} (Q_2, \varpi_2) &= \left(\frac{1}{2}\right)^2 \odot (P_0, 1) \oplus 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \odot (P_1, -\omega) \\ &\quad \oplus \left(\frac{1}{2}\right)^2 \odot (P_2, 1) \\ &= (A_1, \frac{1}{4}) \oplus (B, -\frac{\omega}{2}) \oplus (C_1, \frac{1}{4}) \\ &= (I_B, \frac{1}{2}) \oplus (B, -\frac{\omega}{2}) \end{aligned}$$

By replacing O with B and ω with $-\omega$, Definition 2 yields:

$$\begin{aligned} \overrightarrow{B \gamma_P(\frac{1}{2})} &= \frac{1}{\frac{1}{2} - \frac{1}{2}\omega} \left(\frac{1}{4} \overrightarrow{B P_0} + \frac{1}{4} \overrightarrow{B P_2} \right) \\ &= \frac{1}{\frac{1}{2} - \frac{1}{2}\omega} \left(\frac{1}{4} \overrightarrow{B A_1} + \frac{1}{4} \overrightarrow{B C_1} \right) \\ &= \frac{1}{\frac{1}{2} - \frac{1}{2}\omega} \cdot \frac{1}{2} \overrightarrow{B I_B} \end{aligned}$$

and $\gamma_P(\frac{1}{2})$ is the barycenter of the points $(I_B, \frac{1}{2})$ and $(B, -\frac{\omega}{2})$, so:

$$(Q_2, \varpi_2) = \left(\gamma_P(\frac{1}{2}), \frac{1-\omega}{2} \right) = \left(B_1, \frac{1-\omega}{2} \right)$$

and the tangent to the ellipse at $\gamma_P(\frac{1}{2})$ is the line $(B_1 C) = (AC)$.

The arc of the ellipse from A_1 to C_1 passing through B_1 is computed. To get the other part, the homographic parameter change with $a_h = 1$, $b_h = 0$, $c_h = 1$, and $d_h = -1$ is applied which gives the following homographic function:

$$h : u \mapsto \frac{1-u}{(1-u)-u} = \frac{1-u}{1-2u} \quad (21)$$

which is increasing on $[0, \frac{1}{2}[$ to $[1, +\infty[$ and on $]\frac{1}{2}, 1]$ to $] -\infty, 0]$, see Figure 9.

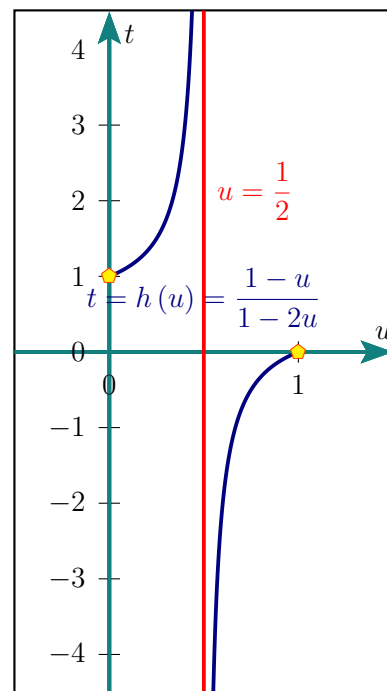


Figure 9: Graph of the parameter change function given by the homographic, Formula (21). Source: created by the authors.

Then:

$$(R_0, \varrho_0) = 1^2 \odot (P_2, 1) = (C_1, 1)$$

and:

$$(R_1, \varrho_1) = (-1) \odot (P_1, -\omega) = (P_1, \omega) = (B, \omega)$$

and:

$$(R_2, \varrho_2) = (-1)^2 \odot (P_0, 1) = (P_0, 1) = (A_1, 1)$$

■

D Proof of Theorem 3

Let γ_P be the Bézier curve in standard form with control points $P_0 = A_1$, $P_1 = B$, and $P_2 = C_1$ and weight ω .

$$\overrightarrow{O\gamma_P(t)} = \frac{B_0(t)}{\omega_f(t)} \overrightarrow{OA_1} + \omega \frac{B_1(t)}{\omega_f(t)} \overrightarrow{OB} + \frac{B_2(t)}{\omega_f(t)} \overrightarrow{OC_1}$$

that is,

$$\overrightarrow{A\gamma_P(t)} = \frac{B_0(t)}{\omega_f(t)} \overrightarrow{AA_1} + \omega \frac{B_1(t)}{\omega_f(t)} \overrightarrow{AB} + \frac{B_2(t)}{\omega_f(t)} \overrightarrow{AC_1}$$

The point $\gamma_P(t)$ belongs to the segment $[AC]$ if and only if:

$$\det(\overrightarrow{A\gamma_P(t)}, \overrightarrow{AC}) = 0$$

that is,

$$\det\left(\frac{B_0(t)}{\omega_f(t)} \overrightarrow{AA_1} + \omega \frac{B_1(t)}{\omega_f(t)} \overrightarrow{AB} + \frac{B_2(t)}{\omega_f(t)} \overrightarrow{AC_1}, \overrightarrow{AC}\right) = 0$$

which simplifies to:

$$\det(B_0(t) \overrightarrow{AA_1} + \omega B_1(t) \overrightarrow{AB} + B_2(t) \overrightarrow{AC_1}, \overrightarrow{AC}) = 0$$

and further to:

$$\begin{aligned} & \det(B_0(t) \overrightarrow{AA_1}, \overrightarrow{AC}) \\ & + \omega \det(B_1(t) \overrightarrow{AB}, \overrightarrow{AC}) \\ & + \det(B_2(t) \overrightarrow{AC_1}, \overrightarrow{AC}) = 0 \end{aligned}$$

which becomes:

$$\begin{aligned} & (1-t)^2 \det(\overrightarrow{AA_1}, \overrightarrow{AC}) \\ & + 2\omega t(1-t) \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ & + t^2 \det(\overrightarrow{AC_1}, \overrightarrow{AC}) = 0 \end{aligned}$$

Expanding and simplifying, this leads to:

$$\begin{aligned} & \det(\overrightarrow{AA_1}, \overrightarrow{AC}) t^2 \\ & + \det(\overrightarrow{AC_1}, \overrightarrow{AC}) t^2 \\ & - 2\omega \det(\overrightarrow{AB}, \overrightarrow{AC}) t^2 \\ & + 2\omega \det(\overrightarrow{AB}, \overrightarrow{AC}) t \\ & - 2 \det(\overrightarrow{AA_1}, \overrightarrow{AC}) t \\ & + \det(\overrightarrow{AA_1}, \overrightarrow{AC}) = 0 \end{aligned} \quad (22)$$

For the curve γ_P to be tangent to the segment $[AC]$, Equation (22) must have a double root. Defining:

$$\begin{cases} \alpha_0 &= \det(\overrightarrow{AA_1}, \overrightarrow{AC}) + \det(\overrightarrow{AC_1}, \overrightarrow{AC}) \\ \alpha_1 &= -2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ \beta_0 &= 2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ \beta_1 &= -2 \det(\overrightarrow{AA_1}, \overrightarrow{AC}) \\ \delta_0 &= \det(\overrightarrow{AA_1}, \overrightarrow{AC}) \end{cases}$$

Equation (22) becomes:

$$(\alpha_0 + \alpha_1 \omega) t^2 + (\omega \beta_0 + \beta_1) t + \delta_0 = 0 \quad (23)$$

with discriminant:

$$\begin{aligned} \Delta' &= (\omega \beta_0 + \beta_1)^2 - 4(\alpha_0 + \alpha_1 \omega) \delta_0 \\ &= (\omega \beta_0 - 2\delta_0)^2 - 4(\alpha_0 + \alpha_1 \omega) \delta_0 \\ &= \omega^2 \beta_0^2 - 4\delta_0 \omega \beta_0 + 4\delta_0^2 - 4\alpha_0 \delta_0 - 4\alpha_1 \omega \delta_0 \\ &= \beta_0^2 \omega^2 - 4(\beta_0 + \alpha_1) \delta_0 \omega + 4\delta_0^2 - 4\alpha_0 \delta_0 \end{aligned}$$

Since $\beta_0 = -\alpha_1$, the discriminant simplifies to:

$$\Delta' = \beta_0^2 \omega^2 + 4(\delta_0 - \alpha_0) \delta_0$$

So:

$$\begin{aligned} \Delta' &= 0 \\ \iff \beta_0^2 \omega^2 + 4(\delta_0 - \alpha_0) \delta_0 &= 0 \\ \iff \beta_0^2 \omega^2 &= 4(\alpha_0 - \delta_0) \delta_0 \\ \iff \omega^2 &= \frac{4}{\beta_0^2} (\alpha_0 - \delta_0) \delta_0 \end{aligned}$$

which yields two opposite solutions:

$$\begin{cases} \omega_+ &= \frac{2}{|\beta_0|} \sqrt{(\alpha_0 - \delta_0) \delta_0} \\ \omega_- &= -\frac{2}{|\beta_0|} \sqrt{(\alpha_0 - \delta_0) \delta_0} \end{cases}$$

Now,

$$\alpha_0 - \delta_0 = \det(\overrightarrow{AC_1}, \overrightarrow{AC})$$

which gives:

$$\begin{cases} \omega_+ &= \frac{\sqrt{\det(\overrightarrow{AC_1}, \overrightarrow{AC}) \det(\overrightarrow{AA_1}, \overrightarrow{AC})}}{|\det(\overrightarrow{AB}, \overrightarrow{AC})|} \\ \omega_- &= -\frac{\sqrt{\det(\overrightarrow{AC_1}, \overrightarrow{AC}) \det(\overrightarrow{AA_1}, \overrightarrow{AC})}}{|\det(\overrightarrow{AB}, \overrightarrow{AC})|} \end{cases}$$

Note that:

$$\alpha_0 = \det(\overrightarrow{AA_1} + \overrightarrow{AC_1}, \overrightarrow{AC}) = 2 \det(\overrightarrow{AB_2}, \overrightarrow{AC})$$

where B_2 is the midpoint of segment $[A_1C_1]$. From Equation (23), the point $\gamma_P(t_0)$ lies on segment $[AC]$, with:

$$t_0 = -\frac{\omega\beta_0 + \beta_1}{2(\alpha_0 + \alpha_1\omega)}$$

i.e.,

$$t_0 = -\frac{\omega \det(\overrightarrow{AB}, \overrightarrow{AC}) - \det(\overrightarrow{AA_1}, \overrightarrow{AC})}{2(\det(\overrightarrow{AB_2}, \overrightarrow{AC}) - \omega \det(\overrightarrow{AB}, \overrightarrow{AC}))}$$

E Proof of Theorem 4

The following holds:

$$\det(\overrightarrow{A\gamma_P(t)}, \overrightarrow{AC}) = 0$$

which implies:

$$\det(\overrightarrow{AB} + \overrightarrow{B\gamma_P(t)}, \overrightarrow{AC}) = 0$$

and equivalently:

$$\det(\overrightarrow{AB}, \overrightarrow{AC}) + \det(\overrightarrow{B\gamma_P(t)}, \overrightarrow{AC}) = 0 \quad (24)$$

Then:

$$\begin{aligned} & \det(\overrightarrow{B\gamma_P(t)}, \overrightarrow{AC}) \\ &= \det((1-t)^2 t_A \overrightarrow{BC} + t^2 k_2 \overrightarrow{AB}, \overrightarrow{AC}) \\ &= (1-t)^2 t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + t^2 k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ &= (t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + k_2 \det(\overrightarrow{AB}, \overrightarrow{AC})) t^2 \\ &\quad - 2t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) t + t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) \end{aligned}$$

and Equation (24) leads to the following second-degree equation:

$$\begin{aligned} & t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) t^2 \\ &+ k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) t^2 \\ &- 2t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) t \\ &+ t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ &+ \det(\overrightarrow{AB}, \overrightarrow{AC}) = 0 \end{aligned} \quad (25)$$

whose reduced discriminant is:

$$\begin{aligned} \Delta' &= (-t_A \det(\overrightarrow{BC}, \overrightarrow{AC}))^2 \\ &- t_A^2 \det(\overrightarrow{BC}, \overrightarrow{AC}) \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ &- k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ &- t_A^2 \det(\overrightarrow{BC}, \overrightarrow{AC}) \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ &- k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \det(\overrightarrow{AB}, \overrightarrow{AC}) \end{aligned}$$

The parabola has only one point of contact with $[AB]$ if $\Delta' = 0$, which leads to:

$$\begin{aligned} & (t_A \det(\overrightarrow{BC}, \overrightarrow{AC}))^2 \\ &= t_A^2 \det(\overrightarrow{BC}, \overrightarrow{AC}) \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ &+ k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \det(\overrightarrow{AB}, \overrightarrow{AC}) \\ &+ t_A^2 \det(\overrightarrow{BC}, \overrightarrow{AC}) \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ &+ k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) \det(\overrightarrow{AB}, \overrightarrow{AC}) \end{aligned}$$

that is:

$$\begin{aligned} & t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + k_2 \det(\overrightarrow{AB}, \overrightarrow{AC}) = \\ & \frac{(t_A \det(\overrightarrow{BC}, \overrightarrow{AC}))^2}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})} \end{aligned}$$

and the solution is:

$$k_2 = \frac{-t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})}$$

Since Equation (25) has a double root, after some calculations, this root is:

$$t_0 = \frac{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}$$

and the point of contact between the ray $[AC]$ and the parabola γ_P is the point $\gamma_P(t_0)$.

To prove that the line (AC) is tangent to the parabola generated by γ_P , a homographic reparameterization with $a_h = 0$, $b_h = t_0$, and $c_h = d_h = 1$ is used. Table 2 gives the control points:

$$(Q_0, \varpi_0) = 1^2 \odot (P_0, 1) = (P_0, 1)$$

and $\gamma_P(t_0)$ has weight:

$$(1-t_0)^2 \cdot 1 + 2t_0(1-t_0) \cdot 1 + t_0^2 \cdot 1 = (1-t_0+t_0)^2 = 1$$

It remains to determine Q_1 with weight ϖ_1 . Then:

$$\begin{aligned} (Q_1, \varpi_1) &= (1-t_0) \odot (P_0, 1) \oplus t_0 \odot (P_1, 1) \\ &= (P_0, 1-t_0) \oplus (P_1, t_0) \\ &= (P_0, 1-t_0) \oplus (B, t_0) \end{aligned}$$

that is, $\varpi_1 = 1$. It remains to show that $Q_1 = C$. After simplifications, it yields:

$$1-t_0 = \frac{-\det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}$$

Thus, Q_1 is the barycenter of P_0 with weight:

$$\frac{-\det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}$$

and B with weight:

$$\frac{t_A \det(\overrightarrow{BC}, \overrightarrow{AC}) + \det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})}$$

That is, after multiplying both weights by $t_A \det(\overrightarrow{BC}, \overrightarrow{AC})$, the point Q_1 is defined by:

$$\overrightarrow{BQ_1} = \frac{-\det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})} \overrightarrow{BP_0}$$

that is:

$$\overrightarrow{BQ_1} = \frac{-\det(\overrightarrow{AB}, \overrightarrow{AC})}{t_A \det(\overrightarrow{BC}, \overrightarrow{AC})} t_A \overrightarrow{BC}$$

so:

$$\overrightarrow{BQ_1} = \frac{-\det(\overrightarrow{AB}, \overrightarrow{AC})}{\det(\overrightarrow{BC}, \overrightarrow{AC})} \overrightarrow{BC}$$

Now:

$$\begin{aligned} & \det(\overrightarrow{AB}, \overrightarrow{AC}) + \det(\overrightarrow{BC}, \overrightarrow{AC}) \\ = & \det(\overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{AC}) \\ = & \det(\overrightarrow{AC}, \overrightarrow{AC}) \\ = & 0 \end{aligned}$$

so:

$$-\det(\overrightarrow{AB}, \overrightarrow{AC}) = \det(\overrightarrow{BC}, \overrightarrow{AC})$$

which shows that $Q_1 = C$. ■

F Proof of Theorem 5 and of Corollary 2

F.1 Proof of Theorem 5

Let λ and μ be two strictly positive real numbers. A branch of the hyperbola $\mathcal{H}$ is modeled by a Bézier curve whose control mass points are $\overrightarrow{P_0} = \lambda \overrightarrow{AB}$ with weight 0, $P_1 = A$ with weight 1, and $\overrightarrow{P_2} = \mu \overrightarrow{AC}$ with weight 0. The vector $\overrightarrow{P_0}$ is obtained for $t = 0$, the vector $\overrightarrow{P_2}$ is obtained for

$t = 1$, and for $t \in]0, 1[$, the point $M(t)$ on the Bézier curve is defined, for any point O , by:

$$\begin{aligned} \overrightarrow{OM(t)} &= \frac{(1-t)^2}{2t(1-t)} \lambda \overrightarrow{AB} \\ &+ \frac{2t(1-t)}{2t(1-t)} \overrightarrow{OP_1} \\ &+ \frac{t^2}{2t(1-t)} \mu \overrightarrow{AC} \end{aligned}$$

which simplifies to:

$$\overrightarrow{AM(t)} = \frac{1-t}{2t} \lambda \overrightarrow{AB} + \frac{t}{2(1-t)} \mu \overrightarrow{AC}$$

Let $u_0 \in]0, 1[$. The parameters λ and μ are computed such that the line (BC) is tangent to the Bézier curve γ_P at $M(u_0)$. To achieve this, the Bézier curve γ_Q is constructed by performing a homographic parameter change on γ_P , according to Theorem 7, such that:

$$\begin{cases} \overrightarrow{Q_0} &= \overrightarrow{P_0} \\ Q_1 &= B \\ Q_2 &\in [BC] \end{cases}$$

The homographic parameter changes are $a_h = 0$, $b_h = u_0$, $c_h = d_h = 1$, that is:

$$h(u) = \frac{0 \times (1-u) + u_0 u}{1 \times (1-u) + 1 \times u} = u_0 u$$

and then:

$$(Q_0, \varpi_0) = (P_0, \omega_0)$$

Using Formula (20), computations leads to:

$$\begin{aligned} (Q_1, \varpi_1) &= (1-u_0) \odot (\overrightarrow{P_0}, 0) \oplus u_0 \odot (P_1, 1) \\ &= ((1-u_0)\lambda \overrightarrow{AB}, 0) \oplus (A, u_0) \\ &= (A_B, u_0) \end{aligned}$$

where:

$$\overrightarrow{AA_B} = \frac{(1-u_0)\lambda}{u_0} \overrightarrow{AB}$$

and $A_B = B$ if and only if:

$$\frac{(1-u_0)\lambda}{u_0} = 1 \Leftrightarrow \lambda = \frac{u_0}{1-u_0}$$

thus:

$$\overrightarrow{P_0} = \frac{u_0}{1-u_0} \overrightarrow{AB}$$

Similarly, it holds that:

$$\begin{aligned}
 (Q_2, \varpi_2) &= (1 - u_0)^2 \odot (\vec{P}_0, 0) \\
 &\oplus 2u_0(1 - u_0) \odot (P_1, 1) \\
 &\oplus u_0^2 \odot (\vec{P}_2, 0) \\
 &= (1 - u_0)^2 \odot \left(\frac{u_0}{1 - u_0} \vec{AB}, 0 \right) \\
 &\oplus (A, 2u_0(1 - u_0)) \\
 &\oplus u_0^2 \odot (\mu \vec{AC}, 0) \\
 &= (u_0(1 - u_0) \vec{AB}, 0) \\
 &\oplus (A, 2u_0(1 - u_0)) \\
 &\oplus (\mu u_0^2 \vec{AC}, 0) \\
 &= (u_0(1 - u_0) \vec{AB} + \mu u_0^2 \vec{AC}, 0) \\
 &\oplus (A, 2u_0(1 - u_0)) \\
 &= (A_C, 2u_0(1 - u_0))
 \end{aligned}$$

where:

$$\vec{AA_C} = \frac{u_0(1 - u_0)}{2u_0(1 - u_0)} \vec{AB} + \frac{\mu u_0^2}{2u_0(1 - u_0)} \vec{AC}$$

which simplifies to:

$$\vec{AA_C} = \frac{1}{2} \vec{AB} + \frac{\mu u_0}{2(1 - u_0)} \vec{AC}$$

and the point A_C belongs to the segment $[BC]$ if and only if:

$$\frac{\mu u_0}{2(1 - u_0)} = \frac{1}{2} \iff \mu = \frac{1 - u_0}{u_0}$$

hence:

$$\vec{P}_2 = \frac{1 - u_0}{u_0} \vec{AC}$$

and $M(u_0)$ is the midpoint of the segment $[BC]$.
■

F.2 Proof of Corollary 2

Formula (12), leads to:

$$\frac{u_0}{1 - u_0} \|\vec{AB}\| = \frac{1 - u_0}{u_0} \|\vec{AC}\|$$

The relation $\|\vec{AC}\| = \|\vec{AB}\|$ implies:

$$\frac{u_0}{1 - u_0} = \frac{1 - u_0}{u_0} \iff u_0^2 = (1 - u_0)^2$$

$$\iff u_0 = \frac{1}{2}$$

The relation $\|\vec{AC}\| \neq \|\vec{AB}\|$, leads to:

$$\begin{aligned}
 u_0^2 \|\vec{AB}\| &= (1 - u_0)^2 \|\vec{AC}\| \\
 &= (1 - 2u_0 + u_0^2) \|\vec{AC}\|
 \end{aligned}$$

which leads to the quadratic equation:

$$u_0^2 (\|\vec{AC}\| - \|\vec{AB}\|) - 2u_0 \|\vec{AC}\| + \|\vec{AC}\| = 0$$

The reduced discriminant is:

$$\begin{aligned}
 \Delta' &= \|\vec{AC}\|^2 - (\|\vec{AC}\| - \|\vec{AB}\|) \|\vec{AC}\| \\
 &= \|\vec{AC}\|^2 - (\|\vec{AC}\|^2 - \|\vec{AB}\| \|\vec{AC}\|) \\
 &= \|\vec{AB}\| \|\vec{AC}\|
 \end{aligned}$$

and the solutions are:

$$\begin{cases} u_{0,a} = \frac{\|\vec{AC}\| + \sqrt{\|\vec{AB}\| \|\vec{AC}\|}}{\|\vec{AC}\| - \|\vec{AB}\|} \\ u_{0,b} = \frac{\|\vec{AC}\| - \sqrt{\|\vec{AB}\| \|\vec{AC}\|}}{\|\vec{AC}\| - \|\vec{AB}\|} \end{cases}$$

- 1) If $\|\vec{AC}\| - \|\vec{AB}\| < 0$, then $u_{0,a} < 0$ and $u_{0,b} > 0$ and:

$$\iff u_{0,b} < 1$$

$$\iff \|\vec{AC}\| - \sqrt{\|\vec{AB}\| \|\vec{AC}\|} > \|\vec{AC}\| - \|\vec{AB}\|$$

$$\iff \sqrt{\|\vec{AB}\| \|\vec{AC}\|} < \|\vec{AB}\|$$

which is always true.

- 2) If $\|\vec{AC}\| - \|\vec{AB}\| > 0$, then $u_{0,a} > 0$ and $u_{0,b} > 0$ and:

$$u_{0,a} < 1$$

$$\iff \|\vec{AC}\| + \sqrt{\|\vec{AB}\| \|\vec{AC}\|} < \|\vec{AC}\| - \|\vec{AB}\|$$

$$\iff \sqrt{\|\vec{AB}\| \|\vec{AC}\|} < -\|\vec{AB}\|$$

which is false.

$$u_{0,b} < 1$$

$$\iff \|\vec{AC}\| - \sqrt{\|\vec{AB}\| \|\vec{AC}\|} < \|\vec{AC}\| - \|\vec{AB}\|$$

$$\Leftrightarrow -\sqrt{\|\vec{AB}\| \|\vec{AC}\|} < -\|\vec{AB}\|$$

$$\Leftrightarrow \sqrt{\|\vec{AB}\| \|\vec{AC}\|} > \|\vec{AB}\|$$

$$\Leftrightarrow \|\vec{AB}\| \|\vec{AC}\| > \|\vec{AB}\|^2$$

$$\Leftrightarrow \|\vec{AC}\| > \|\vec{AB}\|$$

$$\Leftrightarrow \|\vec{AC}\| - \|\vec{AB}\| > 0$$

In both cases, the solution is $u_{0,b}$.



Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0

https://creativecommons.org/licenses/by/4.0/deed.en_US