

Exponential Laws for Mixed-Order Ultradifferentiable Mappings

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Abstract: - We introduce the mixed-order classes $C_{[\mathcal{M}]}^{r,s}$ by combining mixed differentiability with the weight-matrix framework, in both the Beurling and Roumieu cases. At finite order we prove a chain rule under the Faà-di-Bruno property. The currying map is a linear injection. If r and s are finite, the class coincides with $C^{r,s}$ and the exponential law reduces to the mixed-order law without weights. Continuity of the inverse on the image holds when a lower weight bound is available at a single index.

Key-Words: -Mixed-order differentiability, Ultradifferentiable mappings, Weight matrices, Exponential law, Faà-di-Bruno formula, Convenient calculus, Beurling and Roumieu classes, Chain rule.

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1 Introduction

1.1 Background and motivation

The exponential law for mapping spaces is a fundamental tool in infinite-dimensional analysis and global geometry. It asserts a natural isomorphism between spaces of functions on a product and spaces of functions taking values in function spaces. In the convenient setting of global analysis [1], this isomorphism is well-understood for smooth mappings (C^∞) under appropriate topological hypotheses.

However, many natural problems require different degrees of differentiability in different variables. A full calculus for $C^{r,s}$ -mappings (mappings with r derivatives in the first factor and s in the second) was developed and the corresponding exponential law was established [2]. This calculus was further extended to multiple products in [3]. Parallel to this, the theory of ultradifferentiable functions (initiated in the seminal work [4]) intermediate between C^∞ and real-analytic [5], [6], which provides a unified framework for both sequence-defined and function-defined classes.

1.2 The gap in the literature and the main technical difficulty

Despite these advances, a unified theory for *mixed-order ultradifferentiable mappings* has been missing. Combining the $C^{r,s}$ -calculus with the weight matrices [5], [6] cannot be achieved by a formal combination of the existing theories.

The main technical difficulty lies in the

combinatorial proliferation of the mixed Faà-di-Bruno formula and the distinct analytical requirements of the exponential law. In the single-variable ultradifferentiable setting, the Faà-di-Bruno property (FB) controls the stability of the class under composition. In the mixed setting, computing the mixed derivative $\partial_1^p \partial_2^q (g \circ f)$ gives rise to numerous mixed combinatorial terms. Controlling these cross-terms requires estimates compatible with the Faà-di-Bruno derived sequence associated with the weight matrix.

The two mechanisms are used in separate arguments. The chain rule uses (FB) at finite order. The exponential law is proved here as a linear injection.

1.3 Main results and structure of the paper

Building on these observations, we develop a calculus for mixed-order ultradifferentiable mappings. Our main contributions are as follows:

1. We introduce the class $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, treating the Beurling and Roumieu cases separately (Definition 3.1).
2. We prove a mixed chain rule (Theorem 3.8) by combining the Faà-di-Bruno formula with the Faà-di-Bruno property of the underlying weight matrix, exploiting the derived sequence to absorb the combinatorial cross-terms.
3. The currying map is a linear injection (Theorem 4.4).
4. If r and s are finite, the class coincides with

$C^{r,s}$ and the exponential law reduces to the mixed-order law without weights (Theorem 4.7).

The paper is organised as follows. Section 2 recalls the necessary background on weight matrices and ultradifferentiable classes. Section 3 develops the calculus of $C^{r,s}_{[\mathcal{M}]}$ -mappings, including the Schwarz theorem and the chain rule. Section 4 treats the currying map. Section 5 discusses mapping spaces.

1.4 Notation and conventions

Throughout this paper, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. All locally convex spaces are assumed to be Hausdorff and over $\mathbb{K}$. For a locally convex space E , we denote by cP_E a family of seminorms generating its topology. The space of continuous linear maps from E to F is denoted by $L(E, F)$, and the space of continuous k -linear maps $E^k \rightarrow F$ is denoted by $L^k(E, F)$.

For multi-indices $\alpha \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$.

Key notation for weight matrices. Given a weight matrix $\mathcal{M} = \{M^x : x > 0\}$, we write $(M^x)_k^\circ$ for the *Faà-di-Bruno derived sequence* of M^x , defined in Equation (1):

$$(M^x)_k^\circ := \max \left\{ M_{\alpha_1}^x \dots M_{\alpha_j}^x : j \in \mathbb{N}, \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \dots + \alpha_j = k \right\}, \quad k \in \mathbb{N}_0, \quad (1)$$

with the convention $(M^x)_0^\circ := 1$. This sequence captures the combinatorial structure of the Faà-di-Bruno formula for compositions.

Throughout, we denote by (MG) the *moderate growth property* and by (FB) the *Faà-di-Bruno property* of a weight matrix; both notions are introduced and motivated in Definition 2.12 below.

2 Preliminaries

2.1 Locally convex spaces and the convenient setting

We work in the convenient setting of global analysis as developed in [1]. For a comprehensive treatment, we refer the reader to this standard reference.

Definition 2.1 (Mackey-Cauchy sequence, [1, Definition 1.1.1]). *A sequence (x_n) in a locally convex space E is called Mackey-Cauchy if there exists a bounded set $B \subseteq E$ and a sequence $(\mu_{n,k})_{n,k \in \mathbb{N}}$ in $\mathbb{K}$ converging to 0 such that $x_n - x_k \in \mu_{n,k} B$ for all $n, k \in \mathbb{N}$.*

Definition 2.2 (Mackey-completeness, [1, Definition 1.1.2]). *A locally convex space E is called Mackey-complete (or c^∞ -complete) if every Mackey-Cauchy sequence converges.*

Every sequentially complete locally convex space is Mackey-complete, and every Mackey-complete space is complete in the sense of uniform structures if it is metrizable. The category of Mackey-complete locally convex spaces is the appropriate setting for convenient analysis.

Definition 2.3 (c^∞ -topology, [1, Definition 1.2.2]). *For a locally convex space E , the c^∞ -topology on E is the final topology with respect to all smooth curves $c : \mathbb{R} \rightarrow E$. A subset $U \subseteq E$ is called c^∞ -open if $c^{-1}(U)$ is open in $\mathbb{R}$ for every smooth curve c .*

The c^∞ -topology is finer than the original locally convex topology and coincides with it for Fréchet spaces and, more generally, for all locally convex spaces that are bornological and whose bornology has a countable basis [1, Theorem 2.2.8].

2.2 Differentiability in locally convex spaces

We employ the notion of C^k -differentiability in the sense of the C_c^k -theory [7], which is standard in convenient analysis.

Definition 2.4 (Directional derivative, [2, Definition 2.1]). *Let E and F be locally convex spaces, $U \subseteq E$ open, and $f : U \rightarrow F$ a map. The directional derivative of f at $x \in U$ in the direction $h \in E$ is defined as*

$$df(x)(h) = \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} \quad (2)$$

whenever the limit exists. The map f is called differentiable if $df(x)(h)$ exists for all $x \in U$ and $h \in E$.

Definition 2.5 (C^k -maps, [2, Definition 2.2]). *Let E and F be locally convex spaces, $U \subseteq E$ open, and $k \in \mathbb{N}_0 \cup \{\infty\}$. A map $f : U \rightarrow F$ is called C^k if it is differentiable and the iterated directional derivatives*

$$d^{(j)}f(x)(h_1, \dots, h_j) \quad (3)$$

exist for all $j \leq k$, and the maps $d^{(j)}f : U \times E^j \rightarrow F$ are continuous. For $k = 0$, C^0 simply means continuity.

An equivalent characterization is that f is C^k if and only if it is C^1 and the derivative $df : U \rightarrow L(E, F)$ is C^{k-1} [2, Proposition 2.3].

2.3 Mixed differentiability: $C^{r,s}$ -maps

We now recall the notion of mixed differentiability introduced in [2].

Definition 2.6 ($C^{r,s}$ -maps, [2, Definition 3.1]). *Let E_1, E_2, F be locally convex spaces, let $U \subseteq E_1$ and*

$V \subseteq E_2$ be c^∞ -open, and let $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A map $f : U \times V \rightarrow F$ is called $C^{r,s}$ if for each $x \in U$, the map $f_x : V \rightarrow F$, $y \mapsto f(x, y)$ is C^s , and the map

$$\partial_2^j f : U \times V \times E_2^j \rightarrow F, \quad (x, y, \eta_1, \dots, \eta_j) \mapsto d^{(j)}(f_x)(y)(\eta_1, \dots, \eta_j) \quad (4)$$

is $C^{r,0}$ for all $j \leq s$ (interpreted as continuity when $j = 0$).

This definition is inductive in nature: a map is $C^{r,s}$ if it is C^s in the second variable for each fixed first variable, and the derivatives with respect to the second variable are C^r in the first variable. It was proved that this definition is symmetric in r and s when both are finite [2, Proposition 3.3].

Theorem 2.7 (Schwarz theorem for $C^{r,s}$ -maps, [2, Theorem 3.7]). *Let $f : U \times V \rightarrow F$ be $C^{r,s}$ with $r, s \geq 2$. Then the mixed derivatives are symmetric: for all $i \leq r$, $j \leq s$, $x \in U$, $y \in V$, $\xi_1, \dots, \xi_i \in E_1$, $\eta_1, \dots, \eta_j \in E_2$, we have*

$$\partial_1^i \partial_2^j f(x, y)(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_j) = \partial_1^i \partial_2^j f(x, y)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}, \eta_{\tau(1)}, \dots, \eta_{\tau(j)}) \quad (5)$$

for all permutations $\sigma \in S_i$, $\tau \in S_j$.

The main result of [2] is the following exponential law.

Theorem 2.8 (Exponential law for $C^{r,s}$ -maps, [2, Theorem 3.28]). *Let E_1, E_2, F be locally convex spaces, and let $U \subseteq E_1$, $V \subseteq E_2$ be convex subsets with dense interior. For $r, s \in \mathbb{N}_0 \cup \{\infty\}$, the map*

$$\Phi : C^{r,s}(U \times V, F) \rightarrow C^r(U, C^s(V, F)), \quad \Phi(\gamma)(x)(y) = \gamma(x, y) \quad (6)$$

is a linear topological embedding. If $U \times V \times E_1 \times E_2$ is a k -space or V is locally compact, then Φ is an isomorphism of topological vector spaces.

2.4 Ultradifferentiable classes and weight matrices

We now turn to ultradifferentiable classes. The convenient setting for Denjoy–Carleman classes was developed in a series of works [8], [9], [10]. The modern treatment via weight matrices was developed in [5], [6].

Definition 2.9 (Weight sequence, [6, Definition 2.1]). *A weight sequence is a sequence $M = (M_k)_{k \in \mathbb{N}_0}$ of positive real numbers with $M_0 = 1$. It is called:*

- log-convex if $M_k^2 \leq M_{k-1} M_{k+1}$ for all $k \in \mathbb{N}$;
- strongly log-convex if $M_k/k!$ is log-convex;
- of moderate growth if there exists $C > 0$ such that $M_{j+k} \leq C^{j+k} M_j M_k$ for all $j, k \in \mathbb{N}_0$.

Definition 2.10 (Weight matrix, [6, Definition 2.1]). *A weight matrix $\mathcal{M} = \{M^x : x \in \mathbb{R}_{>0}\}$ is a family of weight sequences indexed by $x > 0$ satisfying:*

- $M_0^x = 1$ for all $x > 0$;
- M^x is increasing in x : if $x \leq y$, then $M_k^x \leq M_k^y$ for all k ;
- each M^x is log-convex.

Readers familiar with sequence-defined ultradifferentiable classes may wonder how those fit into the weight matrix formalism. The correspondence is straightforward:

- **Classical Roumieu sequence** $M = (M_k)$: Embeds as the constant matrix $M_k^x := M_k$ for all $x > 0$.
- **Classical Beurling sequence** $M = (M_k)$: Embeds via the scaled matrix $M_k^x := x^k M_k$ (sometimes $x^k M_k/k!$ appears, depending on the normalisation convention).
- **Weight function** ω : Embeds via the matrix $M_k^x := \exp(x\omega^*(k))$, where $\omega^*(t) = \sup_{s \geq 0} (\omega(s) - st)$ is the Young conjugate.

For a thorough discussion we refer to [6, Section 3].

Weight matrices provide a unified framework for ultradifferentiable classes. For a given weight matrix $\mathcal{M}$, we define the associated ultradifferentiable classes as follows.

Definition 2.11 (Ultradifferentiable classes $\mathcal{E}_{[\mathcal{M}]}$, [6, Definition 3.1]). *Let $\mathcal{M}$ be a weight matrix, $U \subseteq \mathbb{R}^d$ open, and F a locally convex space. For a compact set $K \subset U$, define the seminorm*

$$\|f\|_{K, M^x} = \sup_{\substack{\alpha \in \mathbb{N}_0^d \\ x \in K}} \frac{\|\partial^\alpha f(x)\|_F}{M_{|\alpha|}^x}. \quad (7)$$

Then:

- The Roumieu-type class $\mathcal{E}_{\{\mathcal{M}\}}(U, F)$ consists of all $f \in C^\infty(U, F)$ such that for each compact $K \subset U$, there exists $x > 0$ with $\|f\|_{K, M^x} < \infty$.
- The Beurling-type class $\mathcal{E}_{(\mathcal{M})}(U, F)$ consists of all $f \in C^\infty(U, F)$ such that for each compact $K \subset U$ and each $x > 0$, we have $\|f\|_{K, M^x} < \infty$.

We write $\mathcal{E}_{[\mathcal{M}]}$ to denote either the Beurling or Roumieu case simultaneously, but we will be careful to specify the case when needed in proofs.

This definition follows the standard convention for weight matrices; see [6, Definition 3.1]. Note that the parameter x in M^x plays the role of the scaling

factor that appears as h in alternative formulations using weight functions.

The following properties are crucial for the convenient setting.

Definition 2.12 (Conditions on weight matrices, [6, Section 4]). Let $\mathcal{M} = \{M^x : x > 0\}$ be a weight matrix and $m_k^x := M_k^x/k!$. We say that $\mathcal{M}$ satisfies:

- the moderate growth property, denoted (MG), if for every $x > 0$ there exist $y > 0$ and $C > 0$ such that

$$M_{j+k}^x \leq C^{j+k} M_j^y M_k^y \quad \text{for all } j, k \in \mathbb{N}_0; \quad (8)$$

- the Beurling-type Faà-di-Bruno property, denoted (FB-B), if for every $x_0 > 0$ there exist $y_0 > 0$ and $C > 0$ such that, writing $(m^{y_0})_k^\circ$ for the Faà-di-Bruno derived sequence of m^{y_0} ,

$$(m^{y_0})_k^\circ := \max \{m_j^{y_0} m_{\alpha_1}^{y_0} \cdots m_{\alpha_j}^{y_0} : j \in \mathbb{N}, \alpha_1, \dots, \alpha_j \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k\}, \quad (9)$$

with the convention $(m^{y_0})_0^\circ := 1$, one has

$$(m^{y_0})_k^\circ \leq C^k m_k^{x_0} \quad \text{for all } k \in \mathbb{N}_0. \quad (10)$$

- the Roumieu-type Faà-di-Bruno property, denoted (FB-R), if for every $y_0 > 0$ there exist $x_0 > 0$ and $C > 0$ such that

$$(m^{y_0})_k^\circ \leq C^k m_k^{x_0} \quad \text{for all } k \in \mathbb{N}_0. \quad (11)$$

We say that $\mathcal{M}$ satisfies the Faà-di-Bruno property (FB) if it satisfies both (FB-B) and (FB-R).

Remark 2.13 (Why both directions of (FB) are needed). The two directions of the Faà-di-Bruno property correspond to the distinct logical structures of the Beurling and Roumieu cases:

- **Condition** (MG) controls how a single weight M_{j+k}^x splits into a product of two weights of smaller total order—precisely the manipulation that occurs when one passes from a seminorm on γ to the corresponding seminorm on $\Phi(\gamma)$ in the proof of the exponential law (Section 4).
- **Condition** (FB-B) is used in the Beurling case: since Beurling classes require bounds at every index $x_0 > 0$, we are free to choose the auxiliary index y_0 after the target x_0 is fixed. This is precisely the direction $\forall x_0 \exists y_0$.
- **Condition** (FB-R) is used in the Roumieu case: since Roumieu classes only guarantee bounds at some indices y_1, y_2 (which are data, not choices), once $y_0 := \max(y_1, y_2)$ is fixed by this data, we must find a target index x_0 dominating the composition. This is precisely the direction $\forall y_0 \exists x_0$.

Neither direction implies the other in general. However, both directions hold automatically in all standard examples: constant matrices from a single strongly log-convex sequence, Beurling-scaled matrices $M_k^x = x^k M_k$, and matrices associated with a regular weight function via Young conjugation (see [11, Proposition 4.4] and [12, Section 2]).

The precise relationship between (FB-B) and (FB-R) under (MG) remains an open question; see Open Problem 6.3.

In the present paper we assume both directions whenever required, and the precise hypothesis is stated explicitly at the beginning of each theorem that goes beyond a routine application. Related results on the Borel map in the mixed Beurling setting were obtained in [13].

Theorem 2.14 (Composition in ultradifferentiable classes, [11, Theorem 6.3]). Let $\mathcal{M}$ be a weight matrix satisfying the Faà-di-Bruno property. Then the classes $\mathcal{E}_{[\mathcal{M}]}$ are stable under composition: if $f \in \mathcal{E}_{[\mathcal{M}]}(U, \mathbb{R}^m)$ and $g \in \mathcal{E}_{[\mathcal{M}]}(V, \mathbb{R})$ with $f(U) \subseteq V$, then $g \circ f \in \mathcal{E}_{[\mathcal{M}]}(U, \mathbb{R})$. Moreover, the composition map is continuous in the appropriate topologies.

2.5 Function space topologies

For spaces of differentiable maps, we employ the C^k -compact-open topology (also called the topology of uniform convergence of derivatives on compact sets). For a detailed discussion, see [1], [2].

Definition 2.15 (C^k -compact-open topology, [2, Section 2.2]). Let E, F be locally convex spaces, $U \subseteq E$ open, and $k \in \mathbb{N}_0 \cup \{\infty\}$. The space $C^k(U, F)$ is equipped with the locally convex topology generated by seminorms

$$p_{K, \rho, \ell}(f) = \sup_{\substack{x \in K \\ h_1, \dots, h_\ell \in B}} \rho(d^\ell f(x)(h_1, \dots, h_\ell)) \quad (12)$$

where $K \subset U$ is compact, $\ell \leq k$, $B \subset E$ is bounded, and ρ is a continuous seminorm on F .

For $C^{r,s}$ -spaces, we use a mixed variant of this topology; see [2, Section 3.2] for details.

Remark 2.16. The ultradifferentiable bounds introduced in Definition 2.11 are compatible with the C^k -compact-open topology: a sequence (f_n) converges in $\mathcal{E}_{[\mathcal{M}]}$ if and only if it converges in the C^∞ -compact-open topology and the ultradifferentiable seminorms $\|f_n\|_{K, M^x}$ remain uniformly bounded. This compatibility ensures that the topological and the growth-condition aspects of the theory reinforce each other in the proofs of the exponential law (Section 4).

For the mixed spaces $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, the topology is defined by the family of seminorms that control the derivatives in each variable separately, augmented with the weight matrix bounds. Specifically, a fundamental system of seminorms is given as follows.

For the Roumieu case $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, we define for each compact set $K \subset U \times V$, each continuous seminorm q on F , each $x > 0$, and each pair (p, j) with $p \leq r, j \leq s$:

$$p_{K,q,x,p,j}(f) := \sup_{\substack{(a,b) \in K \\ \xi \in B_1^p, \eta \in B_2^j}} \frac{q(\partial_1^p \partial_2^j f(a, b)(\xi, \eta))}{M_{p+j}^x},$$

where B_1 and B_2 range over all bounded subsets of E_1 and E_2 , respectively. The topology is the locally convex topology generated by the family of all such seminorms.

For the Beurling case $C_{(\mathcal{M})}^{r,s}(U \times V, F)$, the topology is generated by the same family of seminorms, but with the quantifier changed: the seminorms are required to be finite for every $x > 0$. Equivalently, the Beurling space is the projective limit (over $x > 0$) of the spaces equipped with the seminorms using a single weight M^x .

This topology is compatible with the C^k -compact-open topology: a sequence (f_n) converges in $C_{[\mathcal{M}]}^{r,s}$ if and only if it converges in the classical $C^{r,s}$ -topology (i.e., uniformly on compact sets together with all derivatives up to orders r, s) and the ultradifferentiable seminorms remain uniformly bounded (with respect to the appropriate quantifiers). This compatibility is essential for the proof of the exponential law (Theorem 4.7), where we repeatedly pass between convergence in the source space $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ and the target space $C_{[\mathcal{M}]}^r(U, C_{[\mathcal{M}]}^s(V, F))$.

For a detailed discussion of function space topologies in the convenient setting, we refer to [1].

3 Calculus of $C_{[\mathcal{M}]}^{r,s}$ -mappings

3.1 Definition and basic properties

We now introduce the central object of this paper: mixed-order ultradifferentiable mappings. We treat the Beurling and Roumieu cases separately to avoid ambiguity in the quantifiers.

Definition 3.1 ($C_{[\mathcal{M}]}^{r,s}$ -mappings). Let E_1, E_2, F be locally convex spaces, let $U \subseteq E_1$ and $V \subseteq E_2$ be c^∞ -open, and let $r, s \in \mathbb{N}_0 \cup \{\infty\}$. We denote by $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ the class of all $C^{r,s}$ -mappings $f : U \times V \rightarrow F$ whose mixed derivatives satisfy

the ultradifferentiable bounds specified below. We emphasise that no regularity hypothesis on $\mathcal{M}$ is imposed at the level of the definition; the properties (MG), (FB-B), and (FB-R) of Definition 2.12 are added explicitly in the statements where they are exploited. This separation allows different structural properties of $C_{[\mathcal{M}]}^{r,s}$ to be established under different assumptions on the weight matrix.

A map $f : U \times V \rightarrow F$ is called $C_{[\mathcal{M}]}^{r,s}$ of Roumieu type if:

1. For each $x \in U$, the map $f_x : V \rightarrow F, y \mapsto f(x, y)$ belongs to $C_{[\mathcal{M}]}^s(V, F)$.
2. The map $\partial_2^j f : U \times V \times E_2^j \rightarrow F$, defined by
$$\partial_2^j f(x, y)(\eta_1, \dots, \eta_j) = d^{(j)}(f_x)(y)(\eta_1, \dots, \eta_j),$$
(13) belongs to $C_{[\mathcal{M}]}^{r,0}(U \times V \times E_2^j, F)$ for all $j \leq s$.
3. For each compact $K_1 \subset U$, each compact $K_2 \subset V$, and each continuous seminorm q on F , there exist $x > 0$ and $C > 0$ such that for all $p \leq r, j \leq s$, and all bounded sets $B_1 \subset E_1, B_2 \subset E_2$,

$$\sup_{\substack{a \in K_1, b \in K_2 \\ \xi \in B_1^p, \eta \in B_2^j}} q(\partial_1^p \partial_2^j f(a, b)(\xi, \eta)) \leq C^{p+j} M_{p+j}^x$$
(14)

A map f is called $C_{(\mathcal{M})}^{r,s}$ of Beurling type if it satisfies conditions (1) and (2) above, and condition (3) is replaced by: for every compact $K_1 \subset U$, every compact $K_2 \subset V$, every continuous seminorm q on F , and every $x > 0$, there exists $C > 0$ such that the bound in (14) holds.

We write $C_{[\mathcal{M}]}^{r,s}$ to denote either case when the distinction is not important, but in proofs we will treat them separately.

Remark 3.2 (Choice of weight in the definition). The weight M_{p+j}^x in condition (14) controls the total derivative order $p + j$. Under the moderate growth condition (MG), the weight M_{p+j}^x is equivalent to the product weight $M_p^x M_j^x$ up to constants, so the class $C_{[\mathcal{M}]}^{r,s}$ would be the same under either choice. We use M_{p+j}^x in the definition to avoid imposing (MG) at the definitional level, and to maintain compatibility with the standard ultradifferentiable classes when $r = s$.

Remark 3.3 (Consistency with the weight matrix definition). When $r = s = \infty$, the class $C_{[\mathcal{M}]}^{\infty, \infty}(U \times V, F)$ coincides with the class $\mathcal{E}_{[\mathcal{M}]}(U \times V, F)$ defined in Definition 2.11. Indeed, in this case, the mixed

derivatives $\partial_1^p \partial_2^j f$ reduce to the total derivatives $d^{p+j} f$, and the bound becomes

$$\sup_{(x,y) \in K} \frac{\|d^{p+j} f(x,y)(\xi,\eta)\|_F}{C^{p+j} M_{p+j}^x} < \infty, \quad (15)$$

which is precisely the condition defining $\mathcal{E}_{[\mathcal{M}]}(U \times V, F)$.

Remark 3.4 (Reduction to classical $C^{r,s}$ -spaces). If $\mathcal{M}$ is the trivial weight matrix (i.e., $M_k^x = C^k$ for some constant $C > 0$ independent of x), then the class $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ reduces to the classical space $C^{r,s}(U \times V, F)$ [2]. In this case, the ultradifferentiable bounds are automatically satisfied for any $C^{r,s}$ -mapping, since the derivatives are bounded on compact sets.

Proposition 3.5 (Basic properties). Let $\mathcal{M}$ be a weight matrix.

1. If $f \in C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, then for each fixed $y \in V$, the map $f^y : U \rightarrow F$, $x \mapsto f(x, y)$ belongs to $C_{[\mathcal{M}]}^r(U, F)$.
2. If $f \in C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ with $r, s \geq 1$, then the partial derivatives commute: $\partial_1 \partial_2 f = \partial_2 \partial_1 f$.
3. The space $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ is a Mackey-complete locally convex space when equipped with the natural seminorms.

Proof. (1) We prove the Roumieu case; the Beurling case is analogous with quantifiers adjusted. Fix $y \in V$ and let $L \subset U$ be compact. Consider the compact set $K = L \times \{y\}$ (or more generally, $L \times \tilde{L}$ for a suitable compact $\tilde{L}$ containing y). By Definition 3.1(3), there exists $x_0 > 0$ such that bounds hold for all mixed derivatives on K . For $i \leq r$, we have

$$\|d^i f^y(x)(\xi_1, \dots, \xi_i)\| = \|\partial_1^i f(x, y)(\xi_1, \dots, \xi_i)\| \leq C M_i^{x_0} \quad (16)$$

uniformly for $x \in L$ and $\xi_1, \dots, \xi_i$ bounded. Hence $f^y \in C_{\{\mathcal{M}\}}^r(U, F)$. The symmetry in the definition (which will be fully justified by Theorem 3.6) ensures that the order of differentiation does not matter.

(2) This follows from the classical Schwarz theorem for $C^{r,s}$ -maps [2, Theorem 3.7] combined with the fact that the ultradifferentiable bounds are preserved under the limits involved in the proof. For fixed x, y , consider the function $\varphi(s, t) = f(x + s\xi, y + t\eta)$ for small $s, t \in \mathbb{R}$. By the definition of $C_{[\mathcal{M}]}^{r,s}$, φ is $C^{2,2}$ in the classical sense, so $\partial_s \partial_t \varphi(0, 0) = \partial_t \partial_s \varphi(0, 0)$, which gives the desired equality.

(3) Mackey-completeness follows from the corresponding property of C^k -spaces [1, Theorem 1.2.12] and the fact that ultradifferentiable bounds are closed under limits of sequences that are bounded in the appropriate seminorms. If (γ_n) is a Mackey-Cauchy sequence in $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, then for each compact K and each i, j , the sequence of derivatives $\partial_1^i \partial_2^j \gamma_n$ is Mackey-Cauchy in $C(K \times (B_1^i \times B_2^j), F)$ and hence converges uniformly. The limit satisfies the same ultradifferentiable bounds because for Roumieu case we can choose a common x for the whole sequence (using the Mackey-Cauchy property to control the constants), and for Beurling case the bounds hold for every x by pointwise convergence. $\square$

3.2 Mixed derivatives and Schwarz theorem

The symmetry of mixed derivatives is a fundamental property that must be verified in our setting. We provide a complete proof that handles both cases.

Theorem 3.6 (Schwarz theorem for $C_{[\mathcal{M}]}^{r,s}$ -maps). Let $f \in C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ with $r, s \geq 2$. Then for all $i \leq r, j \leq s, x \in U, y \in V, \xi_1, \dots, \xi_i \in E_1, \eta_1, \dots, \eta_j \in E_2$, and for all permutations $\sigma \in S_i, \tau \in S_j$, we have

$$\partial_1^i \partial_2^j f(x, y)(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_j) = \partial_1^i \partial_2^j f(x, y)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}, \eta_{\tau(1)}, \dots, \eta_{\tau(j)}). \quad (17)$$

Moreover, the ultradifferentiable estimates are preserved under this interchange. No condition on the weight matrix $\mathcal{M}$ is required for this result.

Proof. By Definition 3.1, every $C_{[\mathcal{M}]}^{r,s}$ -mapping is, in particular, a $C^{r,s}$ -mapping. Therefore, the classical Schwarz theorem for $C^{r,s}$ -maps [2, Theorem 3.7] applies, and we have the algebraic equality

$$\partial_1^i \partial_2^j f(x, y)(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_j) = \partial_1^i \partial_2^j f(x, y)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}, \eta_{\tau(1)}, \dots, \eta_{\tau(j)}) \quad (18)$$

for all permutations $\sigma \in S_i, \tau \in S_j$.

It remains to verify that the ultradifferentiable estimates are preserved under this interchange.

Preservation of ultradifferentiable estimates (Roumieu case). For a compact set $K \subset U \times V$, the Roumieu condition provides an index $x_0 > 0$ such that the bound holds with $M_{i+j}^{x_0}$. The supremum in the seminorm is taken over the Cartesian product of bounded sets $B_1^i \times B_2^j$, which is trivially invariant under permutation of the arguments within each group. Therefore, the supremum of the norm is unchanged, and the exact same index x_0 and constant apply.

Detailed estimates for the Beurling case. For the Beurling class $C_{(\mathcal{M})}^{r,s}$, the ultradifferentiable bounds

must hold for every weight index $\rho > 0$. Fix an arbitrary index $\rho > 0$ and a compact set $K \subset U \times V$. Since $f \in C_{(\mathcal{M})}^{r,s}(U \times V, F)$, by definition, there exists a constant $C > 0$ such that for all $i \leq r$, $j \leq s$, and all bounded directions $\xi_k \in B_1$, $\eta_l \in B_2$, we have

$$\sup_{(x,y) \in K} \frac{\|\partial_1^i \partial_2^j f(x,y)(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_j)\|_F}{M_{i+j}^\rho} \leq C. \quad (19)$$

The classical symmetry argument yields the exact algebraic equality (18). Since the pointwise norms are equal, and the supremum is taken over the permutation-invariant domain $B_1^i \times B_2^j$, the supremum of the norm is unchanged. Thus, the permuted derivative satisfies the exact same bound:

$$\sup_{(x,y) \in K} \frac{\|\partial_1^i \partial_2^j f(x,y)(\xi_{\sigma(1)}, \dots, \eta_{\tau(j)})\|_F}{M_{i+j}^\rho} \leq C. \quad (20)$$

Crucially, the weight index $\rho > 0$ remains unchanged. Since $\rho > 0$ was chosen arbitrarily, this bound holds for every $\rho > 0$, confirming that the permuted derivatives belong to the same Beurling class $C_{(\mathcal{M})}^{r,s}$ without any shift or loss in the weight index.

Hence the ultradifferentiable estimates are unaffected by the interchange of the mixed derivatives. $\square$

3.3 Chain rule

The chain rule is the most delicate part of the calculus. It requires the composition stability of ultradifferentiable classes, which is precisely characterized by the Faà-di-Bruno property. We provide a complete proof with careful handling of both cases.

Lemma 3.7 (Faà-di-Bruno formula for mixed derivatives). *Let E_1, E_2, F, G be locally convex spaces, and let $U \subseteq E_1$, $V \subseteq E_2$, $W \subseteq F$ be open. Let $f : U \times V \rightarrow F$ be of class $C^{p,q}$ and $g : W \rightarrow G$ be of class C^{p+q} . Then the mixed derivative $\partial_1^p \partial_2^q (g \circ f)$ exists and is given by*

$$\partial_1^p \partial_2^q (g \circ f)(x,y)(\xi, \eta) = \sum_{\pi \in \Pi(p,q)} g^{(|\pi|)}(f(x,y)) \circ \prod_{B \in \pi} \partial_1^{p_B} \partial_2^{q_B} f(x,y)(\xi_B, \eta_B), \quad (21)$$

where $\Pi(p,q)$ denotes the set of partitions of the set $\{1, \dots, p\} \times \{1, \dots, q\}$ into non-empty blocks, each block B having p_B indices from $\{1, \dots, p\}$ and q_B indices from $\{1, \dots, q\}$.

Proof. We argue by induction on the total order $N := p + q$.

If $N = 0$, then

$$g \circ f = g \circ f, \quad (22)$$

and the formula is trivial.

For $N = 1$, there are only the two cases $(p,q) = (1,0)$ and $(0,1)$. The classical chain rule yields

$$\partial_1(g \circ f) = g^{(1)}(f) \circ \partial_1 f, \quad (23)$$

and similarly

$$\partial_2(g \circ f) = g^{(1)}(f) \circ \partial_2 f, \quad (24)$$

which coincide with the stated formula since the corresponding partition contains a single block.

Assume now that the statement holds for every pair (p',q') with $p' + q' < N$, and let $p + q = N$.

Without loss of generality suppose that $p > 0$. (The case $q > 0$ is completely analogous.)

Applying ∂_1 to the induction formula for $\partial_1^{p-1} \partial_2^q (g \circ f)$, the ordinary product rule together with the chain rule gives two kinds of terms.

First, differentiating the outer derivative of g produces

$$g^{(|\pi|+1)}(f) \left(\partial_1 f, \partial_1^{p_B} \partial_2^{q_B} f \right)_{B \in \pi}, \quad (25)$$

which corresponds combinatorially to creating a new singleton block for the new differentiation index.

Second, differentiating one of the factors

$$\partial_1^{p_B} \partial_2^{q_B} f$$

gives

$$g^{(|\pi|)}(f) \left(\partial_1^{p_B+1} \partial_2^{q_B} f, (\partial_1^{p_C} \partial_2^{q_C} f)_{C \neq B} \right), \quad (26)$$

which corresponds to inserting the new differentiation index into the existing block B .

These two operations generate every partition of the new set of differentiation indices exactly once. Indeed, every partition in $\Pi(p,q)$ either contains the new first-variable index as a singleton block or else this index belongs to a unique existing block. Removing that index recovers uniquely a partition in $\Pi(p-1,q)$. Conversely, adjoining the new index by either of the two operations above produces a unique partition of $\Pi(p,q)$. Hence the inductive differentiation reproduces precisely the combinatorial structure of the mixed Faà-di-Bruno expansion.

Since each differentiation is justified by the classical chain rule and the induction hypothesis, summing over all partitions yields

$$\partial_1^p \partial_2^q (g \circ f)(x,y) = \sum_{\pi \in \Pi(p,q)} g^{(|\pi|)}(f(x,y)) \circ \prod_{B \in \pi} \partial_1^{p_B} \partial_2^{q_B} f(x,y), \quad (27)$$

which is exactly the asserted formula.

This argument is the mixed analogue of the classical proof of the Faà-di-Bruno formula given in [1, Section III.3]; the identification of the mixed derivatives with the $C^{r,s}$ -calculus follows from the framework developed in [2]. $\square$

Unlike the one-variable case, the proof of the mixed chain rule requires controlling mixed partitions arising in the Faà-di-Bruno expansion. The derived sequence plays the key role in absorbing the resulting products of weights into a single admissible weight.

Theorem 3.8 (Mixed chain rule). *Let $\mathcal{M}$ be a weight matrix satisfying the Faà-di-Bruno property (FB) of Definition 2.12. Let E_1, E_2, F, G be locally convex spaces, and let $U \subseteq E_1, V \subseteq E_2, W \subseteq F$ be C^∞ -open. Consider mappings*

$$f \in C_{[\mathcal{M}]}^{r,s}(U \times V, F), \quad g \in C_{[\mathcal{M}]}^{r+s}(W, G) \quad (28)$$

such that $f(U \times V) \subseteq W$. Then the composition $g \circ f$ belongs to

$$C_{[\mathcal{M}]}^{r,s}(U \times V, G). \quad (29)$$

Moreover, the composition map $(g, f) \mapsto g \circ f$ is continuous in the appropriate topologies.

Proof. The essential new feature.

Let $p \leq r$ and $q \leq s$. We need to show that $\partial_1^p \partial_2^q (g \circ f)$ exists, is continuous, and satisfies the appropriate ultradifferentiable bounds.

Step 1: Analytical form of the mixed derivative.

By Lemma 3.7, the mixed derivative $\partial_1^p \partial_2^q (g \circ f)$ is given by the mixed Faà-di-Bruno formula (21):

$$\partial_1^p \partial_2^q (g \circ f)(x, y)(\xi, \eta) = \sum_{\pi \in \Pi(p, q)} g^{(|\pi|)}(f(x, y)) \circ \prod_{B \in \pi} \partial_1^{p_B} \partial_2^{q_B} f(x, y)(\xi_B, \eta_B), \quad (30)$$

where $\Pi(p, q)$ denotes the set of partitions of the set $\{1, \dots, p\} \times \{1, \dots, q\}$ into non-empty blocks. Since $f \in C_{[\mathcal{M}]}^{r,s} \subseteq C^{r,s}$ and $g \in C_{[\mathcal{M}]}^{r+s} \subseteq C^{r+s}$, all the derivatives appearing in this formula exist and are continuous by the classical $C^{r,s}$ -calculus [2]. This establishes the analytical form of the mixed derivative.

Step 2: Ultradifferentiable estimates (Roumieu case). Let $K \subset U \times V$ be compact. We need to find an index $x_0 > 0$ and a constant $C > 0$ such that the composition satisfies the bound with $M_{p+q}^{x_0}$.

Since $f \in C_{\{\mathcal{M}\}}^{r,s}$, for the compact set K there exists an index $y_1 > 0$ such that for all $p' \leq r, q' \leq s$,

$$\|\partial_1^{p'} \partial_2^{q'} f(z)(\xi, \eta)\| \leq C_1 M_{p'+q'}^{y_1} \quad (31)$$

uniformly for $z \in K$ and bounded directions.

Similarly, since $f(K) \subset W$ is compact and $g \in C_{\{\mathcal{M}\}}^{r+s}$, there exists an index $y_2 > 0$ such that for all $m \leq r + s$,

$$\|g^{(m)}(w)(\zeta_1, \dots, \zeta_m)\| \leq C_2 M_m^{y_2} \quad (32)$$

uniformly for $w \in f(K)$ and bounded directions.

These indices y_1, y_2 are given to us by the hypothesis $f, g \in \text{Roumieu class}$; we do not choose them.

Set $y'_0 := \max(y_1, y_2)$. By the monotonicity of the weight matrix in the index x (i.e., $M_k^x \leq M_k^y$ for $x \leq y$ and all k), we have $M_k^{y_1} \leq M_k^{y'_0}$ and $M_m^{y_2} \leq M_m^{y'_0}$.

Now, and only now, do we invoke (FB-R), applied at this specific y'_0 : there exist $x_0 > 0$ and $C_0 \geq 1$ such that

$$(m^{y'_0})_k^\circ \leq C_0^k m_k^{x_0} \quad \text{for all } k. \quad (33)$$

This is legitimate because (FB-R) is quantified as “for every y_0 there exists x_0 ,” and y'_0 has already been fixed by the previous paragraph—the quantifier order in Definition 2.12 matches, term for term, the order in which the data becomes available in this proof.

Now, using the Faà-di-Bruno formula and the estimates above, we obtain for any partition $\pi \in \Pi(p, q)$ with $m := |\pi|$:

$$\|g^{(m)}(f(z))\| \cdot \prod_{B \in \pi} \|\partial_1^{p_B} \partial_2^{q_B} f(z)\| \leq C_2^m M_m^{y_2} \cdot C_1^{p+q} \prod_{B \in \pi} M_{p_B+q_B}^{y_1}. \quad (34)$$

Since $y_1, y_2 \leq y'_0$ and the weight matrix is increasing in x , we have $M_k^{y_1} \leq M_k^{y'_0}$ and $M_m^{y_2} \leq M_m^{y'_0}$. Therefore,

$$C_2^m M_m^{y_2} \cdot C_1^{p+q} \prod_{B \in \pi} M_{p_B+q_B}^{y_1} \leq C_2^m C_1^{p+q} M_m^{y'_0} \prod_{B \in \pi} M_{p_B+q_B}^{y'_0}. \quad (35)$$

Since $m = |\pi| \leq p + q$, we may enlarge the constants and write

$$C_2^m C_1^{p+q} \leq C_3^{p+q}, \quad (36)$$

for a suitable constant $C_3 \geq 1$. Converting to normalized weights $m_k = M_k/k!$ and using the definition of the derived sequence, this is bounded by:

$$C_2^m C_1^{p+q} (m^{y'_0})_{p+q}^\circ \cdot (p+q)! \cdot \frac{m! \prod_{B \in \pi} (p_B + q_B)!}{(p+q)!} \leq C_3^{p+q} (m^{y'_0})_{p+q}^\circ \cdot (p+q)!, \quad (37)$$

where we absorbed the finite (depending only on p, q) number of partitions and the combinatorial factor into C_3 .

By the bound obtained from (FB-R), $(m^{y'_0})_{p+q}^\circ \leq C_0^{p+q} m_{p+q}^{x_0}$, so converting back to $M_k = k! m_k$,

$$\|\partial_1^p \partial_2^q (g \circ f)(z)(\xi, \eta)\| \leq C_4 M_{p+q}^{x_0} \quad (38)$$

for a constant $C_4 = C_4(K, p, q)$, summing over the finitely many $\pi \in \Pi(p, q)$. This is exactly the Roumieu-type bound required.

Step 3: Ultradifferentiable estimates (Beurling case). In the Beurling setting, we must show that for every given target index $x_0 > 0$, there exists a constant $C > 0$ such that the composition satisfies the bound with $M_{p+q}^{x_0}$. Let $x_0 > 0$ be given arbitrarily.

We exploit the Beurling-type Faà-di-Bruno property (FB-B). By Definition 2.12, for the target index $x_0 > 0$, there exists an auxiliary index $y_0 > 0$ such that

$$(m^{y_0})^\circ \preceq m^{x_0}, \quad (39)$$

where $m_k^x = M_k^x/k!$ and $(m^{y_0})^\circ$ is the sequence defined by

$$(m^{y_0})_k^\circ := \max \left\{ m_j^{y_0} \prod_{\ell=1}^j m_{\alpha_\ell}^{y_0} : j \in \mathbb{N}, \alpha_\ell \in \mathbb{N}_{>0}, \sum_{\ell=1}^j \alpha_\ell = k \right\}, \quad (m^{y_0})_0^\circ := 1. \quad (40)$$

Derivation of the fundamental absorption estimate. Since $(m^{y_0})^\circ \preceq m^{x_0}$, by the definition of $\preceq$, there exists a constant $C_0 > 0$ such that for every $k \in \mathbb{N}$,

$$(m^{y_0})_k^\circ \leq C_0^k m_k^{x_0}. \quad (A)$$

For any partition $\pi \in \Pi(p, q)$ with $m := |\pi|$ blocks, the product of the corresponding m -sequence weights satisfies

$$m_m^{y_0} \prod_{B \in \pi} m_{p_B+q_B}^{y_0} \leq (m^{y_0})_{p+q}^\circ, \quad (41)$$

because the left-hand side is precisely one of the terms over which the maximum in the definition of $(m^{y_0})_{p+q}^\circ$ is taken. Combining this with (A) and recalling that $m_k = M_k/k!$, we obtain

$$\frac{M_m^{y_0}}{m!} \prod_{B \in \pi} \frac{M_{p_B+q_B}^{y_0}}{(p_B+q_B)!} \leq C_0^{p+q} \frac{M_{p+q}^{x_0}}{(p+q)!}. \quad (42)$$

Multiplying through by the factorial terms, we get

$$M_m^{y_0} \prod_{B \in \pi} M_{p_B+q_B}^{y_0} \leq C_0^{p+q} M_{p+q}^{x_0} \cdot \frac{m! \prod_{B \in \pi} (p_B+q_B)!}{(p+q)!}. \quad (43)$$

The combinatorial factor $\frac{m! \prod_{B \in \pi} (p_B+q_B)!}{(p+q)!}$ is bounded by a constant depending only on p and q (it is the inverse of a multinomial coefficient and is ≤ 1 in fact, since $\frac{(p+q)!}{m! \prod_{B \in \pi} (p_B+q_B)!}$ is an integer). Thus, absorbing this constant into C_0 , we obtain the fundamental absorption estimate:

$$M_m^{y_0} \prod_{B \in \pi} M_{p_B+q_B}^{y_0} \leq C_1^{p+q} M_{p+q}^{x_0}. \quad (B)$$

(We emphasize that this estimate does *not* require any assumption on the relative size of y_0 and x_0 ; it follows purely from the Beurling-type (FB-B) property.)

Applying the Beurling bounds. Since $g \in C_{(\mathcal{M})}^{r+s}(W, G)$ and $f \in C_{(\mathcal{M})}^{r',s'}(U \times V, F)$ belong to the Beurling class, their ultradifferentiable bounds hold for every positive index. In particular, for the auxiliary index $y_0 > 0$ obtained above, there exist constants $C_2, C_3 > 0$ such that for all $m \leq r+s$ and all $p' \leq r, q' \leq s$,

$$\|g^{(m)}(w)(\zeta_1, \dots, \zeta_m)\|_G \leq C_2^m M_m^{y_0} \quad (44)$$

uniformly for $w \in f(K)$, and

$$\|\partial_1^{p'} \partial_2^{q'} f(z)(\xi, \eta)\|_F \leq C_3^{p'+q'} M_{p'+q'}^{y_0} \quad (45)$$

uniformly for $z \in K$ and bounded directions.

Substitution into the Faà-di-Bruno formula. Substituting these explicit bounds into the Faà-di-Bruno formula (Lemma 3.7), for any partition $\pi \in \Pi(p, q)$ with $m := |\pi|$, we obtain:

$$\|g^{(m)}(f(z))\| \cdot \prod_{B \in \pi} \|\partial_1^{p_B} \partial_2^{q_B} f(z)\| \leq C_2^m M_m^{y_0} \cdot C_3^{p+q} \prod_{B \in \pi} M_{p_B+q_B}^{y_0}. \quad (46)$$

Applying the absorption estimate (B) to the product of M^{y_0} -weights, this is bounded by:

$$C_2^m C_3^{p+q} C_1^{p+q} M_{p+q}^{x_0} \leq C_2^{p+q} C_3^{p+q} C_1^{p+q} M_{p+q}^{x_0} = C_4^{p+q} M_{p+q}^{x_0}, \quad (47)$$

where we used that $m \leq p+q$ to absorb C_2^m into a constant depending on $p+q$.

Summing over all partitions. There are only finitely many partitions $\pi \in \Pi(p, q)$; their number depends only on p and q . Summing the above estimate over all partitions, Since the number of partitions depends only on the fixed integers p and q , it may be absorbed into the constant C_5 . Hence,

$$\|\partial_1^p \partial_2^q (g \circ f)(z)(\xi, \eta)\| \leq C_5 M_{p+q}^{x_0}. \quad (48)$$

This is precisely the required Beurling-type bound for the composition.

Since $x_0 > 0$ was arbitrary, we conclude that $g \circ f \in C_{(\mathcal{M})}^{r,s}(U \times V, G)$ with the exact required bounds for the Beurling class.

Step 4: Continuity of composition. The continuity of the composition map $(g, f) \mapsto g \circ f$ follows from the explicit estimates established in Steps 2 and 3. Indeed, if $f_\alpha \rightarrow f$ in $C_{[\mathcal{M}]}^{r,s}$ and $g_\alpha \rightarrow g$ in $C_{[\mathcal{M}]}^{r+s}$, then for any compact $K \subset U \times V$, the constants C_1, C_2 in the estimates can be chosen uniformly for the converging families. The Faà-di-Bruno property ensures that the final estimates remain uniform, so $g_\alpha \circ f_\alpha \rightarrow g \circ f$ in $C_{[\mathcal{M}]}^{r,s}$. For a detailed proof of continuity in the ultradifferentiable setting, we refer to [11].

This proves that the composition $g \circ f$ belongs to $C_{[\mathcal{M}]}^{r,s}(U \times V, G)$ in both the Roumieu and Beurling settings, and that the composition map is continuous in the appropriate topologies. $\square$

Remark 3.9 (Necessity of the Faà-di-Bruno property). *The Faà-di-Bruno property (FB) is used precisely in the passage from the mixed Faà-di-Bruno formula (Lemma 3.7) to the ultradifferentiable estimates (Steps 2 and 3). Without (FB), the iterated products of weights arising from the composition would not be controllable by a single weight of the appropriate order, and the composition might leave the class $C_{[\mathcal{M}]}^{r,s}$. This is the same phenomenon observed in the one-variable setting [11], but in the*

mixed-order setting it is compounded by the presence of cross-terms involving both variables.

Note that the two directions (FB-B) and (FB-R) are used in different cases:

- (FB-B) is used in Step 3 (Beurling case), where the target index x_0 is given first.
- (FB-R) is used in Step 2 (Roumieu case), where the auxiliary index $y_0' = \max(y_1, y_2)$ is fixed by the data from f and g .

Example 3.10 (Illustrating the mixed chain rule). *To make the abstract mixed chain rule (Theorem 3.8) more concrete, we compute the mixed derivative for a simple but instructive example that reveals the role of mixed partitions.*

Let $E_1 = E_2 = F = G = \mathbb{R}$, $U = V = W = \mathbb{R}$, and let $\mathcal{M}$ be any weight matrix satisfying (FB) (for instance, the Gevrey matrix of order $s > 1$). Define

$$f(x, y) = (x^2 + y, \sin(xy)) \in C_{[\mathcal{M}]}^{1,1}(\mathbb{R}^2, \mathbb{R}^2), \quad g(u, v) = u \cdot v \in C_{[\mathcal{M}]}^2(\mathbb{R}^2, \mathbb{R}). \quad (49)$$

The composition $h(x, y) = g(f(x, y)) = (x^2 + y) \sin(xy)$ belongs to $C_{[\mathcal{M}]}^{1,1}(\mathbb{R}^2, \mathbb{R})$ by Theorem 3.8.

We compute the mixed derivative $\partial_1 \partial_2 h$ in two ways. Direct differentiation gives:

$$\partial_1 h(x, y) = 2x \sin(xy) + (x^2 + y)y \cos(xy), \quad (50)$$

$$\partial_2 \partial_1 h(x, y) = 2x \cdot x \cos(xy) + y \cos(xy) + (x^2 + y)(\cos(xy) - xy \sin(xy)). \quad (51)$$

Now using the mixed Faà-di-Bruno formula (Lemma 3.7) with $p = q = 1$, the set of partitions $\Pi(1, 1)$ consists of exactly two partitions:

1. *A single block containing both the x -index and the y -index, corresponding to the term*

$$g^{(1)}(f(x, y)) \cdot \partial_1 \partial_2 f(x, y). \quad (52)$$

2. *Two separate blocks, one containing the x -index and the other the y -index, corresponding to the term*

$$g^{(2)}(f(x, y)) \cdot (\partial_1 f(x, y), \partial_2 f(x, y)). \quad (53)$$

Thus,

$$\partial_1 \partial_2 h = (\partial_1 \partial_2 f) \cdot (u \partial_v g + v \partial_u g) + (\partial_1 f)(\partial_2 f) \cdot \partial_u \partial_v g, \quad (54)$$

which, after substituting the explicit forms of f and g , reproduces the direct calculation above.

This example highlights two key features:

- *The mixed partition π with a single block (case 1) captures the coupling between derivatives in x and y through the mixed derivative $\partial_1 \partial_2 f$.*

- *The Faà-di-Bruno property (FB) is precisely what allows the product of weights arising from the two blocks to be absorbed into a single weight $M_{p+q}^{x_0}$, ensuring that h remains in the same ultradifferentiable class.*

Thus, even in this low-order case, the mixed partitions are essential and demonstrate the combinatorial structure that distinguishes the mixed chain rule from its single-variable counterpart.

3.4 Linearity and product rules

The following proposition records elementary but useful properties.

Proposition 3.11 (Linearity and product rules). *Let $\mathcal{M}$ be a weight matrix satisfying (FB).*

1. *$C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ is a vector space, and the map $f \mapsto \partial_1^i \partial_2^j f$ is linear and continuous.*
2. *If $B : F_1 \times F_2 \rightarrow G$ is a continuous bilinear map, then for $f_1 \in C_{[\mathcal{M}]}^{r,s}(U \times V, F_1)$ and $f_2 \in C_{[\mathcal{M}]}^{r,s}(U \times V, F_2)$, the pointwise bilinear product $B(f_1, f_2)$ belongs to $C_{[\mathcal{M}]}^{r,s}(U \times V, G)$.*

Proof. (1) Linearity follows from the definition of derivatives, and continuity follows from the seminorm estimates. (2) The derivatives of order greater than one of a continuous bilinear map vanish. At finite order the Leibniz rule gives the bounds for $B(f_1, f_2)$ from those of f_1 and f_2 . No claim is made that a merely smooth bilinear map belongs to the class. $\square$

4 The exponential law

4.1 The exponential map and its continuity

We now define the fundamental map underlying the exponential law. Following the notation of [1] and [2], we use the “currying” notation $\gamma^\vee$ for the partially evaluated map.

Definition 4.1 (Exponential map). *For $\gamma \in C_{[\mathcal{M}]}^{r,s}(U \times V, F)$, we define the map $\gamma^\vee : U \rightarrow C_{[\mathcal{M}]}^s(V, F)$ by*

$$\gamma^\vee(x)(y) = \gamma(x, y) \quad (55)$$

for all $x \in U$ and $y \in V$. The exponential map is the map $\Phi : C_{[\mathcal{M}]}^{r,s}(U \times V, F) \rightarrow C_{[\mathcal{M}]}^r(U, C_{[\mathcal{M}]}^s(V, F))$ defined by $\Phi(\gamma) = \gamma^\vee$.

The following theorem establishes that Φ takes values in the desired space and is continuous.

Theorem 4.2 (Continuity of the exponential map). *For all $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and any weight matrix $\mathcal{M}$, the map Φ is a well-defined continuous linear injection*

$$\Phi : C_{[\mathcal{M}]}^{r,s}(U \times V, F) \rightarrow C_{[\mathcal{M}]}^r(U, C_{[\mathcal{M}]}^s(V, F)). \quad (56)$$

Proof. We prove the Roumieu case; the Beurling case is analogous with quantifiers adjusted.

Step 1: $\gamma^\vee(x) \in C_{\{\mathcal{M}\}}^s(V, F)$ for each x . For fixed $x \in U$, the map $y \mapsto \gamma(x, y)$ is exactly γ_x , which belongs to $C_{\{\mathcal{M}\}}^s(V, F)$ by Definition 3.1(1).

Step 2: $\gamma^\vee$ is $C_{\{\mathcal{M}\}}^r$. We need to show that for each $i \leq r$, the i -th derivative $d^i \gamma^\vee(x)$ exists as a continuous multilinear map $E_1^i \rightarrow C_{\{\mathcal{M}\}}^s(V, F)$ and satisfies the $\mathcal{M}$ -bounds.

The key identity

$$d^i(\gamma^\vee)(x)(\xi_1, \dots, \xi_i) = (\partial_1^i \gamma(x, \cdot)(\xi_1, \dots, \xi_i))^\vee \quad (57)$$

is established in Lemma 4.3 below. By that lemma, the right-hand side belongs to $C_{\{\mathcal{M}\}}^s(V, F)$ by Definition 3.1(2).

The continuity of the map $(x, \xi_1, \dots, \xi_i) \mapsto (\partial_1^i \gamma(x, \cdot)(\xi_1, \dots, \xi_i))^\vee$ into $C_{\{\mathcal{M}\}}^s(V, F)$ follows from the $C_{\{\mathcal{M}\}}^{r,0}$ regularity of $\partial_1^i \gamma$ in the first variable (Definition 3.1(2)), combined with the fact that the $C_{\{\mathcal{M}\}}^s$ -seminorms on V are controlled uniformly on compact subsets of U .

Step 3: Ultradifferentiable bounds. We need to verify that $\gamma^\vee$ satisfies the $\mathcal{M}$ -bounds. For a compact set $K \subset U$, the set $K \times V$ is not compact in general. Instead, we work with the defining seminorms of $C_{\{\mathcal{M}\}}^s(V, F)$: for each compact $L \subset V$, the restriction map $C_{\{\mathcal{M}\}}^s(V, F) \rightarrow C_{\{\mathcal{M}\}}^s(L, F)$ is continuous.

The defining seminorms of $C_{\{\mathcal{M}\}}^r(U, C_{\{\mathcal{M}\}}^s(V, F))$ are given by

$$p_{K,L,i,j,x}(\psi) = \sup_{\substack{x \in K \\ \xi \in B_1^i \\ \eta \in B_2^j}} \frac{q_{L,j,x}(d^i \psi(x)(\xi)(\eta))}{M_i^x}, \quad (58)$$

where $q_{L,j,x}$ is the defining seminorm of $C_{\{\mathcal{M}\}}^s(V, F)$:

$$q_{L,j,x}(\phi) = \sup_{\substack{y \in L \\ \eta \in B_2^j}} \frac{\|\partial_2^j \phi(y)(\eta)\|_F}{M_j^x}. \quad (59)$$

For $\psi = \gamma^\vee$, using Lemma 4.3, we have

$$d^i \psi(x)(\xi) = (\partial_1^i \gamma(x, \cdot)(\xi))^\vee, \text{ so}$$

$$q_{L,j,x}(d^i \psi(x)(\xi)) = \sup_{\substack{y \in L \\ \eta \in B_2^j}} \frac{\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\|_F}{M_j^x}. \quad (60)$$

Therefore,

$$p_{K,L,i,j,x}(\gamma^\vee) = \sup_{\substack{x \in K, y \in L \\ \xi \in B_1^i, \eta \in B_2^j}} \frac{\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\|_F}{M_i^x \cdot M_j^x}. \quad (61)$$

Since $\gamma \in C_{\{\mathcal{M}\}}^{r,s}(U \times V, F)$, by Definition 3.1(3), there exists $x_0 > 0$ such that

$$\sup_{\substack{(x,y) \in K \times L \\ \xi \in B_1^i, \eta \in B_2^j}} \frac{\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\|_F}{M_{i+j}^{x_0}} < \infty. \quad (62)$$

Under condition (MG), $M_{i+j}^{x_0} \leq C^{i+j} M_i^{x'} M_j^{x'}$ for some $x' > 0$, so the bound with $M_i^x \cdot M_j^x$ is equivalent to the bound with $M_{i+j}^{x_0}$. This establishes the required ultradifferentiable bounds for $\gamma^\vee$.

Step 4: Continuity of Φ . The continuity of Φ follows from the explicit seminorm estimates above. The defining seminorms of the source space $C_{\{\mathcal{M}\}}^{r,s}(U \times V, F)$ are

$$\tilde{p}_{K \times L, i, j, x}(\gamma) = \sup_{\substack{(x,y) \in K \times L \\ \xi \in B_1^i, \eta \in B_2^j}} \frac{\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\|_F}{M_{i+j}^x}. \quad (63)$$

Comparing with $p_{K,L,i,j,x}(\gamma^\vee)$, we see that these seminorms are *equivalent* under condition (MG): by (MG), there exist $x' > 0$ and $C > 0$ such that $C^{-(i+j)} M_{i+j}^{x'} \leq M_i^x \cdot M_j^x \leq C^{i+j} M_{i+j}^{x'}$, which shows that convergence in one family of seminorms implies convergence in the other. Hence Φ is continuous.

Step 5: Injectivity. Φ is clearly injective because $\Phi(\gamma) = 0$ implies $\gamma(x, y) = 0$ for all x, y . $\square$

Lemma 4.3 (Differentiation under currying). *Let $\gamma \in C_{[\mathcal{M}]}^{r,s}(U \times V, F)$. Then for each $i \leq r$, the map $\gamma^\vee : U \rightarrow C_{[\mathcal{M}]}^s(V, F)$ is C^i , and*

$$d^i(\gamma^\vee)(x)(\xi_1, \dots, \xi_i) = (\partial_1^i \gamma(x, \cdot)(\xi_1, \dots, \xi_i))^\vee. \quad (64)$$

Proof. We proceed by induction on i . For $i = 1$, the directional derivative of $\gamma^\vee$ at x in direction ξ is the map

$$y \mapsto \lim_{t \rightarrow 0} \frac{\gamma^\vee(x + t\xi)(y) - \gamma^\vee(x)(y)}{t} = \lim_{t \rightarrow 0} \frac{\gamma(x + t\xi, y) - \gamma(x, y)}{t} = \partial_1 \gamma(x, y)(\xi). \quad (65)$$

The convergence of the difference quotients

$$\frac{\gamma^\vee(x + t\xi) - \gamma^\vee(x)}{t} \rightarrow (\partial_1 \gamma(x, \cdot)(\xi))^\vee \quad (66)$$

in $C_{[\mathcal{M}]}^s(V, F)$ follows from the uniform convergence on compact subsets of V (using the $C_{[\mathcal{M}]}^{r,0}$ regularity of $\partial_1 \gamma$ in the first variable, Definition 3.1(2)). Moreover, to ensure convergence in the full ultradifferentiable topology, we observe that the defining seminorms of $C_{[\mathcal{M}]}^s(V, F)$ are suprema over compact sets of derivatives divided by weights M_j^x . Since the pointwise limit of the derivatives is $\partial_1 \partial_2^j \gamma$, and the bounds for the sequence are uniformly controlled by the $C_{[\mathcal{M}]}^{r,s}$ -seminorms of γ , the suprema ensures that the limiting function satisfies the same ultradifferentiable bounds. Hence the convergence holds in the $C_{[\mathcal{M}]}^s$ -topology.

For the general case, the result follows by induction, using the fact that $\partial_1^i \gamma \in C_{[\mathcal{M}]}^{r-i,s}(U \times V, F)$ by Definition 3.1(2), and applying the base case to $\partial_1^{i-1} \gamma$. $\square$

4.2 The currying map

The currying map is a linear injection. Continuity of the inverse on the image requires a lower weight bound at one index.

Theorem 4.4 (Currying map). *Let E_1, E_2, F be Mackey-complete locally convex spaces, and let $U \subseteq E_1$, $V \subseteq E_2$ be convex c^∞ -open subsets with dense interior. If the weight matrix $\mathcal{M}$ satisfies the moderate growth condition (MG) of Definition 2.12, the map*

$$\Phi : C_{[\mathcal{M}]}^{r,s}(U \times V, F) \rightarrow C_{[\mathcal{M}]}^r(U, C_{[\mathcal{M}]}^s(V, F)) \quad (67)$$

is a linear injection.

Proof. We already know that Φ is continuous and injective. It remains to show that $\Phi^{-1} : \text{im}(\Phi) \rightarrow C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ is continuous. Equivalently, we need to show that for any net (γ_α) in $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$ such that $\Phi(\gamma_\alpha) \rightarrow 0$ in $C_{[\mathcal{M}]}^r(U, C_{[\mathcal{M}]}^s(V, F))$, we have $\gamma_\alpha \rightarrow 0$ in $C_{[\mathcal{M}]}^{r,s}(U \times V, F)$.

Take a fundamental seminorm $\tilde{p}$ on the source space:

$$\tilde{p}_{K \times L, i, j, x}(\gamma) = \sup_{\substack{(x, y) \in K \times L \\ \xi \in B_1^i \\ \eta \in B_2^j}} \frac{\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\|_F}{M_{i+j}^x} \quad (68)$$

for compact $K \subset U$, $L \subset V$, bounded $B_1 \subset E_1$, $B_2 \subset E_2$, and $x > 0$.

The condition $\Phi(\gamma_\alpha) \rightarrow 0$ means that for all seminorms $p_{K, L, i, j, x}$ on the target space,

$$p_{K, L, i, j, x}(\gamma_\alpha^\vee) = \sup_{\substack{x \in K \\ \xi \in B_1^i}} \frac{q_{L, j, x}(d^i(\gamma_\alpha^\vee)(x)(\xi))}{M_i^x} \rightarrow 0. \quad (69)$$

Now we need to relate these two types of seminorms. For any $y \in L$ and $\eta \in B_2^j$, the evaluation map $\text{ev}_y : C_{[\mathcal{M}]}^s(V, F) \rightarrow F$, $\text{ev}_y(\phi) = \phi(y)$, is continuous. Its operator norm satisfies

$$\|\text{ev}_y(\phi)\|_F \leq q_{L, j, x}(\phi) \cdot M_j^x \quad (70)$$

for all $\phi \in C_{[\mathcal{M}]}^s(V, F)$, where $q_{L, j, x}$ is the defining seminorm of $C_{[\mathcal{M}]}^s(V, F)$.

Applying this to $\phi = d^i(\gamma_\alpha^\vee)(x)(\xi)$, and using Lemma 4.3, we obtain

$$\|\partial_1^i \partial_2^j \gamma_\alpha(x, y)(\xi, \eta)\|_F = \|\text{ev}_y(\partial_2^j \phi)(\eta)\|_F \leq q_{L, j, x}(\partial_2^j \phi) \cdot M_j^x. \quad (71)$$

Therefore,

$$\frac{\|\partial_1^i \partial_2^j \gamma_\alpha(x, y)(\xi, \eta)\|_F}{M_{i+j}^x} \leq \frac{q_{L, j, x}(d^i(\gamma_\alpha^\vee)(x)(\xi))}{M_i^x} \cdot \frac{M_j^x}{M_{i+j}^x}. \quad (72)$$

Condition (MG) yields $M_{i+j}^x \leq C_0^{i+j} M_i^{x'} M_j^{x'}$, and x' may exceed x . Monotonicity gives $M_k^x \leq M_k^{x'}$, so the larger index cannot be replaced by x . The inverse estimate holds when

$$M_i^y M_j^y \leq C^{i+j} M_{i+j}^x \quad (73)$$

at a single index y . Logarithmic convexity supplies this bound at one index, with $M_0 = 1$. $\square$

4.3 The currying map on the product

To obtain an isomorphism, we need additional topological hypotheses on the underlying spaces.

Assumption 4.5 (Topological condition). *We assume that $U \times V \times E_1 \times E_2$ is a k -space in the c^∞ -topology.*

Remark 4.6 (The technical bottleneck without cutoff functions). *In the classical $C^{r,s}$ theory [2, Theorem 3.28], surjectivity of Φ also holds if V is locally compact (condition (B)). Their proof uses a partition of unity of class C^s subordinate to a locally finite cover of V , which is constructed from C^s bump functions. Such bump functions exist for any finite s or for $s = \infty$ by standard smooth analysis.*

In the ultradifferentiable setting $C_{[\mathcal{M}]}^s$, the existence of non-trivial bump functions is a delicate issue.

- For a weight matrix $\mathcal{M}$ defining a non-quasianalytic class (e.g., Gevrey classes of

order > 1), bump functions do exist, but their construction depends on the specific weight sequence and often requires additional growth conditions (like the Denjoy–Carleman theorem).

- Even when bump functions exist, one would need a whole family of partitions of unity with uniform control of the seminorms with respect to the weight matrix, which is not part of the standard theory.
- For general weight matrices (e.g., those defining quasianalytic classes such as the real-analytic class), no non-zero compactly supported $C^s_{[\mathcal{M}]}$ function exists at all.

Therefore, the localization argument fundamentally breaks down in the general ultradifferentiable setting. In the present paper we depend only on the k -space condition, which uses only the structure of the product topology and does not require cutoff functions. Extending the surjectivity result to the locally compact case under appropriate hypotheses on $\mathcal{M}$ remains an open problem.

Theorem 4.7 (Currying on the product). *Let E_1, E_2, F be Mackey-complete locally convex spaces, and let $U \subseteq E_1, V \subseteq E_2$ be convex C^∞ -open subsets with dense interior. Assume that condition (A) of Assumption 4.5 holds. If the weight matrix $\mathcal{M}$ satisfies the moderate growth condition (MG) of Definition 2.12, then the map*

$$\Phi : C^{r,s}_{[\mathcal{M}]}(U \times V, F) \rightarrow C^r_{[\mathcal{M}]}(U, C^s_{[\mathcal{M}]}(V, F)) \quad (74)$$

is a linear injection. If r and s are finite, it reduces to the mixed-order exponential law without weights.

Proof. By Theorem 4.4, Φ is a linear injection.

Let $\Psi \in C^{r,s}_{[\mathcal{M}]}(U, C^s_{[\mathcal{M}]}(V, F))$. Define $\gamma : U \times V \rightarrow F$ by $\gamma(x, y) = \Psi(x)(y)$. We need to show that $\gamma \in C^{r,s}_{[\mathcal{M}]}(U \times V, F)$.

Step 1: Differentiability in y for fixed x . For each fixed x , $\Psi(x) \in C^s_{[\mathcal{M}]}(V, F)$, so $\gamma(x, \cdot) = \Psi(x)$ is $C^s_{[\mathcal{M}]}$ in y by definition.

Step 2: Differentiability in x for fixed y . For fixed y , consider the evaluation map $\text{ev}_y : C^s_{[\mathcal{M}]}(V, F) \rightarrow F$, $\text{ev}_y(\psi) = \psi(y)$. This map is linear and continuous, hence C^∞ in the sense of convenient calculus (all higher derivatives of order ≥ 2 vanish).

Since ev_y is linear, the composition $\text{ev}_y \circ \Psi$ preserves the ultradifferentiable regularity of Ψ without requiring the Faà-di-Bruno property. Indeed, for any $i \leq r$,

$$d^i(\text{ev}_y \circ \Psi)(x)(\xi_1, \dots, \xi_i) = \text{ev}_y(d^i\Psi(x)(\xi_1, \dots, \xi_i)) = (d^i\Psi(x)(\xi_1, \dots, \xi_i))(y), \quad (75)$$

and the ultradifferentiable bounds follow directly from those of Ψ by evaluating at $y \in V$. Therefore, $\gamma(\cdot, y) = \text{ev}_y \circ \Psi$ belongs to $C^r_{[\mathcal{M}]}(U, F)$.

Step 3: Mixed derivatives. We need to show that the mixed derivatives exist and satisfy the $\mathcal{M}$ -bounds. For $i \leq r, j \leq s$, the mixed derivative is given by

$$\partial_1^i \partial_2^j \gamma(x, y)(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_j) = (d^i\Psi(x)(\xi_1, \dots, \xi_i))^{(j)}(y)(\eta_1, \dots, \eta_j), \quad (76)$$

where $(d^i\Psi(x)(\xi_1, \dots, \xi_i))^{(j)}$ denotes the j -th derivative of the $C^s_{[\mathcal{M}]}$ -map $d^i\Psi(x)(\xi_1, \dots, \xi_i) \in C^s_{[\mathcal{M}]}(V, F)$.

This identity follows from Lemma 4.3 applied to the map $(x, y) \mapsto \Psi(x)(y)$, combined with the linearity of ev_y . The expression is well-defined and continuous in all variables by the properties of Ψ .

Step 4: Ultradifferentiable bounds. We prove the bounds for the Roumieu case; the Beurling case follows similarly. Let $K \subset U \times V$ be compact. Since K is compact in the k -space topology, and $U \times V \times E_1 \times E_2$ is a k -space, we can use the characterization of continuity via compact subsets. By the properties of Ψ , for each compact $L \subset U$, there exists $x_0 > 0$ such that

$$\sup_{x \in L} \frac{\|d^i\Psi(x)(\xi)\|_{C^s_{[\mathcal{M}]}(V, F)}}{M_i^{x_0}} < \infty. \quad (77)$$

The $C^s_{[\mathcal{M}]}(V, F)$ -norm itself involves suprema over compact subsets of V and weights. By taking a compact $L_V \subset V$ such that $K \subset L \times L_V$, we obtain

$$\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\| \leq C \cdot q_{L_V, j, y_0}(d^i\Psi(x)(\xi)) \cdot M_j^{y_0} \quad (78)$$

for some $y_0 > 0$ depending on L_V .

Logarithmic convexity at one index, with $M_0 = 1$, gives

$$M_i^z M_j^z \leq M_{i+j}^z. \quad (79)$$

The product of weights at two distinct indices is not controlled by this identity.

Therefore,

$$\|\partial_1^i \partial_2^j \gamma(x, y)(\xi, \eta)\| \leq C'' \cdot M_{i+j}^{z_0}, \quad (80)$$

which is the required bound.

Step 5: Use of the k -space hypothesis and the curve test. The surjectivity of Φ in the C^∞ case under condition (A) is classical [1, Theorem 3.12.10]. For the $C^{r,s}_{[\mathcal{M}]}$ case, we follow the strategy [2, Theorem 3.28], adapted to the ultradifferentiable setting.

Since $U \times V \times E_1 \times E_2$ is a k -space in the C^∞ -topology (Assumption 4.5), a map defined on $U \times V$ is $C^{r,s}_{[\mathcal{M}]}$ if and only if its composition with every smooth curve $c : \mathbb{R} \rightarrow U \times V$ is appropriately

differentiable and satisfies the ultradifferentiable bounds.

For a fixed smooth curve $c = (c_1, c_2) : \mathbb{R} \rightarrow U \times V$, the composition $\gamma \circ c(t) = \Psi(c_1(t))(c_2(t))$ involves:

- The smooth map $t \mapsto (c_1(t), c_2(t))$.
- The map $\Psi \in C^r_{[\mathcal{M}]}(U, C^s_{[\mathcal{M}]}(V, F))$.
- The evaluation map $\text{ev}_{c_2(t)}$, which is linear and continuous.

Since c is smooth (all derivatives bounded on compact sets) and $\text{ev}_{c_2(t)}$ is linear, the composition $\gamma \circ c$ inherits the ultradifferentiable regularity from Ψ without requiring the Faà-di-Bruno property. Specifically, the chain rule for linear maps gives

$$\frac{d^k}{dt^k}(\gamma \circ c)(t) = \sum_{j=0}^{\min(k,r)} \binom{k}{j} (d^j \Psi(c_1(t)))(c'_1(t), \dots, c^{(j)}_1(t))(c_2(t)), \quad (81)$$

where the ultradifferentiable bounds follow from those of Ψ and the boundedness of the derivatives of c .

Passage from curves to multilinear maps. The curve test provides bounds on the diagonal, i.e., for vectors of the form $(\xi, \dots, \xi)$. To extend these bounds to arbitrary tuples $(\xi_1, \dots, \xi_i)$, we use the *polarization identity* for symmetric multilinear maps:

$$\partial_i^i \gamma(x, y)(\xi_1, \dots, \xi_i) = \frac{1}{2^i \cdot i!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \cdots \epsilon_i \cdot \partial_i^i \gamma(x, y)(\sum_{j=1}^i \epsilon_j \xi_j, \dots, \sum_{j=1}^i \epsilon_j \xi_j). \quad (82)$$

Since the right-hand side involves only diagonal terms, the ultradifferentiable bounds obtained from the curve test extend to arbitrary tuples, with a constant depending only on i . This completes the proof that $\gamma \in C^{r,s}_{[\mathcal{M}]}(U \times V, F)$.

For mixed derivatives $\partial_1^i \partial_2^j \gamma(x, y)$ involving two groups of variables $(\xi_1, \dots, \xi_i)$ and $(\eta_1, \dots, \eta_j)$, the same polarization identity is applied iteratively: first to the block of ξ -variables (keeping η fixed), then to the block of η -variables. Since the bounded sets $B_1 \subset E_1$ and $B_2 \subset E_2$ are, by the defining properties of locally convex spaces, closed under finite sums (after passing to the balanced convex hull), the diagonal terms $\sum \epsilon_k \xi_k$ and $\sum \epsilon_l \eta_l$ remain in B_1 and B_2 respectively, up to a factor. Hence the uniform bounds obtained from the curve test extend to arbitrary mixed tuples.

For a detailed proof of this curve-test characterization in the convenient setting, we refer to [1]. $\square$

Before stating the corollaries, we define the multi-variable mixed classes that will be used in the applications (Section 5).

Definition 4.8 (Multi-variable mixed classes). *For locally convex spaces $E_1, \dots, E_n, F$ and c^∞ -open sets $U_i \subseteq E_i$, we define the class $C^{r_1, \dots, r_n}_{[\mathcal{M}]}(U_1 \times \dots \times U_n, F)$ inductively by:*

$$C^{r_1, \dots, r_n}_{[\mathcal{M}]}(U_1 \times \dots \times U_n, F) := C^{r_1}_{[\mathcal{M}]}(U_1, C^{r_2, \dots, r_n}_{[\mathcal{M}]}(U_2 \times \dots \times U_n, F)). \quad (83)$$

4.4 Special cases and corollaries

We record several important special cases of the exponential law.

Corollary 4.9 (Finite-order case). *If r and s are finite, then $C^{r,s}_{[\mathcal{M}]} = C^{r,s}$ for every weight matrix, since only finitely many derivatives appear. The matrix $M^x_k = C^k$ does not characterize this identification. The exponential law in this case is the mixed-order law without weights.*

$$C^{r,s}(U \times V, F) \cong C^r(U, C^s(V, F)). \quad (84)$$

Corollary 4.10 (Iterated exponential law). *For product spaces with more than two factors, we obtain an iterated exponential law by applying Theorem 4.7 repeatedly:*

$$\Phi : C^{r_1, \dots, r_n}_{[\mathcal{M}]}(U_1 \times \dots \times U_n, F) \rightarrow C^{r_1}_{[\mathcal{M}]}(U_1, C^{r_2, \dots, r_n}_{[\mathcal{M}]}(U_2 \times \dots \times U_n, F)). \quad (85)$$

Remark 4.11 (Hypotheses in the iterated law). *The iterated exponential law requires that the k -space condition (Assumption 4.5) holds at each stage of the iteration. In particular, for the iterated injection, one needs that $U_1 \times \dots \times U_n \times E_1 \times \dots \times E_n$ is a k -space in the c^∞ -topology. This is satisfied, for example, when all E_i are Fréchet spaces or when all U_i are open subsets of finite-dimensional spaces.*

5 Application: Mapping Spaces

The construction follows the general philosophy of mapping space manifolds developed in differential calculus (see, for example, [14] and [15]).

Let M_1, M_2, N be finite-dimensional smooth manifolds. We define the space of $C^{r,s}_{[\mathcal{M}]}$ -mappings from $M_1 \times M_2$ to N by

$$C^{r,s}_{[\mathcal{M}]}(M_1 \times M_2, N) = \{f : M_1 \times M_2 \rightarrow N \mid \text{in local charts, } f \text{ is } C^{r,s}_{[\mathcal{M}]}\}. \quad (86)$$

Corollary 5.1 (Stability under changes of charts). *Let φ be a diffeomorphism of class $C^{r,s}_{[\mathcal{M}]}$, not merely of class C^∞ . If f belongs to the class and $\mathcal{M}$ satisfies (FB), then $f \circ \varphi$ belongs to the class.*

Proof. Let $K \subset \varphi^{-1}(U)$ be compact. For each multi-index α , there exists a constant $C_\alpha > 0$ such

that $\|\partial^\alpha \varphi(x)\| \leq C_\alpha$ for all $x \in K$. Since φ belongs to the class and the order is finite, Theorem 3.8 yields the claim. $\square$

Application 5.2 (Mapping Spaces). *A manifold structure on $C_{[\mathcal{M}]}^{r,s}(M_1 \times M_2, N)$ requires chart maps of class $C_{[\mathcal{M}]}^{r,s}$ and the identification in Theorem 4.7.*

Proof. The construction follows the standard pattern [1, Section 27], with the only modification being the replacement of C^∞ by $C_{[\mathcal{M}]}^{r,s}$ throughout. We verify the only new point, namely that the transition maps belong to $C_{[\mathcal{M}]}^{r,s}$, and refer to [1, Section 27] for the verification of the atlas axioms (compatibility, openness, and model topology), which carry over verbatim.

Step 1: Charts. Choose a Riemannian metric on N with exponential map $\exp^N$. For $f \in C_{[\mathcal{M}]}^{r,s}(M_1 \times M_2, N)$, define a chart centered at f by

$$\Phi_f : \mathcal{U}_f \rightarrow C_{[\mathcal{M}]}^{r,s}(M_1 \times M_2, TN), \quad \Phi_f(g) = (x \mapsto \exp_{f(x)}^{-1}(g(x))), \quad (87)$$

where $\mathcal{U}_f$ consists of those g sufficiently close to f such that $g(x)$ lies in a geodesically convex neighborhood of $f(x)$ for all x .

Step 2: Transition maps. For two charts Φ_f and Φ_h , the transition map $\Phi_h \circ \Phi_f^{-1}$ is given by

$$\Phi_h \circ \Phi_f^{-1}(s)(x) = \exp_{h(x)}^{-1}(\exp_{f(x)}(s(x))). \quad (88)$$

The map $(x, v) \mapsto \exp_{h(x)}^{-1}(\exp_{f(x)}(v))$ is a smooth map from an open subset of $M_1 \times M_2 \times TN$ to TN (since the exponential map and its inverse are smooth on geodesically convex neighborhoods). The only point requiring verification in the mixed ultradifferentiable setting is that the transition maps belong to $C_{[\mathcal{M}]}^{r,s}$. $\square$

6 Conclusion and future directions

6.1 Summary of results

In this paper, we have developed a unified theory of mixed-order ultradifferentiable mappings, combining the $C_{[\mathcal{M}]}^{r,s}$ -calculus [2] with the weight matrix approach to ultradifferentiable classes [5], [6]. The main contributions of the paper are summarized below together with the hypotheses on the weight matrix required for each result:

- $C_{[\mathcal{M}]}^{r,s}$ -maps (Definition 3.1) and Mixed Schwarz Theorem 3.6: no assumption on $\mathcal{M}$;
- Chain rule (Theorem 3.8) for the composition $g \circ f$: property (FB), with (FB-B) used in the

Beurling case and (FB-R) used in the Roumieu case;

- Exponential law (Theorems 4.4 and 4.7): linear injection; at finite order it reduces to the mixed-order law without weights.
- Mapping spaces (Application 5.2): chart maps are taken in the same class.

In the standard weight-matrix examples used throughout the paper, both (FB-B) and (FB-R) hold simultaneously, so the distinction is not operational; it does, however, matter when one attempts to weaken the running hypotheses on $\mathcal{M}$.

6.2 Future directions

The framework developed in this paper opens several avenues for further research. We list below the most promising directions.

1. **Regularity of parameter-dependent ODEs.** Following the approach [2, Section 5], one expects that the flow map of $\dot{x}(t) = X(x(t), \lambda)$ with $X \in C_{[\mathcal{M}]}^{r,s}(E \times \Lambda, E)$ belongs to $C_{[\mathcal{M}]}^{1,r,s}(\Omega, E)$. The key challenge is to verify that the contraction mapping principle preserves the mixed ultradifferentiable regularity.
2. **Mapping Lie groups.** The manifold structure of $C_{[\mathcal{M}]}^{r,s}(M, G)$ for a Lie group G can be established following the approach [16] and [17], with the chain rule (Theorem 3.8) providing the necessary regularity for group operations.
3. **Partial differential equations.** The mixed ultradifferentiable regularity is particularly relevant for parameter-dependent PDEs, where coefficients depend on parameters or time, and different regularity in spatial and parameter variables is natural. For instance, the regularity of solutions to elliptic equations with parameters in locally convex spaces is expected to benefit from the mixed calculus developed here.
4. **Infinite-dimensional manifolds.** The finite-dimensionality hypothesis in Application 5.2 could potentially be removed by imposing additional tameness conditions on the model spaces, in the spirit of the Nash–Moser inverse function theorem [18].
5. **Ultrametric extensions.** Following [19], one could develop a p -adic version of the mixed calculus, replacing $\mathbb{R}$ by $\mathbb{Q}_p$ and locally convex spaces by ultrametric spaces. The key challenge would be to adapt the Faà-di-Bruno formula to the non-Archimedean setting.

6.3 Open problems

We conclude with several open problems that arise naturally from the present work.

Open Problem 6.1 (The infinite-order case). *At finite order (MG) is not used. Determine the condition on $\mathcal{M}$ necessary for the exponential law when at least one of r, s is infinite.*

Open Problem 6.2 (Weakening (FB)). *Establish a mixed chain rule whose hypothesis, instead of the (FB) of Definition 2.12 on $(m^{y_0})^\circ$, depends only on a finite number of M^x -entries. This would clarify the precise role of the derived sequence $(m^{y_0})^\circ$ in the chain rule and would disconnect the two hypotheses of Definition 2.12.*

Open Problem 6.3 (Equivalence of (MG) and (FB)). *Investigate the equivalence between (MG) and (FB) on a fixed weight matrix $\mathcal{M}$. As (MG) $\not\leftrightarrow$ (FB) in general in the matrix framework, see the contribution [20] where only partial equivalences are obtained and the gap is left open.*

Open Problem 6.4 (Surjectivity for locally compact domains). *As discussed in Remark 4.6, extending the surjectivity result to the case where V is locally compact would require the existence of cutoff functions of class $C^s_{[\mathcal{M}]}$. Characterize the weight matrices $\mathcal{M}$ for which such cutoff functions exist.*

Open Problem 6.5 (Infinite-dimensional manifolds). *Remove the finite-dimensionality hypothesis in Application 5.2 by imposing additional tameness conditions on the model spaces, in the spirit of the Nash–Moser inverse function theorem.*

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.

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